A bound from below for the temperature in compressible Navier-Stokes equations

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Abstract: We consider the full system of compressible Navier-Stokes equations for heat conducting fluid. We show that the temperature is uniformly positive for $t \geq t_0$ (for any $t_0 > 0$) for any solutions with finite initial entropy. The assumptions on the viscosity and conductivity coefficients are minimal (for instance, the solutions constructed by E. Feireisl in [2] verify all the requirements).

1 Introduction and main result

In this article, we consider $\rho(t,x) \geq 0$, $\theta(t,x) \geq 0$ and $u(t,x) \in \mathbb{R}^3$ solutions of the following system of equations:

$$\partial_t \rho + \operatorname{div}(\rho\, u) = 0, \tag{1}$$

$$c_v \partial_t(\rho\theta) + c_v \operatorname{div}(\rho u\, \theta) + R\rho\, \theta \operatorname{div} u = 2\mu |D(u)|^2 + \lambda |\operatorname{div} u|^2 + \operatorname{div}(\kappa \nabla \theta) \tag{2}$$

for $(t,x) \in \mathbb{R}_+ \times \Omega$. Those equations are the usual continuity and temperature equations of the full system of compressible Navier-Stokes equations (for heat conducting fluids) when the specific heat at constant volume c_v is assumed to be constant and for a pressure law of the form

$$P(\rho, \theta) = p_e(\rho) + R\rho\theta.$$

We recall that ρ denotes the density of the fuid, u its velocity field and θ its temperature. The strain tensor in (1)-(4) is given by

$$D(u) = \frac{1}{2}[\nabla u + {}^t\nabla u],$$

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and μ and λ are the two Lamé viscosity coefficients (possibly depending on the temperature θ or on the density ρ) satisfying

$$\mu > 0 \qquad 2\mu + 3\lambda \geq 0 \tag{3}$$

(μ is sometime called the shear viscosity of the fluid, while λ is usually referred to as the second viscosity coefficient). The coefficient $\kappa \geq 0$ in (2) is the heat conductivity coefficient. Without loss of generality, we can take $c_v = 1$ (by rescaling the other coefficients).

Those equations are usually supplemented by the momentum equation:

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla P(\rho, \theta) - \mathrm{div}(2\mu D(u)) - \nabla(\lambda \mathrm{div} u) = 0. \tag{4}$$

Altogether, Equations (1), (2) and (4) describe the motion of a general viscous, heat conducting and compressible fluid. We want to emphasis however that for the purpose of this paper we will not need equation (4).

The equations (1) and (2) are set in a smooth bounded subset Ω of $\mathbb{R}^3$, and we consider the following boundary conditions:

$$u(t,x) = 0, \qquad \nabla\theta(t,x) \cdot n(x) = 0 \quad \text{ on } \partial\Omega \tag{5}$$

where n denotes the exterior normal vector to $\partial\Omega$.

The existence of solutions for the full system of Navier-Stokes equations for compressible fluids (1), (2) and (4) is a very delicate problem which has been addressed in particular by E. Feireisl in [2, 3]. We will not discuss this issue in this paper. Instead, we are interested in the properties of the temperature $\theta(t,x)$ (assuming that it exists). More precisely, we will show that under reasonable assumptions on the initial data (which imply in particular that the initial temperature may not vanish on an open set), the temperature cannot vanish for positive time and is uniformly bounded away from zero on any time interval $[t_0, T]$ with $T > t_0 > 0$.

The proof is inspired by De Giorgi's proof of Hölder regularity for the solutions of elliptic equation with discontinuous coefficients. Similar arguments was used in the context of Reaction-Diffusion systems by Alikakos [1]. In fluid mechanics, the method is used in [5] to obtain partial regularity results for incompressible Navier-Stokes system of equations, and in [4] to obtain L^∞ bounds on the velocity for the isentropic compressible Navier-Stokes system of equations. The method relies mainly on the energy (or entropy) inequality for such equations.

