Fractional diffusion limit for a kinetic equation in the upper-half space with diffusive boundary conditions

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Abstract

We investigate the fractional diffusion approximation of a kinetic equation set in the upper-half space with diffusive reflection conditions at the boundary. In an appropriate singular limit corresponding to small Knudsen number and long time asymptotic, we derive a fractional diffusion equation with a nonlocal Neumann boundary condition for the density of particles. Interestingly, this asymptotic equation is different from the one derived by L. Cesbron in [8] in the case of specular reflection conditions at the boundary and does not seem to have received a lot of attention previously.

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1 Introduction

The purpose of this paper is to investigate the fractional diffusion approximation of a linear kinetic equation set on a bounded domain with diffusive boundary conditions. The starting point of our

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analysis is the following linear Boltzmann equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q(f) & \text{for } (t,x,v) \in (0,\infty) \times \Omega \times \mathbb{R}^N \\ f(0,x,v) = f_{in}(x,v) & \text{for } (x,v) \in \Omega \times \mathbb{R}^N \end{cases} \tag{1}$$

where Ω is a subset of $\mathbb{R}^N$ and Q is the linear Boltzmann Operator

$$\begin{aligned} Q(f)(v) &= \int_{\mathbb{R}^N} \sigma(v,w)\big[F(v)f(w) - F(w)f(v)\big]\,\mathrm{d}w \\ &= K(f)(v) - \nu(v)f(v). \end{aligned}$$

Throughout this paper, the thermodynamical equilibrium $F(v) = \tilde{F}(|v|^2) \geq 0$ will be a normalized heavy-tail distribution function satisfying

$$\int_{\mathbb{R}^N} F(v)\,\mathrm{d}v = 1, \qquad F(v) \sim \frac{\gamma}{|v|^{N+2s}} \text{ as } |v| \to \infty, \qquad s \in (1/2, 1) \tag{2}$$

and, to avoid unnecessarily complicated notations in the proof, we will assume that the cross section $\sigma(v,w)$ is constant equal to ν_0 throughout the rigorous part of the paper. However, the result holds without modifications if we assume instead that $\sigma(v,w)$ is bounded above and below and symmetric:

$$0 < \sigma_0 \leq \sigma(v,w) \leq \sigma_1, \qquad \sigma(v,w) = \sigma(w,v) \quad \text{for all } (v,w) \in \mathbb{R}^N \times \mathbb{R}^N$$

and if the collision frequency $\nu(v) = \displaystyle\int_{\mathbb{R}^N} \sigma(v,w)F(w)\,\mathrm{d}w$ satisfies $\nu(v) \to \nu_0$ as $|v| \to \infty$.

This kinetic equation models the evolution of a particle distribution function $f(t,x,v) \geq 0$ depending on the time $t > 0$, the position $x \in \Omega$ and the velocity $v \in \mathbb{R}^N$. The left hand side of (1) models the free transport of particles, whereas the operator Q in the right hand side models the diffusive and mass preserving interactions between the particles and the background.

The equation must be supplemented by boundary conditions on $\partial\Omega$. In this paper, we consider **diffusive reflection conditions**, which can be written as:

$$\gamma_- f(t,x,v) = \mathcal{B}[\gamma_+ f](t,x,v) \qquad \forall (x,v) \in \Sigma_- \tag{3}$$

where $\gamma_\pm f$ is the restriction of the trace γf on $\Sigma_\pm := \{(x,v) \in \partial\Omega \times \mathbb{R},\ \pm n(x) \cdot v > 0\}$ with $n(x)$ the outward unit normal vector. To avoid the need of boundary layer analysis, we assume that the boundary operator $\mathcal{B}$ takes the form

$$\mathcal{B}[g](x,v) := \alpha_0 F(v) \int_{w \cdot n(x) > 0} g(x,w)|w \cdot n(x)|\,\mathrm{d}w \qquad \forall (x,v) \in \Sigma_- \tag{4}$$

with the same $F(v)$ as in (2) and with α_0 a normalization constant chosen such that

$$\alpha_0 \int_{v \cdot n < 0} |v \cdot n| F(v)\,\mathrm{d}v = 1$$

for any unit vector n (this integral is well defined since $F(v) \sim \frac{1}{|v|^{N+2s}}$ and $s > 1/2$).

The diffusion approximation of such an equation is obtained by investigating the long time, small mean-free-path asymptotic behavior of f. To this end we introduce the Knudsen number ε and the following rescaling of (1)-(3)

$$\begin{cases} \varepsilon^{2s-1}\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \dfrac{1}{\varepsilon} Q(f^\varepsilon) & \text{for } (t,x,v) \in (0,\infty) \times \Omega \times \mathbb{R}^N, \\ f^\varepsilon(0,x,v) = f_{in}(x,v) & \text{for } (x,v) \in \Omega \times \mathbb{R}^N, \\ \gamma_- f^\varepsilon(t,x,v) = \mathcal{B}[\gamma_+ f^\varepsilon](t,x,v) & \text{for } (t,x,v) \in (0,\infty) \times \Sigma_-. \end{cases} \tag{5}$$

We see that the particular choice of power of ε in front of the time derivative in (5) depends on the equilibrium F. The correct scaling was established in [21] (see also [1, 20, 5]) where it was shown that if Ω is the whole space $\mathbb{R}^N$ then the solution f^ε of (5) converges, as ε goes to zero, to a function

$$f^0(t,x,v) = \rho(t,x)F(v) \in \ker Q$$

where $\rho(t,x)$ solves the following fractional diffusion equation:

$$\begin{cases} \partial_t \rho + \kappa\big(-\Delta\big)^s \rho = 0 & \text{for } (t,x) \in (0,+\infty) \times \mathbb{R}^N, \\ \rho(0,x) = \rho_{in}(x) = \displaystyle\int_{\mathbb{R}^N} f_{in}(x,v)\,\mathrm{d}v & \text{for } x \in \mathbb{R}^N \end{cases}$$

for some $\kappa > 0$. Recall that the fractional Laplacian $\big(-\Delta\big)^s$ is a non-local integro-differential operator which can be defined through its Fourier transform:

$$\mathcal{F}\left(\big(-\Delta\big)^s \rho\right)(\xi) := |\xi|^{2s} \mathcal{F}(\rho)(\xi)$$

or equivalently as a singular integral

$$\big(-\Delta\big)^s \rho(x) = c_{N,s} \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{\rho(x) - \rho(y)}{|x-y|^{N+2s}}\,\mathrm{d}y$$

where $c_{N,s}$ is an explicit constant, see e.g. [13, 18] for more details.

In this paper, though, the equation is set in a subset Ω of $\mathbb{R}^N$. So we expect to derive a fractional diffusion equation confined to the domain Ω. The question at the heart of this paper is to determine the appropriate boundary conditions for this asymptotic equation. When the thermodynamical equilibrium F is a Gaussian (or Maxwellian) distribution then it is well known that the diffusion limit of (5), with $s = 1$, leads to the classical heat equation supplemented with homogeneous Neumann boundary conditions. Interestingly, these boundary conditions are not very sensitive to the type of microscopic boundary conditions. In particular, if instead of (3), we supplement equation (1) with specular reflection conditions (6), or with a combination of diffuse and specular reflections (Maxwell boundary conditions), the limiting boundary conditions are the same homogeneous Neumann boundary conditions mentioned above.

However, the issue of boundary condition is much more delicate with nonlocal operators such as fractional Laplacians. Indeed, these operators are classically associated with alpha-stable Lévy processes (or jump processes). Unlike the Brownian motion, these processes are discontinuous and may exit the domain without touching the boundary. This is the reason why the usual Dirichlet problem for the fractional Laplacian requires a prescribed data in $\mathbb{R}^N \setminus \overline{\Omega}$ rather than just on the boundary $\partial\Omega$ [15]. Neumann boundary value problems correspond to processes that are not allowed to jump outside Ω (sometimes referred to as censored stable processes). Several constructions of such processes are possible. A classical construction consist in cancelling the process after any outside jump and restarting it at the last position inside the set (resurrected processes). This construction (see [6, 16, 17] for details) leads to the *regional fractional laplacian* defined by

$$\big(-\Delta\big)^s_\Omega \rho(x) = c_{N,s} \mathrm{P.V.} \int_{\Omega} \frac{\rho(x) - \rho(y)}{|x-y|^{N+2s}}\,\mathrm{d}y$$

However, other constructions of censored processes are possible. Because of the nonlocal nature of the problem, the choice of boundary conditions for the underlying process typically changes the operator inside the domain. In [3], several such operators are discussed. For instance the process that reaches a position $y \notin \Omega$ can be restarted inside Ω by projecting y onto $\partial\Omega$, or by reflecting y

about $\partial\Omega$ (see discussion below). In [14] a different Neumann problem is obtained by restarting the process from a point $x \in \Omega$ chosen randomly with probability proportional to $|x-y|^{-N-2s}$.

In a recent paper [8], L. Cesbron studied the derivation of fractional diffusion approximation from a kinetic model in a bounded domain with **specular reflection** at the boundary. These conditions read:

$$\gamma_- f^\varepsilon(t,x,v) = \gamma_+ f^\varepsilon(t,x,\mathcal{R}_x v), \quad \mathcal{R}_x(v) = v - 2\big(n(x)\cdot v\big)n(x), \quad (t,x,v) \in (0,T)\times\Sigma_-. \tag{6}$$

In that case, the asymptotic equation reads

$$\begin{cases} \partial_t \rho + (-\Delta)^s_{\mathrm{SR}}\rho = 0 & \text{for } (t,x) \in (0,+\infty)\times\Omega \\ \rho(0,x) = \rho_{in}(x) & \text{for } x \in \Omega \end{cases}$$

where

$$(-\Delta)^s_{\mathrm{SR}}\rho(x) = c_{N,s}\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{\rho(x) - \rho\big(\eta(x,w)\big)}{|w|^{N+2s}}\,\mathrm{d}w \tag{7}$$

where $\eta : \Omega\times\mathbb{R}^N \to \overline{\Omega}$ is the flow of the free transport equation with specular reflection on the boundary. When Ω is the upper-half space, we simply have

$$\eta(x,w) = \begin{cases} x+w & \text{if } x_N + w_N > 0 \\ (x'+w', -x_N - w_N) & \text{if } x_N + w_N < 0 \end{cases}$$

and the underlying alpha stable process is the process which is moved back inside Ω by a mirror reflection about the boundary $\partial\Omega$ upon leaving the domain (see [8, 3]).

Our main result in this paper states that when the boundary conditions at the microscopic level are given by (3), then the asymptotic operator is

$$\mathcal{L}[\rho] = -c_{s,N}\mathrm{P.V.}\int_\Omega \nabla\rho(y)\cdot\frac{y-x}{|x-y|^{N+2s}}\,dy$$

which is neither the regional fractional Laplacian, nor the operator (7) (see (66) for the precise relation between $\mathcal{L}$ and $(-\Delta)^s_\Omega$). Furthermore, this operator can be written in divergence form as $\mathrm{div}D^{2s-1}[\rho]$ where $D^{2s-1}[\rho]$ is a nonlocal gradient of order $2s-1$ (see (9)), and the fractional diffusion equation must be supplemented by the following Neumann type condition

$$D^{2s-1}[\rho]\cdot n = 0 \qquad \text{on } \partial\Omega$$

(see (12)). Note that while the operator D^{2s-1} is non local, the boundary condition itself is only assumed to hold on the boundary $\partial\Omega$. This is thus different from the Nonlocal Neumann problem studied in [14], where the Neumann condition is set in $\mathbb{R}^N\setminus\Omega$.

The main takeaway from this paper is thus that for the fractional diffusion approximation, the limiting operator is very sensitive to the particular choice of microscopic boundary conditions. Note also that unlike (6) where the interaction with the boundary is entirely included in the diffusion operator, here the diffusive boundary condition (3) gives rise to the boundary condition above. This can be seen as a result of the difference in nature of the kinetic boundary conditions: the local-in-velocity specular reflection vs. non-local-in-velocity diffusive condition.

The goal of this paper is to formally explain the derivation of the asymptotic equations for (5) in convex subsets of $\mathbb{R}^N$ and to rigorously prove this derivation when Ω is the upper half-space.

1.1 Main results and outline of the paper

The existence of solutions to equation (5) is a delicate problem because it is difficult to control the trace $\gamma_+ f$ in an appropriate functional space (see [24, 23]). Note that for a given test function $\phi \in \mathcal{D}([0,\infty) \times \overline{\Omega} \times \mathbb{R}^N)$, a smooth solution of (5) will satisfy

$$\begin{aligned}
&- \iiint_{\mathbb{R}^+ \times \Omega \times \mathbb{R}^N} f^\varepsilon \Big(\partial_t \phi + \varepsilon^{1-2s} v \cdot \nabla_x \phi\Big) \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&+ \varepsilon^{1-2s} \iint_{\mathbb{R}^+ \times \Sigma_+} \gamma_+ f^\varepsilon \Big(\gamma_+ \phi - \mathcal{B}^*[\gamma_- \phi]\Big) |v \cdot n| \,\mathrm{d}v\,\mathrm{d}S(x)\,\mathrm{d}t \\
&= \varepsilon^{-2s} \iiint_{\mathbb{R}^+ \times \Omega \times \mathbb{R}^N} f^\varepsilon Q^*(\phi) \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega \times \mathbb{R}} f_{in}(x,v) \phi(0,x,v) \,\mathrm{d}x\,\mathrm{d}v.
\end{aligned}$$

where

$$\mathcal{B}^*[\gamma_- \phi](x,v) = \int_{w \cdot n(x) < 0} \alpha_0 F(w) \gamma_- \phi(w) |w \cdot n(x)| \,\mathrm{d}w.$$

A classical way of defining weak solutions of (5), (3) without having to deal with the trace γf is then the following (see for instance [22]):

Definition 1.1. *We say that a function $f(t,x,v)$ in $L^2_{F^{-1}}((0,\infty) \times \Omega \times \mathbb{R}^N)$ is a weak solution of* (5) *if for every test functions $\phi(t,x,v)$ such that ϕ, $\partial_t \phi$ and $v \cdot \nabla_x \phi$ are $L^2_F((0,\infty) \times \Omega \times \mathbb{R}^N)$ and satisfying the boundary condition*

$$\gamma_+ \phi = \mathcal{B}^*[\gamma_- \phi],$$

the following equality holds:

$$\begin{aligned}
&- \iiint_{\mathbb{R}^+ \times \Omega \times \mathbb{R}^N} f^\varepsilon \Big(\partial_t \phi + \varepsilon^{1-2s} v \cdot \nabla_x \phi\Big) \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \varepsilon^{-2s} \iiint_{\mathbb{R}^+ \times \Omega \times \mathbb{R}^N} f^\varepsilon Q^*(\phi) \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega \times \mathbb{R}} f_{in}(x,v) \phi(0,x,v) \,\mathrm{d}x\,\mathrm{d}v. \qquad (8)
\end{aligned}$$

Here and in the rest of the paper, we used the notation

$$L^2_{F^{-1}}((0,\infty) \times \Omega \times \mathbb{R}^N) = \left\{ f(t,x,v)\,;\ \int_0^\infty \int_\Omega \int_{\mathbb{R}^N} |f(t,x,v)|^2 \frac{1}{F(v)} \,dv\,dx\,dt < \infty \right\}$$

and a similar definition for $L^2_F((0,\infty) \times \Omega \times \mathbb{R}^N)$.

In order to write our main result, we now define the operator

$$D^{2s-1}[u](x) = \gamma \nu_0^{1-2s} \Gamma(2s-1) \int_\Omega (y-x) \cdot \nabla u(y) \frac{y-x}{|y-x|^{N+2s}} \,dy \qquad (9)$$

which is defined pointwise for example if $\nabla u \in L^\infty_{loc}(\Omega) \cap L^1(\Omega)$ (note that we included the constant $\gamma \nu_0^{1-2s}$ which depends F and ν in this definition in order to simplify the notations later on). In particular, if $N = 1$ and $\Omega = \mathbb{R}$, we find

$$\begin{aligned}
D^{2s-1}[u](x) &= \gamma \nu_0^{1-2s} \Gamma(2s-1) \int_{\mathbb{R}} \frac{u'(y)}{|x-y|^{N-2(1-s)}} \,dy \\
&= \gamma \nu_0^{1-2s} c (-\Delta)^{-(1-s)} u'(x),
\end{aligned}$$

for some constant c. So the operator D^{2s-1} can be interpreted as a fractional gradient of order $2s-1\in(0,1)$

Our main result is then the following:

Theorem 1.2. *Assume that Q is given by (32) and that F satisfies (33) with $s\in(1/2,1)$. Let Ω be the upper half space*

$$\Omega=\{x\in\mathbb{R}^N\,;\,x_N>0\}.$$

Assume that $f^\varepsilon(t,x,v)$ is a weak solution of (5) in $(0,\infty)\times\Omega\times\mathbb{R}^N$ in the sense of Definition 1.1 and satisfies the energy inequality (21). Then, up a subsequence, the function $f^\varepsilon(t,x,v)$ converges weakly in $L^\infty(0,\infty;L^2_{F^{-1}}(\Omega\times\mathbb{R}^N))$, as ε goes to 0, to a function $\rho(t,x)F(v)$ where $\rho(t,x)$ satisfies

$$\iiint_{\mathbb{R}^+\times\Omega}\rho(t,x)\Big(\partial_t\psi(t,x)+\mathrm{div}D^{2s-1}[\psi](t,x)\Big)\,\mathrm{d}t\,\mathrm{d}x+\iint_{\Omega}\rho_{in}(x)\psi(0,x)\,\mathrm{d}x=0 \tag{10}$$

for all test function $\psi\in W^{1,\infty}(0,\infty;H^2(\Omega))$, such that $\mathrm{div}D^{2s-1}[\psi]\in L^2(\mathbb{R}_+\times\Omega)$ and

$$D^{2s-1}[\psi]\cdot n=0 \tag{11}$$

We now make several remarks concerning this result:

1. As mentioned in the introduction, the result holds for more general collision operators Q. We restrict ourselves to the simplest case here in order to focus on the novelty of our analysis, which is to deal with the boundary conditions.

2. Equation (10) is the fractional equivalent of the following weak formulation of the usual heat equation with Neumann boundary conditions:

$$\begin{cases}\displaystyle\iiint_{\mathbb{R}^+\times\Omega}\rho\Big(\partial_t\psi(t,x)+\Delta\psi(t,x)\Big)\,\mathrm{d}t\,\mathrm{d}x+\iint_{\Omega}\rho_{in}(x)\psi(0,x)\,\mathrm{d}x=0\\ \text{for all }\psi\in W^{1,\infty}(0,\infty;H^2(\Omega))\text{ such that }\nabla\psi(x)\cdot n(x)=0\text{ on }\partial\Omega.\end{cases}$$

   In particular, condition (11) is the nonlocal equivalent of this classical Neumann boundary condition.