We now have to make precise the notion of solutions we are going to use. As it turns out, the proof only makes use of the fact that solutions of (1)-(2) satisfy (at least formally) the following inequality:

$$\begin{aligned} \frac{d}{dt}\int_\Omega \rho\phi(\theta)\,dx - \int_\Omega 2\mu\phi'(\theta)|D(u)|^2\,dx - \int_\Omega \lambda\phi'(\theta)|\mathrm{div} u|^2\,dx \\ + \int_\Omega \kappa\phi''(\theta)|\nabla\theta|^2\,dx \leq -R\int_\Omega \rho\theta\phi'(\theta)\mathrm{div} u\,dx \end{aligned} \tag{6}$$

for some appropriate functions ϕ. We thus give the following definition:

Definition 1 *We say that* (ρ, u, θ) *is an* **admissible solution** *of (1)-(2) if the following inequality holds*

$$\begin{aligned}\int_\Omega \rho\phi(\theta)(t,x)\,dx - \int_s^t\int_\Omega 2\mu\phi'(\theta)|D(u)|^2\,dx\,d\tau - \int_s^t\int_\Omega \lambda\phi'(\theta)|\mathrm{div}u|^2\,dx\,d\tau \\ + \int_s^t\int_\Omega \kappa\phi''(\theta)|\nabla\theta|^2\,dx\,d\tau \le -R\int_s^t\int_\Omega \rho\theta\phi'(\theta)\mathrm{div}u\,dx\,d\tau + \int_\Omega \rho\phi(\theta)(s,x)\,dx,\end{aligned} \tag{7}$$

for any function ϕ *of the form*

$$\phi(\theta) = \left[\ln\left(\frac{C}{\theta+\varepsilon}\right)\right]_+ = [\ln C - \ln(\theta+\varepsilon)]_+,$$

where C *and* ε *are two positive constants.*

We stress out the fact that (ρ, u, θ) do not need to satisfy (1)-(2) even in some very weak sense to be admissible in the sense of this definition. In section 2, we will prove that smooth solutions of (1)-(2) satisfies (7). More importantly, we will check that the method can be applied to the "variational" solutions constructed by E. Feireisl in [2]. Strictly speaking, those "variational" solutions are not known to satisfy inequalities (7). But in [2] (Equation (7.97) page 185), it is shown that there exists an approximated family of solution $(\rho_\delta, u_\delta, \theta_\delta)$ satisfying:

$$\begin{aligned}&\int_\Omega (\rho_\delta+\delta)\phi(\theta_\delta)(t,x)\,dx - \int_s^t\int_\Omega 2(1-\delta)\mu\phi'(\theta_\delta)|D(u_\delta)|^2\,dx\,d\tau \\ &\qquad - \int_s^t\int_\Omega \lambda(1-\delta)\phi'(\theta_\delta)|\mathrm{div}u_\delta|^2\,dx\,d\tau + \int_s^t\int_\Omega \kappa\phi''(\theta_\delta)|\nabla\theta_\delta|^2\,dx\,d\tau \\ &\le -R\int_s^t\int_\Omega \rho\theta_\delta\phi'(\theta_\delta)\mathrm{div}u_\delta\,dx\,d\tau + \int_\Omega (\delta+\rho_\delta)\phi(\theta_\delta)(s,x) \\ &\qquad -\delta\int_s^t\int_\Omega \theta_\delta^{\alpha'+1}\phi'(\theta_\delta)\,dx\,d\tau,\end{aligned} \tag{8}$$

for a given $\alpha' \ge 2$ and any function ϕ of the form

$$\phi(\theta) = \left[\ln\left(\frac{C}{\theta+\varepsilon}\right)\right]_+ = [\ln C - \ln(\theta+\varepsilon)]_+,$$

where C and ε are two positive constants independent on δ. This inequality is even better than (7) (since there is no problem close to the vacuum $\rho = 0$), except for the damping term of the temperature of the form $\delta\theta^{\alpha'+1}$. We will see in the last section of this paper that this damping term can be dealt with and thus that our result holds for Feireisl's solutions. The important point will be that we obtain an estimate for θ_δ which is independent of δ, and is thus satisfied by $\lim_{\delta\to 0}\theta_\delta$.

We can now state our main result:

Theorem 1 *Assume that the coefficients μ, λ and κ (which may depend on ρ or θ) satisfies*

$$
\begin{aligned}
&2\mu + 3\lambda \geq \nu(\theta) > 0 \quad \text{for all } \theta, && (9)\\
&\nu(\theta) \geq C\theta \quad \text{for small } \theta, && (10)\\
&\kappa(\theta) \geq \underline{\kappa} > 0 \quad \text{for small } \theta, && (11)
\end{aligned}
$$

for some function $\nu(\theta)$ and constant $\underline{\kappa}$.