3. Using the following integration by parts formula (which we will prove in Proposition 2.11):

$$\int_\Omega\mathrm{div}D^{2s-1}[\varphi]\psi\,\mathrm{d}x-\int_\Omega\varphi\,\mathrm{div}D^{2s-1}[\psi]\,\mathrm{d}x=\int_{\partial\Omega}\left[\psi D^{2s-1}[\varphi]\cdot n-\varphi D^{2s-1}[\psi]\cdot n\right]\,\mathrm{d}S(x)$$

   we see that Equation (10) is the weak formulation for the following fractional Neumann boundary problem:

$$\begin{cases}\partial_t\rho-\mathrm{div}D^{2s-1}[\rho]=0 & \text{in }(0,\infty)\times\Omega\\ D^{2s-1}[\rho]\cdot n=0 & \text{in }(0,\infty)\times\partial\Omega\\ \rho(0,x)=\rho_{in}(x) & \text{in }\Omega.\end{cases} \tag{12}$$

4. We need to require that $\mathrm{div}D^{2s-1}[\psi]\in L^2(\mathbb{R}_+\times\Omega)$ in Theorem 1.2, because such a fact is not implied by the condition $\psi\in W^{1,\infty}(0,\infty;H^2(\Omega))$ (which might seem surprising if one thinks of $\mathrm{div}D^{2s-1}$ as a Laplacian of order $s\in(1/2,1)$). We will characterize precisely in Proposition 2.6 the functions such that $\mathrm{div}D^{2s-1}[\psi]\in L^2(\mathbb{R}_+\times\Omega)$ and in particular, we will prove that when $s\geq 3/4$, this condition requires ψ to satisfy the local Neumann boundary condition $\nabla\psi\cdot n=0$ on $\partial\Omega$. This suggests that solutions of (12) also satisfy the classical Neumann boundary conditions at the boundary, though this fact emerges as a consequence of the regularity theory, rather than as a boundary condition necessary to get a unique solution.

5. As explained in the first part of this introduction, we will show that the main operator in (12) is
$$\mathcal{L}[\rho](x) := \gamma\nu_0^{1-2s}\Gamma(2s)\mathrm{P.V.}\int_\Omega \nabla\rho(y)\cdot\frac{y-x}{|y-x|^{N+2s}}\,\mathrm{d}y. \tag{13}$$
Indeed, taking the divergence in (9), we obtain (formally at least)
$$\mathrm{div}D^{2s-1}[\rho] = \mathcal{L}[\rho].$$
We will rigorously justify this formula later on, see Lemma 3.7. We see in particular that when $\Omega = \mathbb{R}^N$, we recover the usual fractional Laplacian of order s in $\mathbb{R}^N$ (up to a constant).

Because equation (12) does not seem to have been studied in details before, we will prove the following theorem:

Theorem 1.3. *For all $\rho_{in} \in L^2(\Omega)$, the evolution problem*
$$\begin{cases} \partial_t\rho - \mathrm{div}D^{2s-1}[\rho] = 0 & \text{in } (0,\infty)\times\Omega,\\ D^{2s-1}[\rho]\cdot n = 0 & \text{on } (0,\infty)\times\partial\Omega,\\ \rho(0,x) = \rho_{in}(x) & \text{in } \Omega. \end{cases} \tag{14}$$
has a unique solution $\rho \in C^0(0,\infty; L^2(\Omega)) \cap L^2(0,\infty; D(\mathcal{L}))$ where
$$D(\mathcal{L}) = \{\varphi \in H^s(\Omega)\,;\, \mathcal{L}[\varphi] \in L^2(\Omega), \quad D^{2s-1}[\varphi]\cdot n = 0 \text{ on } \partial\Omega\}.$$

However, we do not show, in this paper, that the function $\rho(t,x)$ identified in Theorem 1.2 is the unique solution of (14) (note that, once proved, such a uniqueness result implies that the whole sequence f^ε, and not just a subsequence, converges to ρF). To prove such a fact requires additional regularity results for the solutions of (14). Namely, we need the weak solution of (14) - or rather that of the dual problem - to be in $W^{1,\infty}(0,\infty; H^2(\Omega))$ for smooth initial data. This is actually a delicate problem which requires a detailed analysis of the boundary regularity of the solution of (14) and which does not seem to have been addressed so far in the literature. It is the object of the companion paper [9].

Finally, we need to stress that we only rigorously prove the fractional diffusion approximation when Ω is the upper half-space because in this case the boundary values do not interact with each other via the boundary conditions which simplifies some of the arguments in the (already delicate) proof. However the result certainly holds for general convex domains.

Outline of the paper. The rest of the paper is organized as follows: In the second part of this introductory section we will briefly present the main ideas of the proof. Section 2, is devoted to some preliminary results: First we recall some important properties of the solutions of the kinetic equation (5), in particular the existence of weak solutions and the convergence to a thermodynamical equilibrium. We also establish (in Section 2.2) some important properties of the operators D^{2s-1} and $\mathcal{L} = \mathrm{div}D^{2s-1}$, some of which are needed for the proof of our main result, as well as others that are of independent interest. Section 3, is devoted to the proof of the main result, Theorem 1.2. Finally, in Section 4 we study the asymptotic fractional Neumann problem (12) and prove Theorem 1.3.

1.2 Idea of the proof

In this section we explain the main idea of the proof. As in previous works on this topic, e.g. [20, 4, 2], for a given test function $\psi(t,x)$ defined in $[0,\infty)\times\Omega$, we introduce ϕ^ε solution of the auxiliary problem
$$\nu\phi^\varepsilon - \varepsilon v\cdot\nabla_x\phi^\varepsilon = \nu\psi \qquad \text{in } \Omega\times\mathbb{R}^N. \tag{15}$$

When $\Omega = \mathbb{R}^N$ this equation can easily be solved explicitly. In our framework, this transport equation must be supplemented with the boundary condition

$$\gamma_+\phi^\varepsilon(t,x,v) = \mathcal{B}^*[\gamma_-\phi^\varepsilon](t,x,v) \qquad (x,v) \in \Sigma_+. \tag{16}$$

Assuming that we can find such a function ϕ^ε, we note that since ψ does not depend on v, we have $K^*(\psi) = \nu\psi$, and so

$$\begin{aligned} Q^*(\phi^\varepsilon) + \varepsilon v \cdot \nabla_x \phi^\varepsilon &= K^*(\phi^\varepsilon) - \nu\phi^\varepsilon + \varepsilon v \cdot \nabla_x \phi^\varepsilon \\ &= K^*(\phi^\varepsilon) - \nu\psi \\ &= K^*(\phi^\varepsilon - \psi). \end{aligned}$$

Taking ϕ^ε as a test function in (8) (which we can do since ϕ^ε satisfies (16)), we deduce

$$\begin{aligned} -\iiint\limits_{\mathbb{R}^+\times\Omega\times\mathbb{R}^N} f^\varepsilon \partial_t \phi \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t - \iint\limits_{\Omega\times\mathbb{R}} f_{in}(x,v)\phi(0,x,v) &= \varepsilon^{-2s} \iiint\limits_{\mathbb{R}^+\times\Omega\times\mathbb{R}^N} f^\varepsilon K^*(\phi^\varepsilon - \psi)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &= \varepsilon^{-2s} \iiint\limits_{\mathbb{R}^+\times\Omega\times\mathbb{R}^N} K(f^\varepsilon)[\phi^\varepsilon - \psi]\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Next we introduce the decomposition

$$f^\varepsilon(t,x,v) = \rho^\varepsilon(t,x)F(v) + g^\varepsilon(t,x,v), \qquad \rho^\varepsilon(t,x) = \int_{\mathbb{R}^N} f^\varepsilon(t,x,v)\,\mathrm{d}v$$

where we expect $\|g^\varepsilon\| \ll 1$ since f^ε converges to $\ker Q$. Using the fact that $K(F) = \nu F$, we can write

$$K(f^\varepsilon) = \rho^\varepsilon K(F) + K(g^\varepsilon) = \rho^\varepsilon \nu F + K(g^\varepsilon).$$

We thus have

$$\begin{aligned} \varepsilon^{-2s} \iiint\limits_{\mathbb{R}^+\times\Omega\times\mathbb{R}^N} K(f^\varepsilon)[\phi^\varepsilon - \psi]\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t &= \varepsilon^{-2s} \iiint\limits_{\mathbb{R}^+\times\Omega\times\mathbb{R}^N} \rho^\varepsilon \nu(v)F(v)[\phi^\varepsilon - \psi]\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \varepsilon^{-2s} \iiint\limits_{\mathbb{R}^+\times\Omega\times\mathbb{R}^N} K(g^\varepsilon)[\phi^\varepsilon - \psi]\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

The second term in the right hand side should converge to zero, while the first term can be written as

$$\varepsilon^{-2s} \iiint\limits_{\mathbb{R}^+\times\Omega\times\mathbb{R}^N} \rho^\varepsilon \nu(v)F(v)[\phi^\varepsilon - \psi]\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t = \iint\limits_{\mathbb{R}^+\times\Omega} \rho^\varepsilon \tilde{\mathcal{L}}^\varepsilon[\psi]\,\mathrm{d}x\,\mathrm{d}t$$

with (using (15)):

$$\begin{aligned} \tilde{\mathcal{L}}^\varepsilon[\psi](x) &:= \varepsilon^{-2s} \int_{\mathbb{R}^N} \nu(v)F(v)[\phi^\varepsilon(x,v) - \psi(x)]\,\mathrm{d}v \\ &= \varepsilon^{-2s} \int_{\mathbb{R}^N} F(v)\varepsilon v \cdot \nabla_x \phi^\varepsilon(x,v)\,\mathrm{d}v \\ &= \mathrm{div}_x \left(\varepsilon^{1-2s} \int_{\mathbb{R}^N} vF(v)\phi^\varepsilon(x,v)\,\mathrm{d}v \right). \end{aligned} \tag{17}$$

Gathering all those computations, we finally arrive at the weak formulation

$$
\begin{aligned}
-\iiint\limits_{\mathbb{R}^+\times\Omega\times\mathbb{R}^N} f^\varepsilon \partial_t \phi^\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t - \iint\limits_{\Omega\times\mathbb{R}} f_{in}(x,v)\phi^\varepsilon(0,x,v) \\
= \iint\limits_{\mathbb{R}^+\times\Omega} \rho^\varepsilon \tilde{\mathcal{L}}^\varepsilon[\psi]\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^{-2s} \iiint\limits_{\mathbb{R}^+\times\Omega\times\mathbb{R}^N} K(g^\varepsilon)(\phi^\varepsilon - \psi)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{18}
$$

The proof then consists in passing to the limit in this weak formulation. Passing to the limit in the left hand side requires ϕ^ε to converge to ψ strongly in some L^2 space, which is reasonable in view of (15) (note also that since ψ does not depends on v, it trivially satisfies the boundary condition (16)). For the right hand side, we notice that the last term should vanish in the limit since $f^\varepsilon - \rho^\varepsilon F \to 0$, so the main step in the proof is to identify the limit of $\tilde{\mathcal{L}}^\varepsilon[\psi]$ for appropriate test functions ψ.

When $\Omega = \mathbb{R}^N$, this task is greatly simplified by the fact that equation (15) yields an explicit formula for ϕ^ε as a function of ψ. When Ω is a proper subset of $\mathbb{R}^N$, the task is more delicate.

In order to identify the limit of $\tilde{\mathcal{L}}^\varepsilon[\psi]$, we introduce the following operator:

$$
\tilde{D}^{2s-1}_\varepsilon[\psi](x) := \varepsilon^{1-2s} \int_{\mathbb{R}^N} vF(v)[\phi^\varepsilon(x,v) - \psi(x)]\,\mathrm{d}v. \tag{19}
$$

With this notation, we have (using (17) and the fact that $\int_{\mathbb{R}^N} vF(v)\,\mathrm{d}v = 0$):

$$
\begin{aligned}
\tilde{\mathcal{L}}^\varepsilon[\psi](x) &= \mathrm{div}_x\left(\varepsilon^{1-2s}\int_{\mathbb{R}^N} vF(v)\phi^\varepsilon(x,v)\,\mathrm{d}v\right) \\
&= \mathrm{div}_x\left(\varepsilon^{1-2s}\int_{\mathbb{R}^N} vF(v)[\phi^\varepsilon(x,v)-\psi(x)]\,\mathrm{d}v\right) \\
&= \mathrm{div}_x \tilde{D}^{2s-1}_\varepsilon[\psi](x).
\end{aligned}
\tag{20}
$$

The key step in the proof is thus to show that for appropriate test function ψ we have

$$
\tilde{D}^{2s-1}_\varepsilon[\psi] \longrightarrow D^{2s-1}[\psi] \qquad \text{as } \varepsilon \to 0
$$

where D^{2s-1} is the fractional derivative (or gradient) of order $2s-1$ defined by (9), and

$$
\tilde{\mathcal{L}}^\varepsilon[\psi] \longrightarrow \mathrm{div} D^{2s-1}[\psi] \qquad \text{as } \varepsilon \to 0
$$

However, it should be noted that, without further assumptions on ψ, the term

$$
\iint\limits_{\mathbb{R}^+\times\Omega} \rho^\varepsilon \tilde{\mathcal{L}}^\varepsilon[\psi]\,\mathrm{d}x\,\mathrm{d}t.
$$

in (18) should yield, in the limit, an appropriate boundary term as well. So the convergence above will only hold "up to the boundary" if ψ satisfies the following appropriate non-local Neumann boundary condition:

$$
D^{2s-1}[\psi](x)\cdot n(x) = 0\ , \qquad \text{for all } x \in \partial\Omega.
$$

Assuming that all the convergences above holds, we see that passing to the limit in equation (18), using the fact that $f^\varepsilon \to \rho(t,x)F(v)$, yields:

$$
\iiint\limits_{\mathbb{R}^+\times\Omega} \rho\Big(\partial_t\psi(t,x) + \mathrm{div} D^{2s-1}[\psi](t,x)\Big)\,\mathrm{d}t\,\mathrm{d}x + \iint\limits_{\Omega} \rho_{in}(x)\psi(0,x)\,\mathrm{d}x = 0
$$

which is the main claim of Theorem 1.2.

2 Preliminary results

2.1 Entropy inequality and existence of weak solutions for (5)

We end this introduction with a short proof of the classical a priori estimates satisfied by weak solutions of (5), and which are key in showing the convergence of f^ε toward a thermodynamical equilibrium (the kernel of Q):

Lemma 2.1. *Let f_{in} be in $L^2_{F^{-1}}(\Omega\times\mathbb{R}^N)$ and let $f^\varepsilon(t,x,v)$ be a strong solution of (5) satisfying the boundary condition (3). Then f^ε satisfies*

$$\|f^\varepsilon(t)\|^2_{L^2_{F^{-1}}(\Omega\times\mathbb{R}^N)} + \varepsilon^{-2s}\int_0^t \|f^\varepsilon(s)-\rho^\varepsilon(s)F\|^2_{L^2_{F^{-1}}(\Omega\times\mathbb{R}^N)}\,ds \le \|f_{in}(t)\|^2_{L^2_{F^{-1}}(\Omega\times\mathbb{R}^N)} \tag{21}$$

for all $t\geq 0$.

Proof. Multiplying (5) by f^ε/F and integrating with respect to x and v we get

$$\varepsilon^{2s-1}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega\times\mathbb{R}^N}|f^\varepsilon|^2\frac{\mathrm{d}x\,\mathrm{d}v}{F(v)}+\iint_\Sigma|\gamma f^\varepsilon|^2 v\cdot n(x)\frac{\mathrm{d}S(x)\,\mathrm{d}v}{F(v)}=\frac{1}{\varepsilon}\iint_{\Omega\times\mathbb{R}^N}f^\varepsilon Q(f^\varepsilon)\frac{\mathrm{d}x\,\mathrm{d}v}{F(v)}.$$

Inequality (21) thus follows from the following classical inequality (see for instance [20, Lemma A.1]):

$$-\int_{\mathbb{R}^N} f(v)Q(f)(v)\frac{dv}{F(v)} \ge \int_{\mathbb{R}^N}\frac{|f(v)-\rho F(v)|^2}{F(v)}\,dv \qquad \forall f\in L^2_{F^{-1}}(\mathbb{R}^N),\quad \rho=\int_{\mathbb{R}^N} f(v)\,dv$$

and the so-called Darrozès-Guiraud inequality satisfied by operators of the form (4) (see [11]):

$$\int_{v\cdot n(x)<0}|\mathcal{B}[\gamma_+ f_\varepsilon]|^2|v\cdot n(x)|\frac{\mathrm{d}v}{F(v)} \le \int_{v\cdot n(x)>0}|\gamma_+ f_\varepsilon|^2|v\cdot n(x)|\frac{\mathrm{d}v}{F(v)} \qquad \text{a.e. } x\in\partial\Omega$$

which implies

$$\iint_\Sigma |\gamma f_\varepsilon|^2\, v\cdot n(x)\frac{\mathrm{d}S(x)\,\mathrm{d}v}{F(v)}\ge 0.$$

□

We then give the following classical result (which can be proved for instance as in [22]):

Proposition 2.2. *For all $f_{in}\in L^2_{F^{-1}}(\Omega\times\mathbb{R}^N)$ there exists a weak solution of (5) in the sense of Definition 1.1 and satisfying the energy inequality (21).*

Inequality (21) implies that f^ε is bounded in $L^\infty(0,\infty;L^2_{F^{-1}}(\Omega\times\mathbb{R}^N))$ and thus converges, up to a subsequence, $\star$-weak to a function $f^0(t,x,v)$. Note also that

$$\int_\Omega|\rho^\varepsilon|^2\,dx \le \int_\Omega\left|\int_{\mathbb{R}^N} f^\varepsilon\,dv\right|^2 dx \le \int_\Omega\int_{\mathbb{R}^N}\frac{|f^\varepsilon|^2}{F(v)}\,dv\,dx$$

and so ρ^ε converges weakly to $\rho(t,x)=\int_{\mathbb{R}^N} f^0(t,x,v)\,dv$. Finally, (21) also implies that

$$\|f^\varepsilon-\rho^\varepsilon F\|_{L^2_{F^{-1}}(\Omega\times\mathbb{R}^N)}\to 0 \quad \text{ as } \varepsilon\to 0.$$

and so $f^0-\rho F(v)=0$. We deduce:

Corollary 2.3. *Let f^ε be weak solution of (5) provided by Proposition 2.2. Then, up to a subsequence*

$$f^\varepsilon \rightharpoonup \rho(t,x)F(v) \quad \textit{weakly in } L^\infty(0,+\infty;L^2_{F^{-1}}(\Omega\times\mathbb{R}^N))$$

where $\rho(t,x)$ is the weak limit of $\rho^\varepsilon(t,x)=\int_{\mathbb{R}^N} f^\varepsilon\,\mathrm{d}v$.

2.2 Properties of the limiting operators: D^{2s-1} and $\mathcal{L}$

In this section, we establish some important properties of the operators D^{2s-1} and $\mathcal{L}$. First, we need to introduce some classical functional spaces: For $\gamma \in (0,1)$, we denote by $C^\gamma(\Omega)$ the set of Hölder continuous functions satisfying

$$\|\varphi\|_{C^\gamma(\Omega)} = \|\varphi\|_{L^\infty(\Omega)} + [\varphi]_{C^\gamma(\Omega)} < \infty$$

where

$$[\varphi]_{C^\gamma(\Omega)} = \sup_{x,y\in\Omega\times\Omega} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\gamma}$$

We also denote by $C^{1,\gamma}(\Omega)$ the set of functions φ such that $\varphi \in C^1(\Omega)$ and $\nabla\varphi \in C^\gamma(\Omega)$.

Next, for $s \in (0,1)$ we recall that the fractional Sobolev space H^s is defined by (see [12]):

$$H^s(\Omega) = \left\{\varphi \in L^2(\Omega)\,;\ \int_\Omega\int_\Omega \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y < \infty\right\}.$$

It is equipped with the norm:

$$\|\varphi\|^2_{H^s} = \int_\Omega |\varphi(x)|^2\,\mathrm{d}x + \int_\Omega\int_\Omega \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

For $s \in (1,2)$, we also have

$$H^s(\Omega) = \{\varphi \in H^1(\Omega)\,;\ \nabla\phi \in H^{s-1}(\Omega)\}$$

which is equipped with the norm

$$\|\varphi\|^2_{H^s(\Omega)} = \|\varphi\|^2_{H^1(\Omega)} + \|\nabla\varphi\|^2_{H^{s-1}(\Omega)}.$$

Our goal in this section is to prove some results about the operators D^{2s-1} and $\mathcal{L}$ that are used in this paper. We start by noticing that by (9)

$$|D^{2s-1}[\psi](x)| \le C\int_\Omega \frac{|\nabla\psi(y)|}{|y-x|^{N+2s-2}}\,dy.$$

Classical results about Riesz potentials thus implies

Proposition 2.4. *If $\nabla\psi \in L^p(\Omega)$ for some $1 < p < \frac{N}{2-2s}$, then $D^{2s-1}[\psi] \in L^q(\Omega)$ with $q = \frac{N}{N-(2-2s)p}$ and there exists a constant C such that*

$$\|D^{2s-1}[\psi]\|_{L^q(\Omega)} \le C\|\nabla\psi\|_{L^p(\Omega)}.$$

Next, we note that for $x \in \Omega$ and $\varepsilon < x_N$, we can write (using (13)):

$$\begin{aligned}\mathcal{L}[\psi](x) &= \gamma\nu_0^{1-2s}\Gamma(2s)\int_{\Omega\cap B_\varepsilon(x)} \frac{y-x}{|y-x|^{N+2s}}[\nabla\psi(y)-\nabla\psi(x)]\,\mathrm{d}y\\ &\quad+\gamma\nu_0^{1-2s}\Gamma(2s)\int_{\Omega\setminus B_\varepsilon(x)} \frac{y-x}{|y-x|^{N+2s}}\nabla\psi(y)\,\mathrm{d}y.\end{aligned}$$

In particular $\mathcal{L}[\psi](x)$ is well defined for all $x \in \Omega$ if $\psi \in C^{1,\gamma}(\Omega)$ for some $\gamma > 2s-1$ and $\psi \in W^{1,\infty}(\Omega)$. However, when x approaches $\partial\Omega$, the ε becomes very small and it is difficult to get a bound on $\mathcal{L}[u]$ up to the boundary. The next two propositions give necessary and sufficient conditions for such bounds to hold:

Proposition 2.5. *Assume that $\psi \in C^{1,\gamma}(\Omega)$ for some $\gamma > 2s-1$, and $\psi \in W^{1,\infty}(\Omega)$. Then $\mathcal{L}[\psi] \in L^\infty(\Omega)$ if and only if $x_N^{1-2s}\partial_{x_N}\psi(x) \in L^\infty(\Omega)$ and so if and only if $n \cdot \nabla\psi(x) = 0$ on $\partial\Omega$. Furthermore, if ψ satisfies $n \cdot \nabla\psi(x) = 0$ on $\partial\Omega$, then*

$$\|\mathcal{L}[\psi]\|_{L^\infty(\Omega)} \leq C\|\nabla\psi\|_{C^\gamma(\Omega)}.$$

Proof. We use Formula (13) for the operator $\mathcal{L}$ and write:

$$\begin{aligned}\mathcal{L}[\psi](x) &= \gamma\nu_0^{1-2s}\Gamma(2s)\mathrm{P.V.}\int_\Omega \frac{y-x}{|y-x|^{N+2s}} \cdot \nabla\psi(y)\,\mathrm{d}y \\ &= \gamma\nu_0^{1-2s}\Gamma(2s)\int_\Omega \frac{y-x}{|y-x|^{N+2s}} \cdot [\nabla\psi(y) - \nabla\psi(x)]\,\mathrm{d}y \\ &\quad + \gamma\nu_0^{1-2s}\Gamma(2s)\nabla\psi(x) \cdot \mathrm{P.V.}\int_\Omega \frac{y-x}{|y-x|^{N+2s}}\,\mathrm{d}y. \end{aligned} \tag{22}$$

The first term in the right hand side is bounded by

$$C[\nabla\psi]_{C^\gamma}\int_{\Omega\cap B_1(x)} \frac{|y-x|^{1+\gamma}}{|y-x|^{N+2s}}\,\mathrm{d}y + C\|\nabla\psi\|_{L^\infty}\int_{\Omega\setminus B_1(x)} \frac{|y-x|}{|y-x|^{N+2s}}\,\mathrm{d}y \leq C\|\nabla\psi\|_{C^\gamma(\Omega)}$$

(we recall that $s \in (1/2, 1)$ and $1+\gamma > 2s$). Furthermore, a simple computation shows that

$$\mathrm{P.V.}\int_\Omega \frac{y_i - x_i}{|y-x|^{N+2s}}\,\mathrm{d}y = \begin{cases} 0 & i = 1, \dots, N-1 \\ C_{N,s}x_N^{1-2s} & i = N \end{cases}$$

so the second term in the right hand side of (22) is equal to (up to a constant)

$$x_N^{1-2s}\partial_{x_N}\psi(x).$$

It follows that $\mathcal{L}[\psi] \in L^\infty(\Omega)$ if and only if $x_N^{1-2s}\partial_{x_N}\psi(x) \in L^\infty(\Omega)$.

This condition implies that $\partial_{x_N}\psi(x) = 0$ on $\partial\Omega$ since $1-2s < 0$. Furthermore, for such a function, we have

$$|\partial_{x_N}\psi(x)| = |\partial_{x_N}\psi(x) - \partial_{x_N}\psi(x', 0)| \leq C[\nabla\psi]_{C^\gamma(\Omega)}|x_N|^\gamma$$

and so

$$|x^{1-2s}\partial_{x_N}\psi(x)| = |x^{1-2s}||\partial_{x_N}\psi(x) - \partial_{x_N}\psi(x', 0)| \leq C[\nabla\psi]_{C^\gamma(\Omega)}|x_N|^{\gamma+1-2s} \leq C[\nabla\psi]_{C^\gamma(\Omega)} \text{ for } x_N \leq 1.$$

Since

$$|x^{1-2s}\partial_{x_N}\psi(x)| \leq \|\nabla\psi\|_{L^\infty} \text{ for } x_N \geq 1,$$

we deduce

$$\|\mathcal{L}[\psi]\|_{L^\infty(\Omega)} \leq C([\nabla\psi]_{C^\gamma} + \|\nabla\psi\|_{L^\infty}).$$

□

We can also prove a similar result in Sobolev spaces:

Proposition 2.6. *Assume that $\psi \in H^{2s+\beta}(\Omega)$ for some $\beta > 0$. Then the following holds:*

(i) If $2s-1 < 1/2$ (that is $s \in (1/2, 3/4)$), then $\mathcal{L}[\psi] \in L^2(\Omega)$ and

$$\|\mathcal{L}[\psi]\|_{L^2(\Omega)} \leq C\|\psi\|_{H^{2s+\beta}(\Omega)}.$$

(ii) If $2s-1 \geq 1/2$, (that is $s \in [3/4, 1)$) then $\mathcal{L}[\psi] \in L^2(\Omega)$ if and only if

$$\int_\Omega \left| x_N^{1-2s} \partial_{x_N} \psi \right|^2 \, \mathrm{d}x < \infty \tag{23}$$

or, equivalently, if and only if $\partial_{x_N}\psi = 0$ on $\partial\Omega$. When this condition is satisfied, we then have

$$\|\mathcal{L}[\psi]\|_{L^2(\Omega)} \leq C \|\psi\|_{H^{2s+\beta}(\Omega)}. \tag{24}$$

Before starting the proof of this proposition, we recall the following Hardy inequality (see [19, 7]):

Theorem 2.7. *Recall that Ω is the half space $\{(x_1, \dots, x_N)\,;\, x_N > 0\}$. Then for all $s \in (0,1)$ there exists a constant C depending only on s and N such that for all $f \in C_c(\Omega)$:*

$$\int_\Omega \frac{|f(x)|^2}{|x_N|^{2s}} \, \mathrm{d}x \leq C_s \|f\|^2_{\dot{H}^s(\Omega)} \tag{25}$$

Remarks 2.8. *When $s < 1/2$, then $C_c(\Omega)$ is dense in $H^s(\Omega)$ and so (25) holds for all $f \in H^s(\Omega)$. When $s > 1/2$, the closure of $C_c(\Omega)$ in $H^s(\Omega)$ is $H^s_0(\Omega)$, the set of functions in $H^s(\Omega)$ whose trace vanishes at the boundary. In that case (25) holds for all $f \in H^s_0(\Omega)$.*

Proof of Proposition 2.6. We write

$$\begin{aligned}
\mathcal{L}[\psi](x) &= \gamma \nu_0^{1-2s} \Gamma(2s) \int_{\Omega \setminus B_1(x)} \frac{y-x}{|y-x|^{N+2s}} \cdot \nabla\psi(y) \, \mathrm{d}y \\
&\quad + \gamma \nu_0^{1-2s} \Gamma(2s) \int_{\Omega \cap B_1(x)} \frac{y-x}{|y-x|^{N+2s}} \cdot [\nabla\psi(y) - \nabla\psi(x)] \, \mathrm{d}y \\
&\quad + \gamma \nu_0^{1-2s} \Gamma(2s) \nabla\psi(x) \cdot \mathrm{P.V.} \int_{\Omega \cap B_1(x)} \frac{y-x}{|y-x|^{N+2s}} \, \mathrm{d}y \\
&= I_1(x) + I_2(x) + I_3(x).
\end{aligned} \tag{26}$$

The first term in (26) satisfies (since $s \in (1/2, 1)$):

$$\begin{aligned}
\int_\Omega |I_1(x)|^2 \, dx &\leq C \int_\Omega \left(\int_{\Omega \setminus B_1(x)} \frac{1}{|y-x|^{N+2s-1}} |\nabla\psi(y)| \, \mathrm{d}y \right)^2 dx \\
&\leq C \int_\Omega \int_{\Omega \setminus B_1(x)} \frac{1}{|y-x|^{N+2s-1}} |\nabla\psi(y)|^2 \, \mathrm{d}y \, dx \\
&\leq C \int_\Omega |\nabla\psi(y)|^2 \, \mathrm{d}y
\end{aligned}$$

For the second term in (26), we write

$$\begin{aligned}
\int_\Omega |I_2(x)|^2 \, dx &\leq C \int_\Omega \left(\int_{\Omega \cap B_1(x)} \frac{|\nabla\psi(y) - \nabla\psi(x)|}{|y-x|^{N+2s-1}} \, \mathrm{d}y \right)^2 dx \\
&\leq C \int_\Omega \int_{\Omega \cap B_1(x)} \frac{|\nabla\psi(y) - \nabla\psi(x)|^2}{|y-x|^{N+2(2s-1+\beta)}} \, \mathrm{d}y \int_{\Omega \cap B_1(x)} \frac{1}{|y-x|^{N-2\beta}} \, dy \, dx \\
&\leq C \int_\Omega \int_\Omega \frac{|\nabla\psi(y) - \nabla\psi(x)|^2}{|y-x|^{N+2(2s-1+\beta)}} \, \mathrm{d}y \, dx \\
&\leq C \|\nabla\psi\|^2_{H^{2s-1+\beta}(\Omega)} \leq C \|\psi\|^2_{H^{2s+\beta}(\Omega)}
\end{aligned}$$

Finally, for the last term in (26), we note that

$$\text{P.V.} \int_{\Omega \cap B_1(x)} \frac{y_i - x_i}{|y-x|^{N+2s}}\, \mathrm{d}y = \begin{cases} 0 & \text{if } i = 1, \dots, N-1 \\ \displaystyle\int_{|z|<1,\ z_N > -x_N} \frac{z_N}{|z|^{N+2s}}\, \mathrm{d}z & \text{if } i = N \end{cases}$$

and we have the following Lemma:

Lemma 2.9. *The function*

$$h(x_N) = \text{P.V.} \int_{|z|<1,\ z_N>-x_N} \frac{z_N}{|z|^{N+2s}}\, \mathrm{d}z, \qquad x_N > 0$$

satisfies $h(x_N) = 0$ *if* $x_N \geq 1$ *and*

$$C_1 |x_N|^{1-2s} \leq h(x_N) \leq C_2 |x_N|^{1-2s} \qquad \textit{for } 0 < x_N < 1/2$$

and

$$0 \leq h(x_N) \leq C_3 \qquad \textit{for } 1/2 < x_N < 1$$

Postponing the proof for now, we note that this lemma implies that

$$\int_\Omega |x_N|^{2(1-2s)} 1_{\{x_N \leq 1/2\}} |\partial_{x_N} \psi|^2\, \mathrm{d}x \leq \int_\Omega |I_3(x)|^2\, dx \leq \int_\Omega |x_N|^{2(1-2s)} 1_{\{x_N \leq 1\}} |\partial_{x_N}\psi|^2\, \mathrm{d}x$$

Since I_1 and I_2 in (26) are in $L^2(\Omega)$ when $\psi \in H^{2s+\beta}(\Omega)$, we deduce that $\mathcal{L}[\psi]$ belongs to $L^2(\Omega)$ if and only if I_3 is in $L^2(\Omega)$ as well, which is then equivalent to (23).

We can now complete the proof of the Proposition:

(i) When $2s-1 < 1/2$ (that is $s < 3/4$), Hardy's inequality (see Remark 2.8) implies

$$\int_\Omega \frac{|\partial_{x_N}\psi|^2}{|x_N|^{2(2s-1)}}\, \mathrm{d}x \leq C \|\nabla\psi\|^2_{H^{2s-1}(\Omega)}$$

and so $\mathcal{L}[\psi] \in L^2(\Omega)$ without further conditions and the bound on $\|\mathcal{L}\psi]\|_{L^2(\Omega)}$ follows from the bounds on I_1 and I_2 above.

(ii) When $2s - 1 \geq 1/2$, then we proved above that $\mathcal{L}[\psi]$ belongs to $L^2(\Omega)$ if and only if (23) holds.

Furthermore, since $\nabla\psi \in H^{2s-1+\beta}(\Omega)$ with $2s-1+\beta > 1/2$ we see that $\partial_{x_N}\psi$ has a well defined trace in $L^2(\partial\Omega)$ and (23) implies that this trace must vanish since $2(2s-1) \geq 1$. Conversely, if $\partial_{x_N}\psi = 0$ on $\partial\Omega$, then $\partial_{x_N}\psi$ belongs to $H^{2s-1+\beta}_0(\Omega)$, the closure of $C^\infty_0(\Omega)$. Hardy inequality (see Remark (2.8)) thus implies

$$\int_\Omega \frac{|\partial_{x_N}\psi|^2}{|x_N|^{2(2s-1+\beta)}}\, \mathrm{d}x \leq C \|\nabla\psi\|^2_{H^{2s-1+\beta}(\Omega)} \tag{27}$$

and so (23) holds.

We have thus shown that (23) was equivalent to the condition that $\partial_{x_N}\psi = 0$ on $\partial\Omega$. When this condition holds, then the inequality above gives

$$\int_\Omega \frac{|\partial_{x_N}\psi|^2}{|x_N|^{2(2s-1+\beta)}}\, \mathrm{d}x \leq C \|\psi\|^2_{H^{2s+\beta}(\Omega)}$$

and (24) follows. $\square$

Proof of Lemma 2.9. Using symmetry properties, we write:

$$h(x_N) = \int_{z_N > x_N;\, z'^2 + z_N^2 < 1} \frac{z_N}{(z_N^2 + z'^2)^{\frac{N+2s}{2}}} dz$$

Thus, by proceeding to the change of variable $y' = \frac{z'}{z_N}$ we find

$$\begin{aligned}
h(x_N) &= \int_{x_N < z_N < 1} \frac{z_N}{z_N^{N+2s}} z_N^{N-1} \left(\int_{y'^2 + 1 < \frac{1}{z_N{}^2}} \frac{1}{(1+y'^2)^{\frac{N+2s}{2}}} dy' \right) dz_N \\
&\leq \int_{x_N < z_N < 1} z_N^{-2s} dz_N \left(\int_{\mathbb{R}^{N-1}} \frac{1}{(1+y'^2)^{\frac{N+2s}{2}}} dy' \right) \\
&\leq \frac{C}{2s-1} ({x_N}^{1-2s} - 1),
\end{aligned}$$

which gives the desired upper bounds for $0 < x_N < 1$.

On another hand, we clearly have $h(x_N) \geq 0$ and we can also write

$$\begin{aligned}
h(x_N) &\geq \int_{x_N < z_N < \frac{3}{4}} z_N^{-2s} dz_N \left(\int_{y'^2 + 1 < \frac{16}{9}} \frac{1}{(1+y'^2)^{\frac{N+2s}{2}}} dy' \right) \\
&\geq \frac{C}{2s-1} ({x_N}^{1-2s} - (\frac{3}{4})^{1-2s})
\end{aligned}$$

which gives the lower bound when $0 < x_N < \frac{1}{2}$. □

We deduce the following Corollary which is useful in the proof of our main theorem:

Corollary 2.10. *If $\psi \in H^{2s+\beta}(\Omega)$ for some $\beta > 0$ and $\mathcal{L}[\psi] \in L^2(\Omega)$, then*

$$\delta^{-2(2s-1)} \int_\Omega |\partial_{x_N} \psi(t,x)|^2 1_{x_N \leq \delta}\, \mathrm{d}x \leq \|\psi\|^2_{H^{2s+\beta}(\Omega)} \delta^{2\beta'} \tag{28}$$

for some $\beta' > 0$.

Proof. When $2s - 1 < 1/2$ (that is $s < 3/4$), we can take $\beta' < \beta$ such that $2s - 1 + \beta' < 1/2$ and Hardy's inequality implies

$$\int_\Omega |x_N|^{-2(2s-1+\beta')} |\partial_{x_N} \psi|^2 \, \mathrm{d}x \leq C \|\nabla \psi\|^2_{H^{2s-1+\beta'}(\Omega)} \leq C \|\psi\|^2_{H^{2s+\beta}(\Omega)}.$$

Since $\delta^{-2(2s-1+\beta')} \leq |x_N|^{-2(2s-1+\beta')}$ when $x_N \leq \delta$, we deduce

$$\delta^{-2(2s-1+\beta')} \int_\Omega |\partial_{x_N} \psi(t,x)|^2 1_{x_N \leq \delta}\, \mathrm{d}x \leq C \|\psi\|^2_{H^{2s+\beta}(\Omega)}$$

and (28) follows.