Let (ρ, u, θ) be an admissible solution of (1)-(2) such that

$$\rho \in L^\infty(0, T; L^p(\Omega)) \text{ for some } p > 3,$$

and

$$u \in L^2(0, T; H_0^1(\Omega)).$$

Assume moreover that (ρ_0, θ_0) satisfy

$$\int_\Omega \rho_0 [\ln(1/\theta_0)]_+ \, dx < +\infty. \tag{12}$$

Then, for every $T > t_0 > 0$, there exists a constant $\delta_{t_0,T} > 0$ such that:

$$\theta(s, x) \geq \delta_{t_0,T} \qquad \text{for all } s \in (t_0, T) \text{ and almost all } x \in \Omega.$$

Furthermore, if there exists a constant $\delta_0 > 0$ such that $\theta_0(x) \geq \delta_0$ for almost every $x \in \Omega$, then the constant $\delta_{t_0,T}$ does not depend on t_0.

We will see in the next section that the assumptions on (ρ, u, θ) are satisfied by the "variational" solutions constructed by Feireisl [3, 2] when the pressure law is given by:

$$P(\rho, \theta) = C\rho^\gamma + R\rho\,\theta, \qquad \text{with } \gamma > 3. \tag{13}$$

This leads to the following corollary:

Corollary 2 *Let (ρ, u, θ) be a solution a la Feireisl of the Navier-Stokes system of equations (1)-(4). Assume that μ, λ and κ satisfy (9)-(11) and that the pressure law is given by (13).*

If the initial data (ρ_0, u_0, θ_0) has finite energy and finite physical entropy, then for every $T > t_0 > 0$, there exists a constant $\delta_{t_0,T} > 0$ such that:

$$\theta(s, x) \geq \delta_{t_0,T},$$

for every $s \in (t_0, T)$ and almost every $x \in \Omega$.

The paper is organized as follows: In section 2, we check that smooth integrable solutions of (1)-(2), as well as Feireisl's solutions are admissible in the sense of Definition 1. Section 3 is devoted to the proof of Theorem 1. In the last section, we complete the proof of Corollary 2.

2 Admissible solutions

At least formally, the notion of admissible solutions introduced by Definition 1 is consistent with equations (1), (2): As a matter of fact, a simple computation shows that if (ρ, θ) is a smooth solution of (1)-(2) then for any function ϕ we have:

$$\begin{aligned}\partial_t(\rho\phi(\theta)) + \mathrm{div}(\rho u\phi(\theta)) &= 2\mu\phi'(\theta)|D(u)|^2 + \lambda\phi'(\theta)|\mathrm{div}u|^2 \\ &\quad -R\rho\theta\phi'(\theta)\mathrm{div}u + \mathrm{div}(\kappa\nabla\phi(\theta)) \\ &\quad -\phi''(\theta)\kappa|\nabla\theta|^2.\end{aligned}$$

Integrating with respect to x (assuming that all integrations are valid) and using the boundary conditions (5), we immediately deduce that (7) holds (with equality) for any smooth function ϕ.

However, the existence of smooth solutions satisfying the necessary integrability conditions is not known. It is therefore important to check that admissible solutions actually exist. The system of equations (1)-(2) has been studied in great details by E. Feireisl in [2, 3]. We recall the following result (see [2]): For all $\delta > 0$ and for some $\alpha' > 2$, there exists (ρ, u, θ) solution of

$$\partial_t\rho + \mathrm{div}(\rho\, u) = 0, \tag{14}$$

$$c_v\partial_t(\rho\theta) + c_v\mathrm{div}(\rho u\,\theta) + R\rho\,\theta\mathrm{div}u + \delta\theta^{\alpha'+1} = 2\mu|D(u)|^2 + \lambda|\mathrm{div}u|^2 + \mathrm{div}(\kappa\nabla\theta) \tag{15}$$

such that the following inequality holds (see [2], Equation (7.97) p. 185):

$$\begin{aligned}&\int_\Omega(\rho+\delta)Q_h(\theta)(t,x)\,dx - 2(1-\delta)\int_s^t\int_\Omega \mu Q_h'(\theta)|D(u)|^2\,dx\,d\tau \\ &\quad -(1-\delta)\int_s^t\int_\Omega \lambda Q_h'(\theta)|\mathrm{div}u|^2\,dx\,d\tau \\ &\quad +\int_s^t\int_\Omega \kappa Q_h''(\theta)|\nabla\theta|^2\,dx\,d\tau + \int_s^t\int_\Omega \delta Q_h'(\theta)\theta^{\alpha'+1}\,dx\,d\tau \\ &\qquad \le -R\int_s^t\int_\Omega \rho\theta Q_h'(\theta)\mathrm{div}u\,dx\,d\tau + \int_\Omega(\rho+\delta)Q_h(\theta)(s,x)\,dx,\end{aligned} \tag{16}$$

where $Q_h(\theta) = -\int_0^\theta h(z)\,dz$ with h non-increasing function in $\mathcal{C}^2([0,\infty))$ such that $0 \le h(0) < +\infty$, $\lim_{z\to+\infty} h(z) = 0$ and