When $2s - 1 \geq 1/2$, (28) (with $\beta' = \beta$) follows by a similar computation using (27). □

Finally, we prove the following integration by part formula for $\mathrm{div} D^{2s-1}$ (that we will prove to be $\mathcal{L}$):

Proposition 2.11. *Let ψ and φ be functions in $H^{2s+\beta}(\Omega)$ for some $\beta > 0$ such that $\mathcal{L}[\psi]$ and $\mathcal{L}[\varphi] \in L^2(\Omega)$. Then the following integration by parts formula holds:*

$$\int_\Omega \mathrm{div} D^{2s-1}[\varphi]\psi \,\mathrm{d}x - \int_\Omega \varphi \,\mathrm{div} D^{2s-1}[\psi]\,\mathrm{d}x = \int_{\partial\Omega} \left[\psi D^{2s-1}[\varphi]\cdot n - \varphi D^{2s-1}[\psi]\cdot n\right] \mathrm{d}S(x). \quad (29)$$

Note that we can also prove that this formula holds when φ and ψ are in $C^{1,\gamma}(\Omega)$ for some $\gamma > 2s-1$ and satisfies the Neumann condition $\partial_{x_N}\varphi = \partial_{x_N}\psi = 0$ on $\partial\Omega$ (see Proposition 2.5).

Proof of Proposition 2.11. Integrating by parts, we find:

$$\int_\Omega \mathrm{div} D^{2s-1}[\varphi]\psi\,\mathrm{d}x = -\int_\Omega D^{2s-1}[\varphi]\cdot\nabla\psi\,\mathrm{d}x + \int_{\partial\Omega} \psi D^{2s-1}[\varphi]\cdot n\,\mathrm{d}S(x). \quad (30)$$

So, formula (29) follows from the following equality:

$$\int_\Omega D^{2s-1}[\varphi]\cdot\nabla\psi\,\mathrm{d}x = \int_\Omega D^{2s-1}[\psi]\cdot\nabla\varphi\,\mathrm{d}x.$$

This equality is easily proved using the formula (9) for the operator D^{2s-1} since it gives the following symmetric expression:

$$\int_\Omega D^{2s-1}[\varphi]\cdot\nabla\psi\,\mathrm{d}x = \gamma\nu_0^{1-2s}\Gamma(2s-1)\int_\Omega\int_\Omega (y-x)\cdot\nabla\varphi(y)\frac{y-x}{|y-x|^{N+2s}}\cdot\nabla\psi(x)\,dy\,\mathrm{d}x. \quad (31)$$

□

3 Proof of Theorem 1.2

In this section, we rigorously prove the limit presented in the previous section in a particular case: we assume that $\Omega = \mathbb{R}^N_+$ is the upper-half space and that the collision cross-section is constant, so that

$$Q(f)(v) = \nu_0(\rho F(v) - f(v)), \qquad \rho = \int_{\mathbb{R}^N} f(v)\,\mathrm{d}v. \quad (32)$$

Furthermore, we assume that F satisfies

$$\begin{cases} F(v) \in L^\infty, \qquad \displaystyle\int_{\mathbb{R}^N} F(v)\,\mathrm{d}v = 1, \qquad F(v) = F(-v) \\ \left|F(v) - \frac{\gamma}{|v|^{N+2s}}\right| \le \frac{C}{|v|^{N+4s}} \qquad \text{for all } |v|\ge 1. \end{cases} \quad (33)$$

These assumptions on the equilibrium F are motivated by the equilibrium of the fractional Fokker-Planck operator studied in [10].

The basic idea of the proof is to rigorously pass to the limit in (18) (note that when Q is given by (32), the last term in (18) vanishes). To do this, we would like to solve (15)-(16) explicitly, which is difficult because of the boundary condition. Instead, we will construct solutions of the following equation

$$\begin{cases} \nu_0\phi^\varepsilon - \varepsilon v\cdot\nabla_x\phi^\varepsilon = \nu_0\psi(x) & \text{in } \Omega\times\mathbb{R}^N \\ \gamma_+\phi^\varepsilon = \psi(x) & \text{on } \Sigma_+. \end{cases} \quad (34)$$

The function ϕ^ε then satisfies the boundary condition (16) if and only if (see Lemma 3.1 below)

$$\int_{\mathbb{R}^N} vF(v)\left[\phi^\varepsilon(t,x,v) - \psi(t,x)\right]\,\mathrm{d}v\cdot n(x) = 0\,, \qquad \text{for all } x\in\partial\Omega, \quad (35)$$

which leads us to introduce the following operator

$$D_\varepsilon^{2s-1}[\psi](x) := \varepsilon^{1-2s} \int_{\mathbb{R}^N} vF(v)\big[\phi^\varepsilon(t,x,v) - \psi(t,x)\big]\, \mathrm{d}v, \qquad x \in \Omega \tag{36}$$

(note that this operator coincides with the operator $\tilde{D}_\varepsilon^{2s-1}$ of the previous section when ψ satisfies (35), but is otherwise different).

However, condition (35) depends on ε, so this approach would require us to consider a sequence of test function ψ^ε satisfying (35) and converging to ψ when $\varepsilon \to 0$. Since the existence of such a sequence is not clear, we will instead fix a function ψ such that

$$\lim_{\varepsilon \to 0} D_\varepsilon^{2s-1}[\psi] \cdot n = 0$$

and show that the corresponding function ϕ^ε, solution of (34) can be approximated by a function satisfying the boundary conditions (16). This last approximation is the main reason why we only prove our result when Ω is the upper-half space since the construction is significantly simpler in that case.

In the next section, we introduce an extension of ψ to $\mathbb{R}^N \times \mathbb{R}^N$ which will lead to an explicit formula for the solution of (34). We will then proceed with the proof of our main theorem with, in particular, the proof of the convergence of the operators D_ε^{2s-1} and $\mathcal{L}^\varepsilon$.

3.1 Construction of the test functions

Our first task is to explicitly solve equation (34) for a given a test function $\psi(t,x)$ defined in $[0,\infty) \times \overline{\Omega}$. To do that, we first define an extension $\widetilde{\psi}(t,x,v)$ of ψ to $[0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N$ by setting

$$\widetilde{\psi}(t,x,v) = \psi(t,x) \text{ for all } x \in \overline{\Omega} \text{ and all } v \in \mathbb{R}^N \tag{37}$$

and assuming that $\widetilde{\psi}(x,v,t)$ solves

$$\begin{cases} v \cdot \nabla_x \widetilde{\psi}(t,x,v) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times \mathbb{R}^N \\ \widetilde{\psi}(t,x,v) = \psi(t,x) & \text{on } \Sigma_+. \end{cases} \tag{38}$$

This equation states that for fixed (t,v), the function $x \mapsto \widetilde{\psi}(t,x,v)$ is constant along the characteristic lines $\tau \mapsto x + \tau v$ outside the set Ω. Since not all characteristic lines will intersect Σ_+, this does not define $\widetilde{\psi}(x,v)$ uniquely everywhere. However, we will see that the ambiguous points do not play any role in the sequel, so we can set $\widetilde{\psi}(x,v)$ to be zero there. We note the following obvious but important facts about this extension

1. For any $x \in \partial\Omega$, and any $v \in \mathbb{R}^N$ such that $v \cdot n(x) > 0$, we have

$$\widetilde{\psi}(t, x + \tau v, v) = \psi(t,x) \qquad \forall \tau \geq 0. \tag{39}$$

2. If $x \in \overline{\Omega}$, then $\widetilde{\psi}(t,x,\tau v) = \psi(t,x) = \widetilde{\psi}(t,x,v)$ for any $v \in \mathbb{R}^N$ and any $\tau \in \mathbb{R}$. If $x \in \mathbb{R}^N \setminus \Omega$, since the boundary condition in (38) does not depend on v, the function $\tau \mapsto \widetilde{\psi}(t,x,\tau v)$ is constant for any $v \in \mathbb{R}^N$ (check, for instance, that $\widetilde{\psi}(t,x,\tau v)$ is also a solution of (38)). We deduce
$$\widetilde{\psi}(t,x,\tau v) = \widetilde{\psi}(t,x,v) \qquad \forall \tau > 0, \quad \forall (t,x,v) \in (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N.$$

3. This construction is useful for any convex set Ω, but when Ω is the upper-half plane, we can get the following explicit formula:

$$\widetilde{\psi}(t,x,v) = \begin{cases} \psi(t,x) & \text{if } x_N > 0 \\ \psi\left(t, x' - \frac{x_N}{v_N} v', 0\right) & \text{if } x_N < 0, \quad v_N < 0 \end{cases} \tag{40}$$

As noted above, this does not define $\widetilde{\psi}(t,x,v)$ for $x \notin \Omega$ and $v_N > 0$, but these value do not play any role in what follows. We also have

$$\widetilde{\psi}(y, y-x) = \left\{ \begin{array}{ll} \psi(x) & \text{if } y_N < 0 \\ \psi(y) & \text{if } y_N > 0 \end{array} \right. \qquad \text{for all } x \in \partial\Omega. \tag{41}$$

4. Even of ψ is smooth, we do not expect $\widetilde{\psi}$ to be regular near $\partial\Omega$. For example, the function $x \mapsto \widetilde{\psi}(x,v)$ is C^0 (but typically not C^1) at a point $x_0 \in \partial\Omega$ only if $n(x_0)\cdot v > 0$. Nevertheless, it is difficult to show that if $x \mapsto \psi(x)$ is in $C^\alpha(\Omega)$ for some $\alpha \in (0,1)$, then we have

$$|\widetilde{\psi}(y, y-x) - \psi(x)| \le C|y-x|^\alpha \qquad \forall x \in \Omega,\ y \in \mathbb{R}^N. \tag{42}$$

We now have the following Lemma:

Lemma 3.1. *For any test function $\psi(t,x)$ defined in $[0,\infty)\times\overline{\Omega}$, the function*

$$\phi^\varepsilon(t,x,v) = \int_0^\infty e^{-\nu_0 z}\nu_0 \widetilde{\psi}(t, x+\varepsilon v z, v)\,\mathrm{d}z \tag{43}$$

solves (34)*. Furthermore, ϕ^ε satisfies the boundary condition* (16) *if and only if ψ is such that*

$$\int_{\mathbb{R}^N} vF(v)\big[\phi^\varepsilon(t,x,v) - \psi(t,x)\big]\,\mathrm{d}v\cdot n(x) = 0\ , \qquad \textit{for all } x \in \partial\Omega. \tag{44}$$

With ϕ^ε is given by (43), the operator defined in (36) becomes:

$$D^{2s-1}_\varepsilon[\psi](x) = \varepsilon^{1-2s}\int_{\mathbb{R}^N}\int_0^\infty e^{-\nu_0 z}\nu_0 vF(v)[\widetilde{\psi}(x+\varepsilon v z, v) - \psi(x)]\,dz\,\mathrm{d}v. \tag{45}$$

We also introduce the limiting operator

$$D^{2s-1}[\psi](x) := \gamma\nu_0^{1-2s}\Gamma(2s)\int_{\mathbb{R}^N}\left[\widetilde{\psi}(y, y-x) - \psi(x)\right]\frac{y-x}{|y-x|^{N+2s}}\,dy. \tag{46}$$

We will prove in Proposition 3.4 below that $D^{2s-1}_\varepsilon[\psi]$ converges to $D^{2s-1}[\psi]$ and that (46) is equivalent to the formula (9) given in the introduction (and which does not involve the extension $\widetilde{\psi}$).

Proof of Lemma 3.1. Using (37), we easily check that ϕ^ε satisfies (15), and using (39) (this is where the definition of the extension $\widetilde{\psi}$ is crucial), we see that for all $x \in \partial\Omega$ and v such that $v\cdot n(x) > 0$, we have:

$$\gamma_+\phi^\varepsilon(t,x,v) = \int_0^\infty e^{-\nu_0 z}\nu_0\widetilde{\psi}(t, x+\varepsilon v z, v)\,\mathrm{d}z = \int_0^\infty e^{-\nu_0 z}\nu_0\psi(t,x)\,\mathrm{d}z = \psi(t,x).$$

which is the boundary condition in (34). Next, we note that the boundary condition (16) is satisfied if and only if

$$\gamma_+\phi^\varepsilon = \psi(x,t) = \int_{w\cdot n(x)<0}\alpha_0 F(w)\phi^\varepsilon(x,v)|w\cdot n(x)|\,\mathrm{d}w \quad \text{ for all } x \in \partial\Omega.$$

Using the normalization condition $\int_{w\cdot n(x)<0}\alpha_0 F(w)|w\cdot n(x)|\,\mathrm{d}w = 1$, we can rewrite this condition as

$$\int_{w\cdot n(x)<0} wF(w)\big[\phi^\varepsilon(t,x,v)-\psi(t,x)\big]\,\mathrm{d}w\cdot n(x)=0\,,\qquad \text{for all } x\in\partial\Omega. \tag{47}$$

Finally, since $\phi^\varepsilon(t,x,v)=\psi(t,x)$ when $x\in\partial\Omega$ and $v\cdot n(x)>0$, we can extend the integral over all $w\in\mathbb{R}^N$ and write this condition as (44). □

Since equation (44) depends on ε and we want to work with a fixed ψ, we will assume that ψ satisfies the limiting Neumann boundary condition (11):

$$D^{2s-1}[\psi](t,x)\cdot n(x)=0 \qquad \text{for all } x\in\partial\Omega.$$

While this implies that ψ almost satisfies (44) for small ε, it is not enough since we need ϕ^ε to satisfy (16) in order to take it as a test function in (8). We will thus now approximate ϕ^ε by a new function ϕ_0^ε which is an exact solution of the boundary condition (44) (but an approximated solution of the transport equation(15)).

This construction is simpler when Ω is the upper-half space so we restrict ourselves to this case from now on. In particular, using (41) and (46), the Neumann boundary condition (11) can then be written as

$$\int_{\mathbb{R}^N_+}\big[\psi(y)-\psi(x)\big]\frac{(y-x)\cdot n}{|y-x|^{N+2s}}\,dy=0 \qquad \text{for all } x=(x',0)\in\partial\Omega$$

or equivalently

$$\int_{\mathbb{R}^N_+}\big[\psi(x+y)-\psi(x)\big]\frac{y\cdot n}{|y|^{N+2s}}\,dy=0 \qquad \text{for all } x=(x',0)\in\partial\Omega \tag{48}$$

where the outward normal vector n is given by $n=(0,\dots,0,-1)$.

We now introduce the following approximation of ϕ^ε:

$$\phi_0^\varepsilon(t,x,v)=\phi^\varepsilon(t,x,v)+T^\varepsilon(t,x)\chi(v). \tag{49}$$

where χ is a smooth function compactly supported in $\mathbb{R}^N_+$, satisfying

$$\chi(v)=0 \text{ if } v_N<0, \quad \mathcal{B}^*[\chi](v)=\int_{w\cdot n<0}\alpha_0 F(w)\chi(w)|w\cdot n|\,\mathrm{d}w=1.$$

Using (41), we see that the function $\phi^\varepsilon(t,x,v)$ satisfies the boundary condition (16) if and only if

$$\begin{aligned}\psi(t,x)&=\mathcal{B}^*[\phi^\varepsilon(t,x,\cdot)+T^\varepsilon(t,x)\chi(v)]\\&=\mathcal{B}^*[\phi^\varepsilon(t,x,\cdot)]+T^\varepsilon(t,x)\mathcal{B}^*[\chi(v)]\\&=\mathcal{B}^*[\phi^\varepsilon(t,x,\cdot)]+T^\varepsilon(t,x)\qquad \text{for all } x=(x',0)\in\partial\Omega.\end{aligned}$$

So $T^\varepsilon(t,x)$ must satisfy

$$\begin{aligned}T^\varepsilon(t,x)&=-\int_{w\cdot n<0}\alpha_0F(w)(\phi^\varepsilon(t,x,v)-\psi(x))|w\cdot n|\,\mathrm{d}w\\&=\varepsilon^{2s-1}\alpha_0 D_\varepsilon^{2s-1}[\psi](t,x)\cdot n \qquad \forall x=(x',0)\in\partial\Omega.\end{aligned}$$

We thus define

$$\overline{T}^\varepsilon(t,x'):=D_\varepsilon^{2s-1}[\psi](t,x',0)\cdot n \quad \text{ for } x'\in\mathbb{R}^{N-1}$$

and set

$$T^\varepsilon(t,x) = \varepsilon^{2s-1}\alpha_0 \overline{T}^\varepsilon(t,x')e^{-x_N{}^2} \qquad \text{for } x \in \Omega. \tag{50}$$

Note that the addition of the corrector $T^\varepsilon(t,x)\chi(v)$ in (49) guarantees that the function ϕ_0^ε satisfies the boundary condition (16), but it no longer satisfies the transport-like equation (15). However, we will show in Proposition 3.9 that T^ε goes to zero as $\varepsilon \to 0$. The following remark ensures that it is still an admissible test function in the sense of Definition 1.1:

Remarks 3.2. *The definition of ϕ^ε, (43), implies*

$$\iint_{\Omega\times\mathbb{R}^N} |\phi^\varepsilon(t,x,v)|^2 F(v)\,\mathrm{d}x\,\mathrm{d}v \le \iint_{\Omega\times\mathbb{R}^N} \int_0^\infty e^{-\nu_0 z}\nu_0 |\widetilde{\psi}^\varepsilon(t,x+\varepsilon z v, v)|^2 F(v)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}v$$

Using (59) *(proved below), we deduce*

$$\begin{aligned}\iint_{\Omega\times\mathbb{R}^N} |\phi^\varepsilon(t,x,v)|^2 F(v)\,\mathrm{d}x\,\mathrm{d}v &\le \int_{\mathbb{R}^N}\int_0^\infty e^{-\nu_0 z}\nu_0 C(\psi)(1+|v_N|)F(v)\,\mathrm{d}v \\ &\le C(\psi)\end{aligned}$$

for some constant $C(\psi)$ depending on $\|\psi\|_{L^2(\mathbb{R}_+\times\mathbb{R}^N_+)}$ and $\|\psi|_{\partial\Omega}\|_{L^2(\mathbb{R}_+\times\mathbb{R}^{N-1})}$. A similar bound holds for $\partial_t\phi^\varepsilon$ since t is a parameter in the definition of ϕ^ε. Furthermore, Equation (15) *then implies that*

$$\iint_{\Omega\times\mathbb{R}^N} |\varepsilon v\cdot\nabla_x\phi^\varepsilon(t,x,v)|^2 F(v)\,\mathrm{d}x\,\mathrm{d}v < \infty$$

From there, it is easy to check that we can indeed take ϕ_0^ε as a test function in (8).