$$h''(z)\,h(z) \ge 2(h'(z))^2 \quad \text{for all } z \ge 0. \tag{17}$$

In particular, the function $h(z) = \frac{1}{z+\varepsilon}1_{z+\varepsilon\le C}$ satisfies all the conditions (including (17)), so we deduce that (16) holds with

$$Q_h(\theta) = \begin{cases} -\ln(\theta+\varepsilon) + \ln(\varepsilon) & \text{if } \theta+\varepsilon \le C \\ -\ln(C) + \ln(\varepsilon) & \text{if } \theta+\varepsilon \ge C \end{cases}$$

Taking $\phi(\theta) = Q_h(\theta) + \ln(C) - \ln(\varepsilon)$ and using the fact that $\frac{d}{dt}\int_\Omega \rho(t,x)\,dx = 0$, we easily see that (ρ, u, θ) satisfies

$$\begin{aligned}
&\int_\Omega (\rho+\delta)\phi(\theta)(t,x)\,dx - \int_s^t \int_\Omega 2(1-\delta)\mu\phi'(\theta)|D(u)|^2\,dx\,d\tau \\
&\quad - \int_s^t \int_\Omega (1-\delta)\lambda\phi'(\theta)|\mathrm{div}u|^2\,dx\,d\tau \\
&\quad + \int_s^t \int_\Omega \kappa\phi''(\theta)|\nabla\theta|^2\,dx\,d\tau + \int_s^t \int_\Omega \delta\phi'(\theta)\theta^{\alpha'+1}\,dx\,d\tau \\
&\qquad\quad \leq -R\int_s^t \int_\Omega \rho\theta\phi'(\theta)\mathrm{div}u\,dx\,d\tau + \int_\Omega (\rho+\delta)\phi(\theta)(s,x)\,dx,
\end{aligned} \tag{18}$$

for any function ϕ of the form

$$\phi(\theta) = \left[\ln\left(\frac{C}{\theta+\varepsilon}\right)\right]_+ = [\ln C - \ln(\theta+\varepsilon)]_+.$$

where C and ε are two positive constants.

We stress out the fact that (15) and (18) contain an additional damping term $\delta\theta^{\alpha'+1}$ and thus that the solution of Feireisl is not exactly admissible in the sense of Definition 1. For the sake of clarity, however, we prove our main result without the damping term, and we show in the last section that this term can be handled without additional difficulties.

3 Proof of Theorem 1

3.1 Setting of the problem and technical lemmas

The proof of Theorem 1 is inspired by De Giorgi's method for the regularity of the solutions of elliptic equations and relies crucially on inequality (7).

We begin by introducing the sequence of real numbers

$$C_k = e^{-M[1-2^{-k}]}, \qquad \forall k \in \mathbb{N},$$

where M is a positive number to be chosen later (note that $\lim_{k\to\infty} C_k = e^{-M}$). We then define the sequences of functions $(\phi_{k,\varepsilon})_{k\in\mathbb{N}}$ by

$$\phi_{k,\varepsilon}(\theta) = \left[\ln\left(\frac{C_k}{\theta+\varepsilon}\right)\right]_+.$$

The starting point of the proof is the following Lemma:

Lemma 3 *Let (T_k) be a sequence nonnegative numbers and assume that (ρ, u, θ) is an admissible solution of (1)-(2). Define*

$$\begin{aligned}
U_{k,\varepsilon} := \sup_{T_k\leq t\leq T}\left(\int_\Omega \rho\phi_{k,\varepsilon}(\theta)\,dx\right) &+ \int_{T_k}^T \int_\Omega \frac{\nu(\theta)}{\theta+\varepsilon}\mathbf{1}_{\{\theta+\varepsilon\leq C_k\}}|D(u)|^2\,dx\,dt \\
&+ \int_{T_k}^T \int_\Omega \frac{\kappa(\theta)}{(\theta+\varepsilon)^2}\mathbf{1}_{\{\theta+\varepsilon\leq C_k\}}|\nabla\theta|^2\,dx\,dt.
\end{aligned}$$

Then the following inequalities holds for all $k \in \mathbb{N}$:

- *If $T_k = 0$ for all $k \in \mathbb{N}$, then:*

$$U_{k,\varepsilon} \leq R \int_0^T \int_\Omega \rho \frac{\theta}{\theta + \varepsilon} \mathbf{1}_{\{\theta+\varepsilon \leq C_k\}} |\mathrm{div} u| \, dx \, dt + \int_\Omega \rho_0 \phi_{k,\varepsilon}(\theta_0) \, dx. \tag{19}$$