We can now proceed as in Section 1.2: We take the function ϕ_0^ε constructed above as test function in the weak formulation (8) of (5). We obtain:

$$\iiint_{\mathbb{R}_+\times\Omega\times\mathbb{R}^N} f^\varepsilon\partial_t\phi^\varepsilon\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v + \iint_{\mathbb{R}_+\times\Omega} \rho^\varepsilon\mathcal{L}^\varepsilon[\psi]\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \tag{51}$$

$$+ \iiint_{\mathbb{R}_+\times\Omega\times\mathbb{R}^N} f^\varepsilon\Big(\partial_t T^\varepsilon(t,x)\chi(v) + \varepsilon^{1-2s}v\cdot\nabla_x T^\varepsilon(t,x)\chi(v) - \varepsilon^{-2s}Q^*\big[\chi(v)\big]T^\varepsilon(t,x)\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \tag{52}$$

$$+ \iint_{\Omega\times\mathbb{R}^N} f_{in}(x,v)\phi^\varepsilon(0,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0 \tag{53}$$

with

$$\mathcal{L}^\varepsilon[\psi](x) := \varepsilon^{-2s}\int_{\mathbb{R}^N} \nu_0 F(v)[\phi^\varepsilon(x,v) - \psi(x)]\,\mathrm{d}v \tag{54}$$

$$= \varepsilon^{-2s}\int_{\mathbb{R}^N} F(v)\varepsilon v\cdot\nabla_x\phi^\varepsilon(x,v)\,\mathrm{d}v \tag{55}$$

$$= \mathrm{div}_x\left(\varepsilon^{1-2s}\int_{\mathbb{R}^N} vF(v)\phi^\varepsilon(x,v)\,\mathrm{d}v\right) \tag{56}$$

$$= \mathrm{div}_x\left(\varepsilon^{1-2s}\int_{\mathbb{R}^N} vF(v)[\phi^\varepsilon(x,v)-\psi(x)]\,\mathrm{d}v\right)$$

$$= \mathrm{div}_x D_\varepsilon^{2s-1}[\psi](x). \tag{57}$$

This equation differs from (18) in two important ways: Because Q is given by (32), the last term in (18) does not appear in (51) (since $K(g^\varepsilon) = \int_{\mathbb{R}^N} g^\varepsilon(v)\,dv = 0$). On the other hand, the construction of ϕ_0^ε has given rise to the additional terms (52) in the second line and we will need to show that these terms vanishe in the limit (this is where we need ψ to satisfy the Neumann boundary condition). The rest of the proof consist in passing to the limit in this equation.

We conclude this subsection with the following simple lemma which will be useful several times throughout the paper:

Lemma 3.3. *For all $\psi \in H^s(\mathbb{R}^N_+)$, $s > 1/2$, and $v \in \mathbb{R}^N$ we have:*

$$\int_{\mathbb{R}^N_+} |\widetilde{\psi}(x+v,v)|^2\,\mathrm{d}x \le \begin{cases} \int_{\mathbb{R}^N_+} |\psi(x)|^2\,\mathrm{d}x & \text{if } v_N > 0 \\ \int_{\mathbb{R}^N_+} |\psi(x)|^2\,\mathrm{d}x + |v_N| \int_{\mathbb{R}^{N-1}} |\psi(x',0)|^2\,\mathrm{d}x' & \text{if } v_N < 0 \end{cases} \tag{58}$$

In the sequel, we will repeatedly use the inequality (which follows from (58)):

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^N_+} |\widetilde{\psi}(t,x+v,v)|^2\,\mathrm{d}x\,\mathrm{d}t \le C(1+|v_N|)\|\psi\|_{H^s(\mathbb{R}_+\times\mathbb{R}^N_+)}. \tag{59}$$

Proof of Lemma 3.3. If $v_N > 0$, then $x + v \in \Omega$ for all $x \in \Omega$ and so $\widetilde{\psi}(x+v,v) = \psi(x+v)$. We deduce

$$\int_{\mathbb{R}^N_+} |\widetilde{\psi}(x+v,v)|^2\,\mathrm{d}x = \int_{\mathbb{R}^N_+} |\psi(x+v)|^2\,\mathrm{d}x = \int_{\mathbb{R}^N_+, x_N \ge v_N} |\psi(x)|^2\,\mathrm{d}x$$

and the first inequality follows.

If $v_N < 0$, then we write:

$$\begin{aligned} \int_{\mathbb{R}^N_+} |\widetilde{\psi}(x+v,v)|^2\,\mathrm{d}x &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^{N-1}} |\widetilde{\psi}((x'+v', x_N+v_N), v)|^2\,\mathrm{d}x'\,\mathrm{d}x_N \\ &= \int_{-v_N}^{\infty} \int_{\mathbb{R}^{N-1}} |\psi(x+v)|^2\,\mathrm{d}x'\,\mathrm{d}x_N + \int_0^{-v_N} \int_{\mathbb{R}^{N-1}} |\widetilde{\psi}(x+v,v)|^2\,\mathrm{d}x'\,\mathrm{d}x_N \\ &= \int_{\mathbb{R}^N_+} |\psi(x)|^2\,\mathrm{d}x + \int_0^{-v_N} \int_{\mathbb{R}^{N-1}} |\widetilde{\psi}(x+v,v)|^2\,\mathrm{d}x'\,\mathrm{d}x_N. \end{aligned}$$

Next, we note that if $x \in \mathbb{R}^N_+$ and $x + v \in \mathbb{R}^N_-$, then

$$\widetilde{\psi}(x+v,v) = \psi\left(x - \frac{x_N}{v_N} v\right).$$

We deduce

$$\begin{aligned} \int_0^{-v_N} \int_{\mathbb{R}^{N-1}} |\widetilde{\psi}((x'+v', x_N+v_N), v)|^2\,\mathrm{d}x'\,\mathrm{d}x_N &= \int_0^{-v_N} \int_{\mathbb{R}^{N-1}} |\psi\left(x - \frac{x_N}{v_N} v\right)|^2\,\mathrm{d}x'\,\mathrm{d}x_N \\ &= -v_N \int_{\mathbb{R}^{N-1}} |\psi\,(x',0))\,|^2\,\mathrm{d}x'\,\mathrm{d}x_N \end{aligned}$$

and the second inequality in (58) follows (it is in fact an equality). $\square$

3.2 Convergence of the operators

In this section, we carefully define the operators D_ε^{2s-1}, $\mathcal{L}^\varepsilon$ and their limits and prove the main convergence result (Proposition 3.5).

The operators D_ε^{2s-1} and D^{2s-1}. For a given function $\psi(x)$, we recall that we defined the operator D_ε^{2s-1} by (45). After the change of variable $w = vz$, we can also rewrite (45) as

$$D_\varepsilon^{2s-1}[\psi](x) = \varepsilon^{1-2s} \int_{\mathbb{R}^N} vF_0(v)[\widetilde{\psi}(x+\varepsilon v, v) - \psi(x)]\,\mathrm{d}v \tag{60}$$

where

$$F_0(v) = \int_0^\infty e^{-\nu_0 z}\nu_0 F(v/z) z^{-N-1}\,\mathrm{d}z. \tag{61}$$

We will prove:

Proposition 3.4. *For all functions $\psi(x) \in L^\infty(0,\infty; H^1(\Omega))$, we have*

$$D_\varepsilon^{2s-1}[\psi] \to D^{2s-1}[\psi] \text{ in } L^2(\Omega)\text{-strong}$$

as $\varepsilon \to 0$, where the fractional gradient D^{2s-1} is defined by (46) *or, equivalently, by* (9).

The operators $\mathcal{L}^\varepsilon$ and $\mathcal{L}$. We recall that $\mathcal{L}^\varepsilon$ is defined by (54):

$$\mathcal{L}^\varepsilon[\psi](x) := \varepsilon^{-2s} \int_{\mathbb{R}^N} \nu_0 F(v)[\phi^\varepsilon(x,v) - \psi(x)]\,\mathrm{d}v$$

and using (43) and the change of variable $w = vz$, we find

$$\begin{aligned}
\mathcal{L}^\varepsilon[\psi](x) &= \varepsilon^{-2s}\nu_0 \int_{\mathbb{R}^N}\int_0^\infty e^{-\nu_0 z}\nu_0 F(v)[\widetilde{\psi}(x+\varepsilon vz, v) - \psi(x)]\,\mathrm{d}v \\
&= \varepsilon^{-2s}\int_{\mathbb{R}^N} F_1(v)[\widetilde{\psi}(x+\varepsilon v, v) - \psi(x)]\,\mathrm{d}v
\end{aligned} \tag{62}$$

where F_1 is defined by

$$F_1(v) = \int_0^\infty e^{-\nu_0 z}\nu_0^2 F(v/z) z^{-N}\,\mathrm{d}z. \tag{63}$$

We also define the corresponding asymptotic operator:

$$\mathcal{L}[\psi](x) := \gamma\,\nu_0^{1-2s}\Gamma(2s+1)\mathrm{P.V.}\int_{\mathbb{R}^N} [\widetilde{\psi}(y, y-x) - \psi(x)]\frac{1}{|y-x|^{N+2s}}\,\mathrm{d}y. \tag{64}$$

Since $\widetilde{\psi}(y, y-x) = \psi(y)$ for y in Ω, the principal value in the right hand side of (64) is defined for all $x \in \Omega$ if ψ is in $C^{1,\beta}(\Omega)$ for some $\beta > 2s-1$. However, even for such functions, $\mathcal{L}[\psi](x)$ is typically singular when $x \to \partial\Omega$. Indeed, we will prove in Proposition 2.5 that if ψ is in $C^{1,\beta}(\Omega)$, then the function $\mathcal{L}[\psi](x)$ remains bounded as $x \to \partial\Omega$ if and only if ψ satisfies the classical Neumann boundary condition

$$\nabla_x\psi \cdot n(x) = 0.$$

A key result in the proof of Theorem 1.2 will be the following:

Proposition 3.5. *For all function $\psi \in L^\infty(0,\infty; H^2(\Omega))$, such that $\mathcal{L}[\psi] \in L^2(\mathbb{R}_+ \times \Omega)$ we have*

$$\mathcal{L}^\varepsilon[\psi](t,x) \longrightarrow \mathcal{L}[\psi](t,x) \qquad \text{in } L^2((0,T)\times\Omega)\text{-strong for all } T > 0.$$

We refer to Proposition 2.6 for a characterization of the functions $\psi \in H^2(\Omega)$ such that $\mathcal{L}[\psi](x) \in L^2(\Omega)$).

We recall that we also have (see (56))

$$\mathcal{L}^\varepsilon[\psi](x) = \mathrm{div}_x D_\varepsilon^{2s-1}[\psi](x)$$

and using Propositions 3.4 and 3.5, we immediately deduce:

Corollary 3.6. *The operator $\mathcal{L}$ defined by* (64) *satisfies*

$$\mathcal{L}[\psi](x) = \mathrm{div} D^{2s-1}[\psi](x)$$

for all function $\psi \in H^2(\Omega)$ such that $\mathcal{L}[\psi](x) \in L^2(\Omega)$.

We will also prove the following result which justify the formula for $\mathcal{L}$ given in the introduction:

Lemma 3.7. *Let $\psi \in C^{1,\beta}(\Omega)$ for some $\beta > 2s-1$. Then*

$$\mathcal{L}[\psi](x) = \gamma\nu_0^{1-2s}\Gamma(2s)\mathrm{P.V.}\int_\Omega \frac{y-x}{|y-x|^{N+2s}} \cdot \nabla_y\psi(y)\,\mathrm{d}y \qquad \forall x \in \Omega. \tag{65}$$

Finally, integrating by parts the formula (13), we can also write the following formula for $\mathcal{L}$

$$\begin{aligned}\mathcal{L}[\psi](x) = {}& \gamma\nu_0^{1-2s}\Gamma(2s+1)\mathrm{P.V.}\int_\Omega \frac{\psi(y)-\psi(x)}{|y-x|^{N+2s}}\,\mathrm{d}y \\ & + \gamma\nu_0^{1-2s}\Gamma(2s)\mathrm{P.V.}\int_{\partial\Omega} \frac{y-x}{|y-x|^{N+2s}} \cdot n(y)[\psi(y)-\psi(x)]\,\mathrm{d}y \end{aligned} \tag{66}$$

which clearly show the relation between the operator $\mathcal{L}$ and the regional fractional laplacian defined in the introduction.

We now turn to the proof of these results.

The proof of Propositions 3.4 and 3.5 use very similar computations. We will only prove the second one in details, since it is clearly the more complicated of the two. Before that, we note that the introduction of the functions F_0 and F_1 above allowed us to eliminate the variable z from the definition of D^{2s-1}_ε and $\mathcal{L}^\varepsilon$. Of course, their behavior for large v is related to that of F. More precisely, we have the following Lemma:

Lemma 3.8. *If F satisfies* (33)*, then the functions F_0 and F_1 defined by* (61) *and* (63) *satisfy:*

$$\left|F_0(v) - \frac{\gamma_0}{|v|^{N+2s}}\right| \leq \frac{C}{|v|^{N+4s}} \qquad \text{for all } |v| \geq 1, \qquad \gamma_0 = \gamma\,\nu_0^{1-2s}\Gamma(2s) \tag{67}$$

$$\left|F_1(w) - \frac{\gamma_1}{|w|^{N+2s}}\right| \leq \frac{C}{|w|^{N+4s}} \qquad \text{for all } |w| \geq 1, \qquad \gamma_1 = \gamma\nu_0^{1-2s}\Gamma(2s+1) \tag{68}$$

Proof of Lemma 3.8. We only prove (67) since the proof of (68) is almost identical. We start by noticing that

$$\int_0^\infty e^{-\nu_0 z}\nu_0 z^{2s-1}\,\mathrm{d}z = \nu_0^{1-2s}\Gamma(2s)$$

and so

$$F_0(v) - \frac{\gamma_0}{|v|^{N+2s}} = \int_0^\infty e^{-\nu_0 z}\nu_0\left[z^{-N-2s}F(v/z) - \frac{\gamma}{|v|^{N+2s}}\right] z^{2s-1}\,\mathrm{d}z.$$

For $|v| \geq 1$, using (33), we deduce

$$\begin{aligned}\left|F_0(v) - \frac{\gamma_0}{|v|^{N+2s}}\right| &\leq \int_0^{|v|} e^{-\nu_0 z}\nu_0\frac{Cz^{2s}}{|v|^{N+4s}}z^{2s-1}\,\mathrm{d}z + C\int_{|v|}^\infty e^{-\nu_0 z}\nu_0 z^{2s-1}\,\mathrm{d}z \\ &\leq \frac{C}{|v|^{N+4s}} + e^{-\nu_0|v|/2}\end{aligned}$$

and the result follows. $\square$

Proof of Proposition 3.5. We denote

$$I_\varepsilon = \iint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+} \big(\mathcal{L}^\varepsilon[\psi](x) - \mathcal{L}[\psi](x)\big)^2 \,\mathrm{d}t\,\mathrm{d}x.$$

We are going to show that $\lim_{\varepsilon\to 0} I^\varepsilon = 0$. Definition (64) and the fact that $\gamma_1 = \gamma\nu_0^{-2s}\Gamma(2s+1)$ (see (68)) imply

$$\begin{aligned}\mathcal{L}[\psi](x) &= \mathrm{P.V.}\int_{\mathbb{R}^N} [\widetilde{\psi}(y, y-x) - \psi(x)] \frac{\gamma_1}{|y-x|^{N+2s}}\,\mathrm{d}y\\ &= \varepsilon^{-2s}\mathrm{P.V.}\int_{\mathbb{R}^N} [\widetilde{\psi}(x+\varepsilon z v, v) - \psi(x)] \frac{\gamma_1}{|v|^{N+2s}}\,\mathrm{d}v.\end{aligned}$$

Using (62), we deduce

$$\mathcal{L}^\varepsilon[\psi](x) - \mathcal{L}[\psi](x) = \varepsilon^{-2s}\mathrm{P.V.}\int_{\mathbb{R}^N} [\widetilde{\psi}(x+\varepsilon v, v) - \psi(x)] \left(F_1(v) - \frac{\gamma_1}{|v|^{N+2s}}\right)\,\mathrm{d}v.$$

Setting $G(v) = F_1(v) - \frac{\gamma_1}{|v|^{N+2s}}$, we thus write

$$\begin{aligned}I_\varepsilon &= \iint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+} \left(\varepsilon^{-2s}\mathrm{P.V.}\int_{\mathbb{R}^N} [\widetilde{\psi}(x+\varepsilon v, v) - \psi(x)]G(v)\,\mathrm{d}v\right)^2 \mathrm{d}t\,\mathrm{d}x\\ &\leq \iint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+} \left(\varepsilon^{-2s}\mathrm{P.V.}\int_{|\varepsilon^\alpha v|<1} [\widetilde{\psi}(x+\varepsilon v, v) - \psi(x)]G(v)\,\mathrm{d}v\right)^2 \mathrm{d}t\,\mathrm{d}x\\ &\quad+ \iint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+} \left(\varepsilon^{-2s}\int_{|\varepsilon^\alpha v|>1} [\widetilde{\psi}(x+\varepsilon v, v) - \psi(x)]G(v)\,\mathrm{d}v\right)^2 \mathrm{d}t\,\mathrm{d}x\\ &= I_\varepsilon^- + I_\varepsilon^+\end{aligned}$$

for some $\alpha \in (0,1)$ to be chosen later.

Note that $G(v)$ is singular near 0, but $G(v)$ decays faster than $F(v)$ as $|v|\to\infty$. Indeed, we have (see (68)) $G(v) \leq C|v|^{-(N+4s)}$. We thus write, using (58)

$$\begin{aligned}I_\varepsilon^+ &\leq \varepsilon^{-4s}\left(\int_{|\varepsilon^\alpha v|>1} G(v)\,\mathrm{d}v\right)\left(\iint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+}\int_{|\varepsilon^\alpha v|>1} [\widetilde{\psi}(x+\varepsilon v, v) - \psi(x)]^2 G(v)\ \mathrm{d}v\,\mathrm{d}t\,\mathrm{d}x\right)\\ &\leq C\varepsilon^{-4s(1-\alpha)}\int_{|\varepsilon^\alpha v|>1}\iint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+} \big[|\widetilde{\psi}(x+\varepsilon v, v)|^2 + |\psi(x)|^2\big]\,\mathrm{d}t\,\mathrm{d}x\,G(v)\,\mathrm{d}v\\ &\leq C(\psi)\varepsilon^{-4s(1-\alpha)}\int_{|\varepsilon^\alpha v|>1} (1+|\varepsilon v|)G(v)\,\mathrm{d}v\\ &\leq C(\psi)\varepsilon^{4s(2\alpha-1)+1-\alpha}\end{aligned}\tag{69}$$

In order to bound I_ε^-, we write

$$\begin{aligned}\widetilde{\psi}(x+\varepsilon v,v)-\psi(x) &= \int_0^1 \frac{d}{d\tau}\widetilde{\psi}(x+\tau\varepsilon v,v)\,\mathrm{d}\tau\\ &= \int_0^1 \varepsilon v\cdot\nabla_x\widetilde{\psi}(x+\tau\varepsilon v,v)\,\mathrm{d}\tau\\ &= \int_0^{\tau_0^\varepsilon(x,v)} \varepsilon v\cdot\nabla_x\psi(x+\tau\varepsilon v)\,\mathrm{d}\tau\end{aligned}$$

where we use the exit time τ_0^ε defined as

$$\tau_0^\varepsilon(x,v)=\sup\{\tau\in[0,1]\,;\,x+\tau\varepsilon v\in\mathbb{R}^N_+\}. \tag{70}$$

Note that $\tau_0^\varepsilon(x,v)>0$ unless $x\in\partial\mathbb{R}^N_+$ and $\tau_0^\varepsilon(x,v)=1$ if $x+\varepsilon v\in\mathbb{R}^N_+$, otherwise $\tau_0^\varepsilon(x,v)=-\frac{x_N}{\varepsilon v_N}$. With one more integration by part, we can also write

$$\widetilde{\psi}(x+\varepsilon v,v)-\psi(x)=\tau_0^\varepsilon(x,v)\varepsilon v\cdot\nabla_x\psi(x)+\int_0^{\tau_0^\varepsilon(x,v)}(\tau_0^\varepsilon(x,v)-\tau)D_x^2\psi(x+\tau\varepsilon v)(\varepsilon v,\varepsilon v)\,\mathrm{d}\tau \tag{71}$$

We can thus write

$$\begin{aligned}\varepsilon^{-2s}\mathrm{P.V.}\int_{|\varepsilon^\alpha v|<1}[\widetilde{\psi}(x+\varepsilon v,v)-\psi(x)]G(v)\,\mathrm{d}v &= \varepsilon^{1-2s}\mathrm{P.V.}\int_{|\varepsilon^\alpha v|<1}\tau_0^\varepsilon(x,v)vG(v)\,\mathrm{d}v\cdot\nabla_x\psi(x)\\ &\quad+\varepsilon^{-2s}\int_{|\varepsilon^\alpha v|<1}\int_0^{\tau_0^\varepsilon(x,v)}(\tau_0^\varepsilon(x,v)-\tau)D_x^2\psi(x+\tau\varepsilon v)(\varepsilon v,\varepsilon v)\,\mathrm{d}\tau\,G(v)\,\mathrm{d}v\end{aligned}$$

We claim that:

$$\left|\varepsilon^{1-2s}\mathrm{P.V.}\int_{|\varepsilon^\alpha v|<1}\tau_0^\varepsilon(x,v)vG(v)\,\mathrm{d}v\cdot\nabla_x\psi(x)\right|\le\varepsilon^{(\alpha-1)(2s-1)}1_{\{x_N\le\varepsilon^{1-\alpha}\}}|\partial_{x_N}\psi(t,x)|. \tag{72}$$

Assuming this for now, we deduce (recall that $\tau_0^\varepsilon(x,v)\le1$):

$$\begin{aligned}I_\varepsilon^- \le&\iint_{\mathbb{R}_+\times\mathbb{R}^N_+}\left|\varepsilon^{(\alpha-1)(2s-1)}\partial_{x_N}\psi(t,x)\right|^2 1_{\{x_N\le\varepsilon^{1-\alpha}\}}\,\mathrm{d}t\,\mathrm{d}x\\ &+\iint_{\mathbb{R}_+\times\mathbb{R}^N_+}\left(\varepsilon^{-2s}\int_{|\varepsilon^\alpha v|<1}\int_0^{\tau_0^\varepsilon(x,v)}|D_x^2\psi(x+\tau\varepsilon v)||\varepsilon v|^2\,\mathrm{d}\tau\,G(v)\,\mathrm{d}v\right)^2\mathrm{d}t\,\mathrm{d}x\\ \le&\iint_{\mathbb{R}_+\times\mathbb{R}^N_+}\left|\varepsilon^{(\alpha-1)(2s-1)}\partial_{x_N}\psi(t,x)\right|^2 1_{\{x_N\le\varepsilon^{1-\alpha}\}}\,\mathrm{d}t\,\mathrm{d}x\\ &+\left(\iint_{\mathbb{R}_+\times\mathbb{R}^N_+}\int_{|\varepsilon^\alpha v|<1}\varepsilon^{2-2s}|v|^2G(v)\,\mathrm{d}v\right)\left(\int_{|\varepsilon^\alpha v|<1}\int_0^{\tau_0^\varepsilon(x,v)}|D_x^2\psi(x+\tau\varepsilon v)|^2\,\mathrm{d}\tau\varepsilon^{2-2s}|v|^2G(v)\,\mathrm{d}v\,\mathrm{d}t\,\mathrm{d}x\right)\\ \le&\,\varepsilon^{-2(1-\alpha)(2s-1)}\iint_{\mathbb{R}_+\times\mathbb{R}^N_+}|\partial_{x_N}\psi(t,x)|^2 1_{\{x_N\le\varepsilon^{1-\alpha}\}}\,\mathrm{d}t\,\mathrm{d}x\\ &+C(\varepsilon^{2-2s}+\varepsilon^{(1-\alpha)(2-2s)})^2\iint_{\mathbb{R}_+\times\mathbb{R}^N_+}|D_x^2\psi(t,x)|^2\,\mathrm{d}t\,\mathrm{d}x\end{aligned} \tag{73}$$

where we used the fact that $|G(v)| \leq C/|v|^{N+2s}$ and so by definition of $G(v)$

$$\int_{|\varepsilon^\alpha v|<1} \varepsilon^{2-2s}|v|^2 G(v)\,\mathrm{d}v \leq C(\varepsilon^{2-2s} + \varepsilon^{(1-\alpha)(2-2s)}).$$

The first term in the right hand side of (73) is not obviously bounded for test functions ψ in $H^2(\Omega)$. However, we will prove in the next section that this term must go to zero when $\varepsilon \to 0$ if ψ is such that $\mathcal{L}[\psi] \in L^2(\mathbb{R}_+ \times \mathbb{R}^N_+)$ (This is the only place in the proof where we make use of this assumption). More precisely, using Corollary 2.10 (Equation (28)), we deduce from (73) that

$$I_\varepsilon^- \leq C(\psi)\varepsilon^{2(1-\alpha)(2-2s)} + o(1).$$

Combing this with (69) we get

$$I^\varepsilon \leq C(\psi)\varepsilon^{4s(2\alpha-1)+1-\alpha} + C(\psi)\varepsilon^{2(1-\alpha)(2-2s)} + o(1)$$

and taking $\alpha \in (1/2, 1)$ yields the result.

It remains to show (72). First, we note that $G(v)$ is even and that $\tau_0^\varepsilon(x, v', v_N) = \tau_0^\varepsilon(x, -v', v_N)$. We deduce

$$\text{P.V.} \int_{|\varepsilon^\alpha v|<1} \tau_0^\varepsilon(x, v) v_i G(v)\,\mathrm{d}v = 0 \qquad \text{for all } i = 1, \dots N-1.$$

Next, we note that if $|\varepsilon v| \leq x_N$, then $x_n + \varepsilon v_N \geq x_N - |\varepsilon v_N| \geq 0$ and so $\tau_0^\varepsilon(x, v) = 1$. We deduce

$$\text{P.V.} \int_{|\varepsilon v|<x_N} \tau_0^\varepsilon(x, v) v_N G(v)\,\mathrm{d}v = \text{P.V.} \int_{|\varepsilon v|<x_N} v_N G(v)\,\mathrm{d}v = 0.$$

Finally, since $|G(v)| \leq C/|v|^{N+2s}$, we have for $x_N < \varepsilon^{1-\alpha}$:

$$\left| \int_{\varepsilon^{\alpha-1}x_N<|\varepsilon^\alpha v|<1} \tau_0^\varepsilon(x, v) v_N G(v)\,\mathrm{d}v \right| \leq C\varepsilon^{2s-1}|x_N|^{1-2s} + C\varepsilon^{\alpha(2s-1)} \leq C\varepsilon^{\alpha(2s-1)}.$$

The last three equations imply (72). $\square$

Proof of Proposition 3.4. Proposition 3.4 is proved in a similar manner. It is simpler of course since it only requires a first order Taylor expansion instead of the second order expansion (71). We just need to check that (46) is equivalent to definition (9). First we note that

$$\sum_{i=1}^N \partial_i \left((y_i - x_i) \frac{y_j - x_j}{|y-x|^{N+2s}} \right) = -(2s-1) \frac{y_j - x_j}{|y-x|^{N+2s}}$$

and so an integration by parts shows that if $\nabla\psi \in L^\infty \cap L^1(\Omega)$, then

$$\begin{aligned} \int_{\mathbb{R}^N} \left[\widetilde{\psi}(y, y-x) - \psi(x)\right] \frac{y-x}{|y-x|^{N+2s}}\, dy &= (2s-1) \int_{\mathbb{R}^N} (y-x) \cdot \nabla_y \widetilde{\psi}(y, y-x) \frac{y-x}{|y-x|^{N+2s}}\, dy \\ &= (2s-1) \int_\Omega (y-x) \cdot \nabla\psi(y) \frac{y-x}{|y-x|^{N+2s}}\, dy. \end{aligned}$$

The result follows by a density argument. $\square$

We end this section with the proof of Lemma 3.7:

Proof of Lemma 3.7. The Lemma can be proved directly by computing the divergence in (9). Alternatively, we can also use the formulation (55) for $\mathcal{L}^\varepsilon$ and (43) to get:

$$\begin{aligned}\mathcal{L}^\varepsilon[\psi](x) &= \varepsilon^{-2s}\int_{\mathbb{R}^N} F(v)\varepsilon v\cdot\nabla_x\phi^\varepsilon(x,v)\,\mathrm{d}v\\ &= \varepsilon^{1-2s}\int_{\mathbb{R}^N} F_0(v)v\cdot\nabla_x\widetilde{\psi}^\varepsilon(x+\varepsilon v,v)\,\mathrm{d}v\\ &= \int_{\mathbb{R}^N}\varepsilon^{-N-2s}F_0\left(\frac{y-x}{\varepsilon}\right)(y-x)\cdot\nabla_x\widetilde{\psi}^\varepsilon(y,y-x)dy.\end{aligned}$$

Since $w\cdot\nabla_x\widetilde{\psi}(y,w)=0$ for all w whenever $y\notin\Omega$, we can write

$$\mathcal{L}^\varepsilon[\psi](x) = \int_\Omega \varepsilon^{-N-2s}F_0\left(\frac{y-x}{\varepsilon}\right)(y-x)\cdot\nabla_x\psi(y)dy.$$

Proceeding as before, we can pass to the limit in this expression to get (13). □

3.3 Control of the boundary correction terms due to definition of the ad hoc test functions

In this section, we show that the additional terms in (51)-(52)-(53) that are due to the corrector T^ε in the definition of ϕ_0^ε vanish in the limit $\varepsilon\to 0$. We recall that T^ε is defined by (50) with

$$\begin{aligned}\bar{T}^\varepsilon(t,x') &= -\int_{w\cdot n<0}\alpha_0 F(w)(\phi^\varepsilon(t,x,v)-\psi(x))|w\cdot n|\,\mathrm{d}w\\ &= \varepsilon^{2s-1}\alpha_0 D_\varepsilon^{2s-1}[\psi](t,x)\cdot n \qquad \forall x=(x',0)\in\partial\Omega.\end{aligned}$$

The main result is the following proposition:

Proposition 3.9. *There exists a constant $C(\psi)$ (depending on the L^2 norms of ψ, $\nabla\psi$, $D^2\psi$ and $\partial_t\psi$) such that for any $\varepsilon>0$ and $t\in\mathbb{R}_+$*

$$\|\partial_t T^\varepsilon(t,x)\|_{L^2(\Omega)} \le C(\psi)\varepsilon^{\frac{4s-1}{2s+1}} \tag{74}$$

$$\|\nabla_x T^\varepsilon(t,x)\|_{L^2(\Omega)} \le C(\psi)\varepsilon^{\frac{4s-1}{2s+1}}. \tag{75}$$

Furthermore, all the terms in (52) *go to zero when* $\varepsilon\to 0$.

The proof of this proposition relies on the following Lemma, which we prove below:

Lemma 3.10. *There exists a constant C depending on $\|\psi\|_{L^2}$ and $\|\nabla\psi\|_{L^2}$ such that for any $\varepsilon>0$ and $t\in\mathbb{R}_+$ we have*

$$\left(\int_{\mathbb{R}^N}|T^\varepsilon(t,x)|^2\,\mathrm{d}x\right)^{1/2} \le C(\psi)\varepsilon^{\frac{4s-1}{2s+1}}. \tag{76}$$

Proof of Proposition 3.9. Since we can differentiate the definition of T^ε with respect to t and x to derive bounds on $\partial_t T^\varepsilon$ and $\nabla_x T^\varepsilon$ similar to (76) the estimates (74), (75) easily follow from (50) although the constant in (75) will naturally also depend on the L^2 norm of the second derivative of ψ.

We now consider the various terms in (52) one by one. Using the a priori estimate (21), we find:

$$\begin{aligned}
&\iiint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+\times\mathbb{R}^N} f^\varepsilon \partial_t T^\varepsilon(t,x)\chi(v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \\
&\qquad \le \Bigg(\iiint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+\times\mathbb{R}^N} \frac{|f^\varepsilon|^2}{F(v)}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v\Bigg)^{1/2}\Bigg(\iint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+}|\partial_t T^\varepsilon(t,x)|^2\,\mathrm{d}t\,\mathrm{d}x\Bigg)^{1/2}\Bigg(\int\limits_{\mathbb{R}^N}\chi^2(v)F(v)\,\mathrm{d}v\Bigg)^{1/2} \\
&\qquad \le C\varepsilon^{\frac{4s-1}{2s+1}} \longrightarrow 0
\end{aligned}$$

and

$$\begin{aligned}
&\iiint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+\times\mathbb{R}^N} f^\varepsilon \varepsilon^{1-2s} v\cdot\nabla_x T^\varepsilon(t,x)\chi(v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \\
&\qquad \le \varepsilon^{1-2s}\Bigg(\iiint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+\times\mathbb{R}^N} \frac{|f^\varepsilon|^2}{F(v)}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v\Bigg)^{1/2}\Bigg(\iint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+}|\nabla_x T^\varepsilon(t,x)|^2\,\mathrm{d}t\,\mathrm{d}x\Bigg)^{1/2}\Bigg(\int\limits_{\mathbb{R}^N}|v|^2\chi^2(v)F(v)\,\mathrm{d}v\Bigg)^{1/2} \\
&\qquad \le C\varepsilon^{1-2s}\varepsilon^{\frac{4s-1}{2s+1}} = C\varepsilon^{\frac{4s(1-s)}{2s+1}} \longrightarrow 0
\end{aligned}$$

and finally (using the fact that $Q[F](v)=0$),

$$\begin{aligned}
&\iiint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+\times\mathbb{R}^N} f^\varepsilon \varepsilon^{-2s} Q^*\big[\chi(v)\big]T^\varepsilon(t,x)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \\
&\qquad = \iiint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+\times\mathbb{R}^N} (f^\varepsilon-\rho^\varepsilon F)\varepsilon^{-2s} Q^*\big[\chi(v)\big]T^\varepsilon(t,x)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \\
&\qquad \le \varepsilon^{-2s}\Bigg(\iiint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+\times\mathbb{R}^N} \frac{(f^\varepsilon-\rho^\varepsilon F)^2}{F(v)}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v\Bigg)^{1/2}\Bigg(\iint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+}|T^\varepsilon(t,x)|^2\,\mathrm{d}t\,\mathrm{d}x\Bigg)^{1/2}\Bigg(\int\limits_{\mathbb{R}^N}Q^*\big[\chi(v)\big]F(v)\,\mathrm{d}v\Bigg)^{1/2} \\
&\qquad \le C\varepsilon^{-2s}\varepsilon^{s}\varepsilon^{\frac{4s-1}{2s+1}} = C\varepsilon^{\frac{(1-s)(2s-1)}{2s+1}} \longrightarrow 0.
\end{aligned}$$

□

We complete this section with the proof of Lemma 3.10:

Proof of Lemma 3.10. Using the definition of T^ε and (60), we write, for $x=(x',x_N)\in\Omega$:

$$\begin{aligned}
T^\varepsilon(x) &= \alpha_0\int_{\mathbb{R}^N_+} v\cdot nF_0(v)[\psi(x'+\varepsilon v)-\psi(x')]\,\mathrm{d}v\,e^{-x_N^2} \\
&= \varepsilon^{2s-1}\alpha_0\int_{\mathbb{R}^N_+}[\psi(x'+y)-\psi(x')]\frac{1}{\varepsilon^{N+2s}}F_0\left(\frac{y}{\varepsilon}\right)y\cdot n\,\mathrm{d}y\,e^{-x_N^2}.
\end{aligned}$$

Furthermore, since ψ satisfies (48), we can write:

$$T^\varepsilon(x) = \varepsilon^{2s-1}\alpha_0\int_{\mathbb{R}^N_+}[\psi(x'+y)-\psi(x')]\left[\frac{1}{\varepsilon^{N+2s}}F_0\left(\frac{y}{\varepsilon}\right)-\frac{\gamma_0}{|y|^{N+2s}}\right]y\cdot n\,e^{-x_N^2}\,\mathrm{d}y$$

and we write $G_0^\varepsilon(y) = \frac{1}{\varepsilon^{N+2s}} F_0\left(\frac{y}{\varepsilon}\right) - \frac{\gamma_0}{|y|^{N+2s}}$.
To estimate the L^2 norm of T^ε, we split the integral with respect to y in two, for $|y| < \varepsilon^\alpha$ and $|y| > \varepsilon^\alpha$ for some $\alpha \in (0,1)$ to be determined later. First, we write

$$\begin{aligned}
&\left(\int_{|y|<\varepsilon^\alpha, y_N>0} [\psi(x'+y) - \psi(x')] y \cdot n\, G_0^\varepsilon(y)\,\mathrm{d}y\right)^2 \\
&\quad = \left(\int_{|y|<\varepsilon^\alpha, y_N>0} \int_0^1 \nabla\psi(x'+\tau y)\cdot y\,\mathrm{d}\tau\,(y\cdot n) G_0^\varepsilon(y)\,\mathrm{d}y\right)^2 \\
&\quad \leq \int_{|y|<\varepsilon^\alpha, y_N>0} \int_0^1 |\nabla\psi(x'+\tau y)|^2 |y|^2\, |G_0^\varepsilon(y)|\,\mathrm{d}\tau\,\mathrm{d}y \\
&\qquad \times \int_{|y|<\varepsilon^\alpha} |y|^2\, |G_0^\varepsilon(y)|\,\mathrm{d}y
\end{aligned}$$

where, since $|v|^{N+2s}F(v) \in L^\infty$ and thus $F_0(y) \leq \frac{C}{|y|^{N+2s}}$:

$$\int_{|y|<\varepsilon^\alpha} |y|^2\, |G_0^\varepsilon(y)|\,\mathrm{d}y \leq \int_{|y|<\varepsilon^\alpha} |y|^2 \frac{C}{|y|^{N+2s}}\,\mathrm{d}z = C\varepsilon^{\alpha(2-2s)}. \tag{77}$$

We thus have:

$$\begin{aligned}
&\left(\int_{|y|<\varepsilon^\alpha, y_N>0} [\psi(x'+y) - \psi(x')] y \cdot n\, G_0^\varepsilon(y)\,\mathrm{d}y\right)^2 \\
&\quad \leq C\varepsilon^{\alpha(2-2s)} \int_{|y|<\varepsilon^\alpha, y_N>0} \int_0^1 |\nabla\psi(x'+ty)|^2 |y|^2\, |G_0^\varepsilon(y)|\,\mathrm{d}t\,\mathrm{d}y.
\end{aligned} \tag{78}$$