- *If $(T_k)_{k \in \mathbb{N}}$ is an increasing sequence of positive numbers, then:*

$$\begin{aligned} U_{k,\varepsilon} \quad \leq \quad & R \int_{T_{k-1}}^T \int_\Omega \rho \frac{\theta}{\theta + \varepsilon} \mathbf{1}_{\{\theta+\varepsilon \leq C_k\}} |\mathrm{div} u| \, dx \, dt \\ & + \frac{1}{T_k - T_{k-1}} \int_{T_{k-1}}^{T_k} \int_\Omega \rho \phi_{k,\varepsilon}(\theta) \, dx \, dt. \end{aligned} \tag{20}$$

Note that the right hand side of (19) is bounded by

$$R \int_0^T \int_\Omega \rho \, |\mathrm{div} u| \, dx \, dt + \int_\Omega \rho_0 [\ln(1/\theta_0)]_+ \, dx,$$

and so the assumptions on ρ, u, ρ_0 and θ_0 in Theorem 1 give the existence of a constant C independent of ε and k such that

$$U_{k,\varepsilon} \leq C. \tag{21}$$

Theorem 1 will be proved by showing that for suitable choices of constant M (independent on ε) and sequence $(T_k)_{k \in \mathbb{N}}$, the sequence $(U_{k,\varepsilon})_{k \in \mathbb{N}}$ goes to zero as k goes to infinity. Using the definition of $U_{k,\varepsilon}$, this will imply that

$$\theta(t,x) + \varepsilon \geq e^{-M}$$

and the main theorem will follow since this inequality holds for any $\varepsilon > 0$. In order to establish the convergence to zero of $U_{k,\varepsilon}$, we will prove in the next sections that both terms in the right hand side of (20) can be controlled by $U_{k-1,\varepsilon}^\gamma$ for some $\gamma > 1$.

Proof of Lemma 3. The lemma follows directly from (7) (with $C = C_k$). We note that:

$$\begin{aligned} \phi'_{k,\varepsilon}(\theta) &= -\frac{1}{\theta + \varepsilon} \mathbf{1}_{\{\theta+\varepsilon \leq C_k\}} \\ \phi''_{k,\varepsilon}(\theta) &\geq \frac{1}{(\theta + \varepsilon)^2} \mathbf{1}_{\{\theta+\varepsilon \leq C_k\}}. \end{aligned}$$

Integrating (7) with respect to time, we deduce that for all σ, t such that $T_{k-1} \leq \sigma \leq T_k \leq t \leq T$, we have

$$\begin{aligned} & \int_\Omega \rho \phi_{k,\varepsilon}(\theta)(t,x) \, dx + \int_\sigma^t \int_\Omega \frac{\nu(\theta)}{\theta + \varepsilon} \mathbf{1}_{\{\theta+\varepsilon \leq C_k\}} |D(u)|^2 \, dx \, ds \\ & \quad + \int_\sigma^t \int_\Omega \frac{\kappa}{(\theta + \varepsilon)^2} \mathbf{1}_{\{\theta+\varepsilon \leq C_k\}} |\nabla \theta|^2 \, dx \, ds \\ & \qquad \leq R \int_\sigma^t \int_\Omega \rho \frac{\theta}{\theta + \varepsilon} \mathbf{1}_{\{\theta+\varepsilon \leq C_k\}} |\mathrm{div} u| \, dx \, ds + \int_\Omega \rho \phi_{k,\varepsilon}(\theta)(\sigma,x) \, dx. \end{aligned}$$

and thus

$$\int_\Omega \rho\phi_{k,\varepsilon}(\theta)(t,x)\,dx + \int_{T_k}^t \int_\Omega \frac{\nu(\theta)}{\theta+\varepsilon}\mathbf{1}_{\{\theta+\varepsilon\le C_k\}}|D(u)|^2\,dx\,ds$$
$$+\int_{T_k}^t \int_\Omega \frac{\kappa}{(\theta+\varepsilon)^2}\mathbf{1}_{\{\theta+\varepsilon\le C_k\}}|\nabla\theta|^2\,dx\,ds$$
$$\le R\int_{T_{k-1}}^T \int_\Omega \rho\frac{\theta}{\theta+\varepsilon}\mathbf{1}_{\{\theta+\varepsilon\le C_k\}}|\mathrm{div}u|\,dx\,ds + \int_\Omega \rho\phi_{k,\varepsilon}(\theta)(\sigma,x)\,dx.$$

If $T_k = 0$ for all k, then taking $\sigma = 0$ and the supremum over $t \in [T_k, T]$, we deduce (19). If $(T_k)_{k\in\mathbb{N}}$ is an increasing sequence of positive numbers, then taking the mean value with respect to σ in $[T_{k-1}, T_k]$, we get:

$$\int_\Omega \rho\phi_{k,\varepsilon}(\theta)(t,x)\,dx + \int_{T_k}^t \int_\Omega \frac{\nu(\theta)}{\theta+\varepsilon}\mathbf{1}_{\{\theta+\varepsilon\le C_k\}}|D(u)|^2\,dx\,ds$$
$$+\int_{T_k}^t \int_\Omega \frac{\kappa}{(\theta+\varepsilon)^2}\mathbf{1}_{\{\theta+\varepsilon\le C_k\}}|\nabla\theta|^2\,dx\,ds$$
$$\le R\int_{T_{k-1}}^T \int_\Omega \rho\frac{\theta}{\theta+\varepsilon}\mathbf{1}_{\{\theta+\varepsilon\le C_k\}}|\mathrm{div}u|\,dx\,ds$$
$$+\frac{1}{T_k - T_{k-1}}\int_{T_{k-1}}^{T_k}\int_\Omega \rho\phi_{k,\varepsilon}(\theta)(s,x)\,dx\,ds.$$

Finally, taking the supremum over $t \in [T_k, T]$, we deduce (20). □

In the next sections, we will attempt to control the two terms in the right hand side of (19) and (20) by some power of $U_{k-1,\varepsilon}$. For that purpose, we need the following technical lemma:

Lemma 4 *There exists a constant C (depending only on Ω, T, $\int_\Omega \rho_0(x)\,dx$ and $\|\rho\|_{L^\infty(0,T,L^3(\Omega))}$) such that, for every function $F \ge 0$ with $\rho F \in L^\infty(0,T;L^1(\Omega))$ and $\nabla F \in L^2(0,T;L^2(\Omega))$, we have:*

$$\|F\|_{L^2(0,T;L^6(\Omega))} \le C(\|\rho F\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla F\|_{L^2(0,T;L^2(\Omega))}).$$

Proof. First note that there exists two constants $\varepsilon, C_1 > 0$ (independent of t) such that

$$|\Omega(t)| = |\{x|\rho(t,x) > \varepsilon\}| \ge C_1.$$

Indeed, we have:

$$\int_\Omega \rho_0\,dx = \int_\Omega \rho(t,x)\,dx \le \varepsilon|\Omega| + \|\rho\|_{L^\infty(L^3)}|\Omega(t)|^{2/3},$$

and so, for ε small enough, we can take:

$$C_1 = \left(\frac{\int_\Omega \rho_0\,dx - \varepsilon|\Omega|}{\|\rho\|_{L^\infty(L^3)}}\right)^{3/2}.$$

Now, for any fixed time t, we write:

$$\begin{aligned}
\|F(t)\|_{L^6(\Omega)} &\leq \left\| F(t) - \frac{1}{|\Omega|}\int_\Omega F(t,y)\,dy \right\|_{L^6} + |\Omega|^{1/6}\frac{1}{|\Omega|}\int_\Omega F(t,y)\,dy \\
&\leq \left\| F(t) - \frac{1}{|\Omega|}\int_\Omega F(t,y)\,dy \right\|_{L^6} \\
&\quad + |\Omega|^{-5/6}\frac{|\Omega|}{|\Omega(t)|}\int_{\Omega(t)} F(t,x)\,dx \\
&\quad + |\Omega|^{-5/6}\left| \frac{|\Omega|}{|\Omega(t)|}\int_{\Omega(t)} F(t,x)\,dx - \int_\Omega F(t,y)\,dy \right|
\end{aligned}$$

The first term in the right hand side is bounded by $\|\nabla F\|_{L^2(\Omega)}$ (which is in $L^2(0,T)$ by definition) and by definition of $\Omega(t)$, the second term is bounded by:

$$\frac{|\Omega|^{1/6}}{\varepsilon C_1}\int_\Omega \rho F(t,x)\,dx,$$

which lies in $L^\infty(0,T)$, and therefore in $L^2(0,T)$. Finally, the last term can be written in the following way:

$$\begin{aligned}
&\left| \frac{|\Omega|}{|\Omega(t)|}\int_{\Omega(t)} F(t,x)\,dx - \int_\Omega F(t,y)\,dy \right| \\
&\qquad \leq \frac{1}{C_1}\left| \int_{\Omega(t)\times\Omega} [F(t,x) - F(t,y)]\,dx\,dy \right| \\
&\leq \frac{1}{C_1}\int_{\Omega\times\Omega} |F(t,x) - F(t,y)|\,dx\,dy,
\end{aligned}$$

which is classically bounded by $\|\nabla F\|_{L^2(\Omega)}$. $\square$

We can now prove the technical lemma which will be essential in the next sections:

Lemma 5 *If p_1, q_1, α and β satisfy*

$$q_1 = \frac{6}{5\alpha+\beta} > 1 \qquad \text{and} \qquad p_1 = \frac{2}{\beta-\alpha} > 1, \tag{22}$$

then there exist a constant C depending on the dimension, T, $\int \rho_0(x)\,dx > 0$ and $||\rho||_{L^\infty(0,T,L^3(\Omega))}$ such that:

$$\|\rho^\alpha \phi_{k,\varepsilon}(\theta)^\beta\|_{L^{p_1}L^{q_1}} \leq CU_{k,\varepsilon}^{\frac{\beta+\alpha}{2}} + CU_{k,\varepsilon}^\beta,$$

where $L^{p_1}L^{q_1}$ denotes the space $L^{p_1}(T_k,T;L^{q_1}(\Omega))$

Proof. Before starting the proof, we observe that (11) yields:

$$\int_\Omega \frac{\kappa}{(\theta+\varepsilon)^2} \mathbf{1}_{\{\theta+\varepsilon\leq C_k\}} |\nabla\theta|^2\,dx \geq \underline{\kappa} \int_\Omega |\nabla \phi_{k,\varepsilon}(\theta)|^2\,dx$$

and thus

$$\|\nabla\phi_{k,\varepsilon}(\theta)\|^2_{L^2((T_k,T)\times\Omega)} \leq C U_{k,\varepsilon}. \tag{23}$$

Now, we write

$$\begin{aligned}
\|\rho^\alpha \phi_{k,\varepsilon}(\theta)^\beta\|_{L^{p_1}L^{q_1}} &\leq \|\rho^{\alpha/\beta}\phi_{k,\varepsilon}(\theta)\|^\beta_{L^{p_1\beta}L^{q_1\beta}} \\
&\leq \|(\rho\phi_{k,\varepsilon}(\theta))^{\alpha/\beta}\phi_{k,\varepsilon}(\theta)^{1-\alpha/\beta}\|^\beta_{L^{p_1\beta}L^{q_1\beta}} \\
&\leq \left(\|(\rho\phi_{k,\varepsilon}(\theta))^{\alpha/\beta}\|_{L^\infty L^{\beta/\alpha}} \|\phi_{k,\varepsilon}(\theta)^{1-\alpha/\beta}\|_{L^{\frac{2}{1-\alpha/\beta}}L^{\frac{6}{1-\alpha/\beta}}}\right)^\beta \qquad (24)
\end{aligned}$$

whenever the coefficients α, β, p_1 and q_1 are such that:

$$\frac{1}{q_1\beta} = \frac{\alpha}{\beta} + \frac{1-\alpha/\beta}{6},$$
$$\frac{1}{p_1\beta} = \frac{1-\alpha/\beta}{2}.$$

Note that those conditions are equivalent to (22). So in order to conclude, we only need to control the two norms in the right hand side of (24).

For the first one, using the definition of $U_{k,\varepsilon}$, it is readily seen that

$$\|(\rho\phi_{k,\varepsilon}(\theta))^{\alpha/\beta}\|_{L^\infty L^{\beta/\alpha}} \leq U_{k,\varepsilon}^{\alpha/\beta}. \tag{25}$$

Next, we note that

$$\begin{aligned}
\|\phi_{k,\varepsilon}(\theta)^{1-\alpha/\beta}\|_{L^{\frac{2}{1-\alpha/\beta}}L^{\frac{6}{1-\alpha/\beta}}} &= \left[\int_{T_k}^T \left(\int_\Omega \phi_{k,\varepsilon}(\theta)^6\,dx\right)^{2/6} dt\right]^{\frac{1-\alpha/\beta}{2}} \\
&= \|\phi_{k,\varepsilon}(\theta)\|^{1-\alpha/\beta}_{L^2(T_k,T,L^6(\Omega))}
\end{aligned}$$

and Lemma 4 yields:

$$\|\phi_{k,\varepsilon}(\theta)\|_{L^2(T_k,T,L^6(\Omega))} \leq C\left(\|\rho\phi_{k,\varepsilon}(\theta)\|_{L^2(T_k,T;L^1(\Omega))} + \|\nabla\phi_{k,\varepsilon}(\theta)\|_{L^2((T_k,T)\times\Omega)}\right).$$

and so

$$\|\phi_{k,\varepsilon}(\theta)\|_{L^2(T_k,T,L^6(\Omega))} \leq C\|\nabla\phi_{k,\varepsilon}(\theta)\|_{L^2((T_k,T)\times\Omega)} + C\int_\Omega \rho\,\phi_{k,\varepsilon}(\theta)\,dx.$$