For the integral over $|y| > (\varepsilon z)^\alpha$ we write

$$\begin{aligned}
&\left(\int_{|y|>\varepsilon^\alpha, y_N>0} [\psi(x'+y) - \psi(x)] y \cdot n\, G_0^\varepsilon(y)\,\mathrm{d}y\right)^2 \\
&\leq \left(\int_{|y|>\varepsilon^\alpha} [\psi(x'+y) - \psi(x)] |y|\, |G_0^\varepsilon(y)|\,\mathrm{d}y\right)^2 \\
&\leq \left(\int_{|y|>\varepsilon^\alpha, y_N>0} [\psi(x'+y) - \psi(x')]^2 |y|\, |G_0^\varepsilon(y)|\,\mathrm{d}y\right)\left(\int_{|y|>\varepsilon^\alpha} |y|\, |G_0^\varepsilon(y)|\,\mathrm{d}y\right).
\end{aligned}$$

Using (67), we get

$$|G_0^\varepsilon(y)| = \left|\frac{1}{\varepsilon^{N+2s}} F_0\left(\frac{y}{\varepsilon}\right) - \frac{\gamma_0}{|y|^{N+2s}}\right| \leq C\frac{\varepsilon^{2s}}{|y|^{N+4s}}$$

for all $|y| \geq \varepsilon^\alpha$, we deduce

$$\begin{aligned}
\int_{|y|>\varepsilon^\alpha} |y|\, |G_0^\varepsilon(y)|\,\mathrm{d}y &\leq C\varepsilon^{2s} \int_{|z|>\varepsilon^\alpha} |y| \frac{1}{|y|^{N+4s}}\,\mathrm{d}y \\
&\leq C\varepsilon^{2s-\alpha(4s-1)}
\end{aligned} \tag{79}$$

and so

$$\left(\int_{|y|>\varepsilon^\alpha, y_N>0} [\psi(x'+y)-\psi(x')] y\cdot n\, G_0^\varepsilon(y)\,\mathrm{d}y\right)^2$$
$$\leq C\varepsilon^{2s-\alpha(4s-1)} \int_{|y|>\varepsilon^\alpha, y_N>0} [\psi(x'+y)-\psi(x')]^2 |y|\, |G_0^\varepsilon(y)|\,\mathrm{d}y. \tag{80}$$

Finally, combing (78) and (80) we get

$$\begin{aligned}
&(\varepsilon^{2s-1}\alpha_0)^{-2}\int_{\mathbb{R}^N_+} |T^\varepsilon(x)|^2\,\mathrm{d}x \\
&\quad\leq C\varepsilon^{\alpha(2-2s)}\int_{x_N>0} e^{-2x_N^2}\int_{\mathbb{R}^{N-1}}\int_{|y|<\varepsilon^\alpha, y_N>0}\int_0^1 |\nabla\psi(x'+\tau y)|^2|y|^2\,|G_0^\varepsilon(y)|\,\mathrm{d}\tau\,\mathrm{d}y\,\mathrm{d}x'\,\mathrm{d}x_N \\
&\qquad + C\varepsilon^{2s-\alpha(4s-1)}\int_{x_N>0} e^{-2x_N^2}\int_{\mathbb{R}^{N-1}}\int_{|y|>\varepsilon^\alpha, y_N>0} [\psi(x'+y)-\psi(x')]^2|y|\,|G_0^\varepsilon(y)|\,\mathrm{d}y\,\mathrm{d}x'\,\mathrm{d}x_N \\
&\quad\leq C\varepsilon^{\alpha(2-2s)}\frac{\sqrt{\pi}}{2\sqrt{2}}\int_{|y|<\varepsilon^\alpha, y_N>0}\int_0^1\int_{\mathbb{R}^{N-1}} |\nabla\psi(x'+\tau y)|^2\,\mathrm{d}x'|y|^2\,|G_0^\varepsilon(y)|\,\mathrm{d}\tau\,\mathrm{d}y \\
&\qquad + C\varepsilon^{2s-\alpha(4s-1)}\frac{\sqrt{\pi}}{2\sqrt{2}}\int_{|y|>\varepsilon^\alpha, y_N>0}\int_{\mathbb{R}^{N-1}} [\psi(x'+y)-\psi(x')]^2\,\mathrm{d}x'|y|\,|G_0^\varepsilon(y)|\,\mathrm{d}y \\
&\quad\leq C(\nabla\psi)\varepsilon^{\alpha(2-2s)}\int_{|y|<(\varepsilon z)^\alpha, y_N>0} |y|^2\,|G_0^\varepsilon(y)|\,\mathrm{d}y \\
&\qquad + C(\psi)\varepsilon^{2s-\alpha(4s-1)}\int_{|y|>\varepsilon^\alpha, y_N>0} |y|\,|G_0^\varepsilon(y)|\,\mathrm{d}y
\end{aligned}$$

where, up to constants,

$$C(\psi) = \int_{\mathbb{R}^{N-1}} |\psi(x')|^2\,\mathrm{d}x' \text{ and } \quad C(\nabla\psi) = \int_{\mathbb{R}^{N-1}} |\nabla\psi(x')|^2\,\mathrm{d}x'.$$

Using (77) and (79), we deduce

$$\left(\int_{\mathbb{R}^{N-1}} |T^\varepsilon(x)|^2\,\mathrm{d}x\right)^{1/2} \leq C(\psi)(\varepsilon^{2s-1}\alpha_0)\left[\varepsilon^{\alpha(2-2s)}+\varepsilon^{2s-\alpha(4s-1)}\right]$$

and we see that we need to take $\alpha = \frac{2s}{1+2s}$ to get (76). □

3.4 Derivation of the asymptotic equation

We can now complete the proof of Theorem 1.2. Recall that we only need to pass to the limit in (51)-(52)-(53). We proved in Section 3.3 above that (52) vanish in the limit.

Furthermore, the weak convergence of ρ^ε (Lemma 2.1) and the strong convergence of $\mathcal{L}^\varepsilon[\psi]$ (Proposition 3.5) immediately implies

Proposition 3.11. *For all $\psi \in L^\infty(0,\infty; H^2(\Omega))$, satisfying (11) the following limit hold:*

$$\lim_{\varepsilon\to 0}\iint_{\mathbb{R}_+\times\mathbb{R}^N_+} \rho^\varepsilon\mathcal{L}^\varepsilon[\psi]\,\mathrm{d}t\,\mathrm{d}x = \iint_{\mathbb{R}_+\times\mathbb{R}^N_+} \rho\,\mathcal{L}[\psi]\,\mathrm{d}t\,\mathrm{d}x$$

where the operator $\mathcal{L}$ is defined by (64).

The convergence of the last two terms (the time derivative and the initial condition term) will follow from the weak convergence of f^ε to ρF and the following Lemma applied to $\phi^\varepsilon(0,x,v)$ (for the initial condition (53)) and to $\partial_t\phi^\varepsilon$ (for the first term in (51)):

Lemma 3.12. *For all test function $\psi\in L^\infty(0,\infty;H^1(\Omega))$, the following holds:*

$$\lim_{\varepsilon\to 0}\iiint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+\times\mathbb{R}^N}\left|\phi^\varepsilon(t,x,v)-\psi(t,x)\right|^2 F(v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v=0$$

Proof of Lemma 3.12. To prove the lemma, we first write

$$\begin{aligned}|\phi^\varepsilon(t,x,v)-\psi(t,x)|^2&=\left|\int_0^\infty e^{-\nu_0 z}\nu_0\big(\widetilde{\psi}(t,x+\varepsilon vz,z)-\psi(t,x)\big)\,\mathrm{d}z\right|^2\\&\leq\int_0^\infty e^{-\nu_0 z}\nu_0\left|\widetilde{\psi}(t,x+\varepsilon vz,v)-\psi(t,x)\right|^2\mathrm{d}z\end{aligned}$$

and so

$$\begin{aligned}&\iiint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+\times\mathbb{R}^N}\left|\phi^\varepsilon(t,x,v)-\psi(t,x)\right|^2F(v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v\\
&\quad\leq\iiint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+\times\mathbb{R}^N}\int_0^\infty e^{-\nu_0 z}\nu_0|\widetilde{\psi}(t,x+\varepsilon vz,v)-\psi(t,x)|^2\,F(v)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leq\iiint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+\times\mathbb{R}^N}\int_0^\infty e^{-\nu_0 z}\nu_0|\widetilde{\psi}(t,x+\varepsilon w,w)-\psi(t,x)|^2\,F(w/z)z^{-N}\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leq\iiint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+\times\mathbb{R}^N}|\widetilde{\psi}(t,x+\varepsilon w,w)-\psi(t,x)|^2\,F_1(w)\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

where F_1 is given by (63).

We now write

$$\begin{aligned}&\iiint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+\times\mathbb{R}^N}\left|\widetilde{\psi}(t,x+\varepsilon v,\varepsilon v)-\psi(t,x)\right|^2F_1(v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v\\
&\qquad=\int_{|\varepsilon v|<1}\iint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+}\left|\widetilde{\psi}(t,x+\varepsilon v,\varepsilon v)-\psi(t,x)\right|^2F_1(v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v\\
&\qquad\quad+\int_{|\varepsilon v|>1}\iint\limits_{\mathbb{R}_+\times\mathbb{R}^N_+}\left|\widetilde{\psi}(t,x+\varepsilon v,\varepsilon v)-\psi(t,x)\right|^2F_1(v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v\end{aligned}$$

To bound the integral over $|\varepsilon v|<1$ we take advantage of the regularity of ψ to write, using Taylor and τ_0^ε defined in (70):

$$\begin{aligned}
&\int_{|\varepsilon v|<1} \iint_{\mathbb{R}_+\times\mathbb{R}^N_+} \left(\widetilde{\psi}(t,x+\varepsilon v,\varepsilon v)-\psi(t,x)\right)^2 F_1(v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v\\
&\qquad = \int_{|\varepsilon v|<1} \iint_{\mathbb{R}_+\times\mathbb{R}^N_+} \left(\int_0^{\tau_0^\varepsilon(x,v)} \varepsilon v\cdot\nabla_x\psi(t,x+\tau\varepsilon v)\,\mathrm{d}\tau\right)^2 F_1(v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v\\
&\qquad \leq \int_0^1\int_{|\varepsilon v|<1} |\varepsilon v|^2 F_1(v) \iint_{\mathbb{R}_+\times\mathbb{R}^N_+} |\nabla_x\psi(t,x+\tau\varepsilon v)|^2 1_{\{\tau\leq\tau_0^\varepsilon(x,v)\}}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&\qquad \leq \int_0^1\int_{|\varepsilon v|<1} |\varepsilon v|^2 F_1(v) C(\psi)\,\mathrm{d}v\,\mathrm{d}\tau\\
&\qquad \leq C(\psi)\varepsilon^{2s}
\end{aligned}$$

And for the integral over $|\varepsilon v|>1$ we use the decay of F to write (using (59)):

$$\begin{aligned}
&\int_{|\varepsilon v|>1} \iint_{\mathbb{R}_+\times\mathbb{R}^N_+} \left(\widetilde{\psi}(t,x+\varepsilon v,\varepsilon v)-\psi(t,x)\right)^2 F_1(v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v\\
&\qquad \leq \int_{|\varepsilon v|>1} C(\psi)(1+\varepsilon|v|)F_1(v)\,\mathrm{d}v\\
&\qquad \leq C(\psi)\varepsilon^{2s}\int_{|w|>1}(1+|w|)\frac{C}{|w|^{N+2s}}\,\mathrm{d}w.
\end{aligned}$$

This completes the proof of Lemma 3.12. □

4 Well posedness of the asymptotic equation

4.1 A functional framework for D^{2s-1}

Using the fact that D^{2s-1} is the limit of the D^{2s-1}_ε [I don't think that we use that in the proof?] we can improve on Proposition 2.4 and prove the following result, which will be useful in the Proof of Theorem 1.3:

Proposition 4.1. *If $\psi\in H^{2s-1+\beta}(\Omega)$ for some $\beta>0$, then $D^{2s-1}[\psi]\in(L^2(\Omega))^N$.*

In particular, since $s>2s-1$ when $s<1$, we deduce that if $\psi\in H^s(\Omega)$, then $D^{2s-1}[\psi]\in L^2(\Omega)$.

Proof. We recall the definition (46) of D^{2s-1}:

$$\begin{aligned}
D^{2s-1}[\psi](x) &= \gamma\nu_0^{1-2s}\Gamma(2s)\int_{\mathbb{R}^N}\left[\widetilde{\psi}(y,y-x)-\psi(x)\right]\frac{y-x}{|y-x|^{N+2s}}\,dy\\
&= \gamma\nu_0^{1-2s}\Gamma(2s)\int_{\Omega}\left[\psi(y)-\psi(x)\right]\frac{y-x}{|y-x|^{N+2s}}\,dy\\
&\qquad + \gamma\nu_0^{1-2s}\Gamma(2s)\int_{\mathbb{R}^N\setminus\Omega}\left[\widetilde{\psi}(y,y-x)-\psi(x)\right]\frac{y-x}{|y-x|^{N+2s}}\,dy
\end{aligned}$$

Recalling that Ω is the upper-half space $y_N > 0$, we can use (40) to write

$$\widetilde{\psi}(y, y-x) = \psi(y' - \frac{y_N}{y_N - x_N}(y' - x'), 0) \qquad \forall x \in \Omega,\ y \in \mathbb{R}^N \setminus \Omega.$$

We now do a change of variable $z' = y' - \frac{y_N}{y_N - x_N}(y' - x')$. Denoting $z = (z', 0)$, we check that

$$y - x = \frac{x_N - y_N}{x_N}(z - x)$$

and so

$$\begin{aligned}\int_{\mathbb{R}^N \setminus \Omega} \left[\widetilde{\psi}(y, y-x) - \psi(x)\right] \frac{y-x}{|y-x|^{N+2s}}\, dy &= \int_{\mathbb{R}^{N-1}} \int_{-\infty}^{0} \left[\psi(z', 0) - \psi(x)\right] \frac{z-x}{|z-x|^{N+2s}} \left(\frac{x_N - y_N}{x_N}\right)^{-2s} dy_N\, dz' \\ &= \frac{x_N}{2s-1} \int_{\partial\Omega} \left[\psi(z) - \psi(x)\right] \frac{z-x}{|z-x|^{N+2s}} dS(z)\end{aligned}$$

We deduce

$$\begin{aligned}D^{2s-1}[\psi](x) = \gamma \nu_0^{1-2s} \Gamma(2s) \int_{\Omega} \left[\psi(y) - \psi(x)\right] \frac{y-x}{|y-x|^{N+2s}}\, dy \\ + \gamma \nu_0^{1-2s} \Gamma(2s-1) x_N \int_{\partial\Omega} \left[\psi(z) - \psi(x)\right] \frac{z-x}{|z-x|^{N+2s}} dS(z).\end{aligned} \tag{81}$$

Note that the last term can also be split as

$$\begin{aligned}x_N \int_{\partial\Omega} \left[\psi(z) - \psi(x)\right] \frac{z-x}{|z-x|^{N+2s}} dS(z) = x_N \int_{\partial\Omega} \left[\psi(z', 0) - \psi(x', 0)\right] \frac{z-x}{|z-x|^{N+2s}} dS(z) \\ + x_N \int_{\partial\Omega} \left[\psi(x', 0) - \psi(x', x_N)\right] \frac{z-x}{|z-x|^{N+2s}} dS(z).\end{aligned}$$

where

$$x_N \int_{\mathbb{R}^{N-1}} \frac{z_i - x_i}{|z-x|^{N+2s}}\, dz' = \begin{cases} 0 & \text{if } i = 1, \dots, N-1 \\ -x_N^{1-2s} \int_{\mathbb{R}^{N-1}} \frac{1}{(z'^2+1)^{\frac{N+2s}{2}}}\, dz' & \text{if } i = N \end{cases}.$$

We thus need to show that the three terms

$$\begin{aligned}I_1 &= \int_{\Omega} \frac{|\psi(y) - \psi(x)|}{|y-x|^{N+2s-1}}\, dy \\ I_2 &= x_N \int_{\partial\Omega} \frac{|\psi(z', 0) - \psi(x', 0)|}{|z-x|^{N+2s-1}} dS(z) \\ I_3 &= x_N^{1-2s} |\psi(x', 0) - \psi(x', x_N)|\end{aligned}$$

are bounded in $L^2(\Omega)$ by $\|\psi\|_{H^{2s-1+\beta}(\Omega)}$. The first term is obvious and the last follows from Hardy's inequality. For the second term, we write

$$
\begin{aligned}
&\int_\Omega |I_2(x)|^2\,dx \\
&\le \int_\Omega x_N^2 \left(\int_{\mathbb{R}^{N-1}} \frac{|\psi(z',0)-\psi(x',0)|}{|(z'-x')^2+x_N^2|^{\frac{N+2s-1}{2}}} dz' \right)^2 dx \\
&\le \int_{\Omega\cap\{x_N<1\}} x_N^2 \int_{\mathbb{R}^{N-1}} \frac{|\psi(z',0)-\psi(x',0)|^2}{|z'-x'|^{N+4s-4+\beta'}} dz' \int_{\mathbb{R}^{N-1}} \frac{|z'-x'|^{N+4s-4+\beta'}}{|(z'-x')^2+x_N^2|^{N+2s-1}} dz' dx \\
&\quad + \int_{\Omega\cap\{x_N>1\}} x_N^2 \int_{\mathbb{R}^{N-1}} \frac{|\psi(z',0)-\psi(x',0)|^2}{|(z'-x')^2+x_N^2|^{\frac{N-1+2\beta}{2}}} dz' \int_{\mathbb{R}^{N-1}} \frac{1}{|(z'-x')^2+x_N^2|^{\frac{N+4s-1-2\beta}{2}}} dz'\,dx \\
&\le \int_{\Omega\cap\{x_N<1\}} x_N^{-1+\beta'} \int_{\mathbb{R}^{N-1}} \frac{|\psi(z',0)-\psi(x',0)|^2}{|z'-x'|^{N+4s-4+\beta'}} dz'\,dx \int_{\mathbb{R}^{N-1}} \frac{|z'|^{N+4s-4}}{|z'^2+1|^{N+2s-1}} dz' \\
&\quad + C\int_1^\infty \int_{\mathbb{R}^{N-1}} x_N^{2-4s+2\beta} \int_{\mathbb{R}^{N-1}} \frac{|\psi(z',0)-\psi(x',0)|}{|(z'-x')^2+x_N^2|^{\frac{N-1+2\beta}{2}}} dz'\,dx'\,dx_N \int_{\mathbb{R}^{N-1}} \frac{1}{|z'^2+1^2|^{\frac{N+4s-1-2\beta}{2}}} dz' \\
&\le C\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|\psi(z',0)-\psi(x',0)|^2}{|z'-x'|^{N+4s-4+\beta'}} dz'\,dx' \\
&\quad + C\int_1^\infty \int_{\mathbb{R}^{N-1}} x_N^{2-4s+2\beta} \int_{\mathbb{R}^{N-1}} \frac{|\psi(z',0)-\psi(x',0)|^2}{|(z'-x')^2+x_N^2|^{\frac{N-1+2\beta}{2}}} dz'\,dx'\,dx_N
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_1^\infty \int_{\mathbb{R}^{N-1}} x_N^{2-4s+2\beta} \int_{\mathbb{R}^{N-1}} \frac{|\psi(z',0)-\psi(x',0)|^2}{|(z'-x')^2+x_N^2|^{\frac{N-1+2\beta}{2}}} dz'\,dx'\,dx_N \\
&\qquad \le 2\int_1^\infty x_N^{2-4s+2\beta} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|\psi(z',0)|^2}{|(z'-x')^2+x_N^2|^{\frac{N-1+2\beta}{2}}} dz'\,dx'\,dx_N.
\end{aligned}
$$

The result follows.

□

4.2 Proof of Theorem 1.3

The proof of Theorem 1.3 relies on Hille-Yoshida theorem. The first step, which will occupy most of this section is thus devoted to the proof of the well-posedness of the stationary problem:

$$
\begin{cases} \varphi(x) - \mathcal{L}[\varphi](x) = g(x) & \text{for all } x\in\Omega, \\ D^{2s-1}[\varphi](x)\cdot n(x) = 0 & \text{for all } x\in\partial\Omega. \end{cases} \tag{82}
$$

Using (30), we see that classical solutions of (82) satisfy

$$
\int_\Omega \varphi(x)\psi(x)\,\mathrm{d}x + \int_\Omega D^{2s-1}[\varphi](x)\cdot\nabla\psi(x)\,\mathrm{d}x = \int_\Omega g(x)\psi(x)\,\mathrm{d}x \tag{83}
$$

for all $\psi\in\mathcal{D}(\overline{\Omega})$, and using (31), we can write this as

$$
\int_\Omega \varphi(x)\psi(x)\,\mathrm{d}x + \gamma\nu_0^{1-2s}\Gamma(2s-1)\int_\Omega\int_\Omega (y-x)\cdot\nabla\varphi(y)(y-x)\cdot\nabla\psi(x)\frac{dy\,dx}{|y-x|^{N+2s}} = \int_\Omega g(x)\psi(x)\,\mathrm{d}x \tag{84}
$$

We thus introduce the following bilinear symmetric form:

$$a(\varphi,\psi)=\int_\Omega \varphi(x)\psi(x)\,\mathrm{d}x+\gamma\nu_0^{1-2s}\Gamma(2s-1)\int_\Omega\int_\Omega (y-x)\cdot\nabla\varphi(y)(y-x)\cdot\nabla\psi(x)\frac{dy\,dx}{|y-x|^{N+2s}}. \quad (85)$$

This form is bilinear and symmetric. It is clearly well defined for instance if φ and ψ are in $H^1(\Omega)$, but we are going to show that is can be extended to the space $H^s(\Omega)$.

Indeed, we will show the following proposition:

Proposition 4.2. *The bilinear form $a(\varphi,\psi)$ defined by* (85) *satisfies*

$$\begin{cases} a(\varphi,\psi)\le C\|\varphi\|_{H^s}\|\psi\|_{H^s} & \text{for all } \varphi,\ \psi\ \in H^1(\Omega)\\ a(\varphi,\varphi)\ge c\|\varphi\|^2_{H^s} & \text{for all } \varphi\in H^1(\Omega)\end{cases} \quad (86)$$

for some constants c and C depending only on Ω and s and can thus be extended into a bilinear continuous form on $H^s(\Omega)\times H^s(\Omega)$.

Lax-Milgram's Theorem then implies the existence of a weak solution to (82). More precisely, we have:

Theorem 4.3. *For all g in $L^2(\Omega)$, there exists a unique $\varphi\in H^s(\Omega)$ such that*

$$a(\varphi,\psi)=\int_\Omega g(x)\psi(x)\,\mathrm{d}x \qquad \forall\psi\in H^s(\Omega).$$

We now turn to the proof of Proposition 4.2, which relies on the following lemma, the proof of which is postponed until after the proof of Proposition 4.2:

Lemma 4.4. *For all $\varphi\in H^1(\Omega)$, there holds:*

$$\begin{aligned}\int_\Omega D^{2s-1}[\varphi]\cdot\nabla\varphi\,\mathrm{d}x &= s\gamma\Gamma(2s)\int_\Omega\int_\Omega\frac{[\varphi(x)-\varphi(y)]^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+\gamma\Gamma(2s)\int_\Omega\int_{\partial\Omega}[\varphi(x)-\varphi(y)]^2\frac{y-x}{|y-x|^{N+2s}}\cdot n(y)\,\mathrm{d}S(y)\,\mathrm{d}x. \quad (87)\end{aligned}$$

Proof of Proposition 4.2. We just need to prove that

$$c\|\varphi\|^2_{H^s}\le a(\varphi,\varphi)\le C\|\varphi\|^2_{H^s} \qquad \text{for all } \varphi\in H^1(\Omega) \quad (88)$$

since Cauchy-Schwarz inequality then gives

$$a(\varphi,\psi)^2\le a(\varphi,\varphi)a(\psi,\psi)\le C\|\varphi\|^2_{H^s}\|\psi\|^2_{H^s}.$$

In order to prove (88), we first note that both terms in (87) are non-negative and so we immediately get

$$\begin{aligned}a(\varphi,\varphi)&\ge\int_\Omega|\varphi(x)|^2\,\mathrm{d}x+s\gamma\Gamma(2s)\int_\Omega\int_\Omega\frac{[\varphi(x)-\varphi(y)]^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\ &\ge C\|\varphi\|^2_{H^s}\end{aligned}$$

To prove the other inequality, we need to show that the last term in (87) can be bounded by $\|\varphi\|^2_{H^s}$. First, we write

$$\begin{aligned}\int_\Omega\int_{\partial\Omega}[\varphi(x)-\varphi(y)]^2\frac{y-x}{|y-x|^{N+2s}}\cdot n(y)\,\mathrm{d}S(y)\,\mathrm{d}x &= \int_\Omega\int_{\mathbb{R}^{N-1}}[\varphi(x',x_N)-\varphi(y',0)]^2\frac{x_N}{|y-x|^{N+2s}}\,\mathrm{d}y'\,\mathrm{d}x\\ &\le\int_\Omega\int_{\mathbb{R}^{N-1}}[\varphi(x',x_N)-\varphi(x',0)]^2\frac{x_N}{|y-x|^{N+2s}}\,\mathrm{d}y'\,\mathrm{d}x\\ &\quad+\int_\Omega\int_{\mathbb{R}^{N-1}}[\varphi(x',0)-\varphi(y',0)]^2\frac{x_N}{|y-x|^{N+2s}}\,\mathrm{d}y'\,\mathrm{d}x\end{aligned}$$

Using the fact that

$$\int_{\mathbb{R}^{N-1}} \frac{x_N}{|y-x|^{N+2s}}\,\mathrm{d}y' = \frac{1}{x_N^{2s}} \int_{\mathbb{R}^{N-1}} \frac{1}{|1+|z|^2|^{\frac{N+2s}{2}}}\,\mathrm{d}z$$

and

$$\int_0^\infty \frac{x_N}{|y-x|^{N+2s}}\,\mathrm{d}x_N = \frac{1}{|y'-x'|^{N+2s-2}} \int_0^\infty \frac{t}{(1+t^2)^{\frac{N+2s}{2}}}\,\mathrm{d}t$$

we deduce

$$\begin{aligned}\int_\Omega \int_{\partial\Omega} [\varphi(x)-\varphi(y)]^2 \frac{y-x}{|y-x|^{N+2s}} \cdot n(y)\,\mathrm{d}S(y)\,\mathrm{d}x \leq C \int_\Omega \frac{[\varphi(x',x_N)-\varphi(x',0)]^2}{x_N^{2s}}\,\mathrm{d}x \\ + C \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{[\varphi(x',0)-\varphi(y',0)]^2}{|y-x|^{N+2s-2}}\,\mathrm{d}y'\,\mathrm{d}x'.\end{aligned} \tag{89}$$

The second term in (89) is bounded by a Sobolev trace theorem (note that $N+2s-2 = (N-1) + 2(s-1/2)$). Indeed, we recall the following theorem:

Theorem 4.5 ([13])**.** *For all $\varphi \in H^s(\Omega)$, we have*

$$\|\varphi\|_{H^{s-1/2}(\partial\Omega)} \leq C\|\varphi\|_{H^s(\Omega)}.$$

We deduce

$$\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{[\varphi(x',0)-\varphi(y',0)]^2}{|y-x|^{N+2s-2}}\,\mathrm{d}y'\,\mathrm{d}x' \leq C\|\varphi\|^2_{H^s(\Omega)}. \tag{90}$$

In order to bound the first term in the right hand side of (89), we use the fractional Hardy inequality (Theorem 2.7). Since the function $x \mapsto \varphi(x',x_N) - \varphi(x',0)$ is in $H^s_0(\Omega)$, and $s \in (1/2,1)$, we have:

$$\int_\Omega \frac{[\varphi(x',x_N)-\varphi(x',0)]^2}{x_N^{2s}}\,\mathrm{d}x \leq C\|\varphi\|^2_{H^s(\Omega)}. \tag{91}$$

Combining (89), (90) and (91), we deduce

$$\int_\Omega \int_{\partial\Omega} [\varphi(x)-\varphi(y)]^2 \frac{y-x}{|y-x|^{N+2s}} \cdot n(y)\,\mathrm{d}S(y)\,\mathrm{d}x \leq C\|\varphi\|^2_{H^s(\Omega)}$$

and (87) yields

$$a(\varphi,\varphi) \leq C\|\varphi\|^2_{H^s(\Omega)}$$

which gives (88) and concludes the proof. □

Proof of Lemma 4.4. We use the approximated operator D^{2s-1}_ε to prove this equality. First, we write, for $\varphi \in D(\overline{\Omega})$ (using the definition (60) for D^{2s-1}_ε):

$$\begin{aligned}\int_\Omega D^{2s-1}_\varepsilon[\varphi] \cdot \nabla\varphi\,\mathrm{d}x &= \varepsilon^{1-2s} \int_\Omega \int_{\mathbb{R}^N} F_0(v)[\widetilde{\varphi}(x+\varepsilon v, v) - \varphi(x)]v \cdot \nabla_x\varphi(x)\,\mathrm{d}v\,\mathrm{d}x \\ &= \varepsilon^{-N-2s} \int_\Omega \int_{\mathbb{R}^N} F_0\left(\frac{y-x}{\varepsilon}\right) [\widetilde{\varphi}(y, y-x) - \varphi(x)](y-x) \cdot \nabla_x\varphi(x)\,\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

Since $v \cdot \nabla_v \widetilde{\varphi}(x, v) = 0$ for all x and v, we can write

$$\begin{aligned}
&\int_\Omega D_\varepsilon^{2s-1}[\varphi] \cdot \nabla \varphi \, \mathrm{d}x \\
&\quad = \varepsilon^{-N-2s} \int_\Omega \int_{\mathbb{R}^N} F_0\left(\frac{y-x}{\varepsilon}\right) [\widetilde{\varphi}(y, y-x) - \varphi(x)](y-x) \cdot \nabla_x [\varphi(x) - \widetilde{\varphi}(y, y-x)] \, \mathrm{d}y \, \mathrm{d}x \\
&\quad = -\frac{1}{2}\varepsilon^{-N-2s} \int_\Omega \int_{\mathbb{R}^N} F_0\left(\frac{y-x}{\varepsilon}\right) (y-x) \cdot \nabla_x [\varphi(x) - \widetilde{\varphi}(y, y-x)]^2 \, \mathrm{d}y \, \mathrm{d}x \\
&\quad = \frac{1}{2}\varepsilon^{-N-2s} \int_\Omega \int_{\mathbb{R}^N} \mathrm{div}_x \left(F_0\left(\frac{y-x}{\varepsilon}\right) (y-x) \right) [\varphi(x) - \widetilde{\varphi}(y, y-x)]^2 \, \mathrm{d}y \, \mathrm{d}x \\
&\qquad - \frac{1}{2}\varepsilon^{-N-2s} \int_{\partial\Omega} \int_{\mathbb{R}^N} F_0\left(\frac{y-x}{\varepsilon}\right) (y-x) \cdot n(x) [\varphi(x) - \widetilde{\varphi}(y, y-x)]^2 \, \mathrm{d}y \, \mathrm{d}x
\end{aligned}$$

Now, we split the integrals in y into an integral in Ω and one in $\mathbb{R}^N \setminus \Omega$. Note that in the second term, when $x \in \partial\Omega$ and $y \in \mathbb{R}^N \setminus \Omega$ we have $\widetilde{\varphi}(y, y-x) = \varphi(x)$. We thus obtain

$$\begin{aligned}
\int_\Omega D_\varepsilon^{2s-1}[\varphi] \cdot \nabla \varphi \, \mathrm{d}x = \; & \frac{1}{2}\varepsilon^{-N-2s} \int_\Omega \int_\Omega \mathrm{div}_x \left(F_0\left(\frac{y-x}{\varepsilon}\right) (y-x) \right) [\varphi(x) - \varphi(y)]^2 \, \mathrm{d}y \, \mathrm{d}x \\
& + \varepsilon^{-N-2s} \frac{1}{2} \int_\Omega \int_{\mathbb{R}^N \setminus \Omega} \mathrm{div}_x \left(F_0\left(\frac{y-x}{\varepsilon}\right) (y-x) \right) [\varphi(x) - \widetilde{\varphi}(y, y-x)]^2 \, \mathrm{d}y \, \mathrm{d}x \\
& - \frac{1}{2}\varepsilon^{-N-2s} \int_{\partial\Omega} \int_\Omega F_0\left(\frac{y-x}{\varepsilon}\right) (y-x) \cdot n(x) [\varphi(x) - \varphi(y)]^2 \, \mathrm{d}y \, \mathrm{d}x
\end{aligned}$$

For the second term, we write

$$\begin{aligned}
&\int_\Omega \int_{\mathbb{R}^N \setminus \Omega} \mathrm{div}_x \left(F_0\left(\frac{y-x}{\varepsilon}\right) (y-x) \right) [\varphi(x) - \widetilde{\varphi}(y, y-x)]^2 \, \mathrm{d}y \, \mathrm{d}x \\
&\quad = -\int_\Omega \int_{\mathbb{R}^N \setminus \Omega} \mathrm{div}_y \left(F_0\left(\frac{y-x}{\varepsilon}\right) (y-x) \right) [\varphi(x) - \widetilde{\varphi}(y, y-x)]^2 \, \mathrm{d}y \, \mathrm{d}x \\
&\quad = \int_\Omega \int_{\mathbb{R}^N \setminus \Omega} F_0\left(\frac{y-x}{\varepsilon}\right) (y-x) \cdot \nabla_y [\varphi(x) - \widetilde{\varphi}(y, y-x)]^2 \, \mathrm{d}y \, \mathrm{d}x \\
&\qquad + \int_\Omega \int_{\partial(\mathbb{R}^N \setminus \Omega)} F_0\left(\frac{y-x}{\varepsilon}\right) (y-x) \cdot n(y) [\varphi(x) - \widetilde{\varphi}(y, y-x)]^2 \, \mathrm{d}y \, \mathrm{d}x
\end{aligned}$$

where we recall that the vector n points downward. Using the fact that $v \cdot \nabla_y \widetilde{\varphi}(y, v) = 0$ and $v \cdot \nabla_v \widetilde{\varphi}(y, v) = 0$ for $y \in \mathbb{R}^N \setminus \Omega$, we see that the first term vanishes and we get (after changing the name of the variables)

$$\begin{aligned}
&\int_\Omega \int_{\mathbb{R}^N \setminus \Omega} \mathrm{div}_x \left(F_0\left(\frac{y-x}{\varepsilon}\right) (y-x) \right) [\varphi(x) - \widetilde{\varphi}(y, y-x)]^2 \, \mathrm{d}y \, \mathrm{d}x \\
&\quad = -\int_\Omega \int_{\partial\Omega} F_0\left(\frac{y-x}{\varepsilon}\right) (y-x) \cdot n(x) [\varphi(y) - \varphi(x)]^2 \, \mathrm{d}x \, \mathrm{d}y
\end{aligned}$$

We have thus proved

$$\begin{aligned}
\int_\Omega D_\varepsilon^{2s-1}[\varphi] \cdot \nabla \varphi \, \mathrm{d}x = \; & \frac{1}{2} \int_\Omega \int_\Omega G_\varepsilon(y-x) [\varphi(x) - \varphi(y)]^2 \, \mathrm{d}y \, \mathrm{d}x \\
& - \int_{\partial\Omega} \int_\Omega F_0^\varepsilon(y-x)(y-x) \cdot n(x) [\varphi(x) - \varphi(y)]^2 \, \mathrm{d}y \, \mathrm{d}x \qquad (92)
\end{aligned}$$

where

$$F_0^\varepsilon = \varepsilon^{-N-2s} F_0\left(\frac{v}{\varepsilon}\right)$$

and

$$G_\varepsilon(v) = -\varepsilon^{-N-2s}\mathrm{div}_v\left(vF_0\left(\frac{v}{\varepsilon}\right)\right) = \varepsilon^{-N-2s}G(v/\varepsilon), \qquad G(v) = -\mathrm{div}_v(vF_0(v)).$$

We can now use Proposition 3.4 to pass to the limit in the left hand side of (92). To pass to the limit in the right hand side of (92), we proceed as in the proof of Proposition 3.4 using the fact that

$$F_0^\varepsilon\left(\frac{v}{\varepsilon}\right) \sim \frac{\gamma\Gamma(2s)}{|v|^{N+2s}}, \qquad G_\varepsilon\left(\frac{v}{\varepsilon}\right) \sim \frac{2s\gamma\Gamma(2s)}{|v|^{N+2s}}.$$

□

Finally we have:

Proof of Theorem 1.3. The weak solution φ given by Theorem 4.3 is in $H^s(\Omega)$ and so (by Proposition 4.1), $D^{2s-1}[\varphi] \in L^2(\Omega)$. In particular, φ satisfies (83) for all test function $\psi \in \mathcal{D}(\Omega)$. It follows that the equation

$$\varphi - \mathrm{div} D^{2s-1}[\varphi] = g$$

holds in $\mathcal{D}'(\Omega)$. Since both φ and g are in $L^2(\Omega)$, we deduce

$$\mathcal{L}[\varphi] = \mathrm{div}(D^{2s-1}[\varphi]) \in L^2(\Omega).$$

This implies in particular that the trace $D^{2s-1}[\varphi]\cdot n$ on $\partial\Omega$ is well defined in $H^{-1/2}(\partial\Omega)$ and using (83) again, but this time with test functions $\psi \in \mathcal{D}(\overline{\Omega})$, we deduce that

$$D^{2s-1}[\varphi]\cdot n = 0 \text{ on } \partial\Omega.$$

So if we define the space

$$D(\mathcal{L}) = \{\varphi \in H^s(\Omega)\,;\, \mathcal{L}[\varphi] \in L^2(\Omega), \quad D^{2s-1}[\varphi]\cdot n = 0 \text{ on } \partial\Omega\},$$

we have proved that the equation

$$\varphi - \mathcal{L}[\varphi] = g$$

has a unique solution $\varphi \in D(\mathcal{L})$ for all $g \in L^2(\Omega)$.

Theorem 1.3 now follows from Hille-Yoshida theorem. □

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