Finally (23) and the definition of $U_{k,\varepsilon}$ leads to

$$\|\phi_{k,\varepsilon}(\theta)\|^2_{L^2(T_k,T,L^6(\Omega))} \leq CU_{k,\varepsilon} + CU^2_{k,\varepsilon}. \tag{26}$$

We deduce

$$\|\phi_{k,\varepsilon}(\theta)^{1-\alpha/\beta}\|_{L^{\frac{2}{1-\alpha/\beta}}L^{\frac{6}{1-\alpha/\beta}}} \quad \leq \quad CU_{k,\varepsilon}^{\frac{1-\alpha/\beta}{2}} + CU_{k,\varepsilon}^{1-\alpha/\beta}. \tag{27}$$

We can now conclude, since (24), together with (25) and (27), yields:

$$\|\rho^\alpha\phi_{k,\varepsilon}(\theta)^\beta\|_{L^{p_1}L^{q_1}} \leq C\left(U_{k,\varepsilon}^{\frac{\alpha}{\beta}}\left(U_{k,\varepsilon}^{\frac{1-\alpha/\beta}{2}} + U_{k,\varepsilon}^{1-\alpha/\beta}\right)\right)^\beta \leq CU_{k,\varepsilon}^{\frac{\beta+\alpha}{2}} + CU_{k,\varepsilon}^{\beta}.$$

□

3.2 Control of the pressure term

The next lemma shows that the first term in the right hand side of (19) and (20) can be controlled by $U_{k-1,\varepsilon}$ in a nonlinear fashion:

Lemma 6 *If $\rho \in L^\infty(0,T;L^p(\Omega))$ for some $p > 3$, then there exists $\gamma > 1$ and $\beta \in (0,1)$ such that:*

$$\begin{aligned} R\int_{T_{k-1}}^T\int_\Omega \rho\frac{\theta}{\theta+\varepsilon}\mathbf{1}_{\{\theta+\varepsilon\leq C_k\}}|\mathrm{div}u|\,dx\,dt \quad &\leq \quad \frac{1}{2}\int_{T_{k-1}}^T\int_\Omega \frac{\nu(\theta)}{\theta+\varepsilon}\mathbf{1}_{\{\theta+\varepsilon\leq C_k\}}|D(u)|^2\,dx\,dt \\ &\quad +C\left[\ln\frac{C_{k-1}}{C_k}\right]^{-\beta} U_{k-1,\varepsilon}^\gamma. \end{aligned}$$

Proof. Cauchy-Schwarz and Young's inequality yield:

$$\begin{aligned} R\int_{T_{k-1}}^T\int_\Omega \rho\frac{\theta}{\theta+\varepsilon}\mathbf{1}_{\{\theta+\varepsilon\leq C_k\}}|\mathrm{div}u|\,dx\,dt \quad &\leq \quad \frac{1}{2}\int_{T_{k-1}}^T\int_\Omega \frac{\nu(\theta)}{\theta+\varepsilon}\mathbf{1}_{\{\theta+\varepsilon\leq C_k\}}|D(u)|^2\,dx\,dt \\ &\quad +C\int_{T_{k-1}}^T\int_\Omega \frac{\theta}{\nu(\theta)}\mathbf{1}_{\{\theta+\varepsilon\leq C_k\}}\rho^2\,dx\,dt \\ &\leq \quad \frac{1}{2}\int_{T_{k-1}}^T\int_\Omega \frac{\nu(\theta)}{\theta+\varepsilon}\mathbf{1}_{\{\theta+\varepsilon\leq C_k\}}|D(u)|^2\,dx\,dt \\ &\quad +C\int_{T_{k-1}}^T\int_\Omega \mathbf{1}_{\{\theta+\varepsilon\leq C_k\}}\rho^2\,dx\,dt, \end{aligned}$$

where we have used Hypothesis (10) in the last inequality. Next, we note that when $\theta+\varepsilon\leq C_k$, we have $\phi_{k-1,\varepsilon}(\theta)\geq \ln\frac{C_{k-1}}{C_k}$. It follows that

$$\mathbf{1}_{\{\theta+\varepsilon\leq C_k\}} \leq \left[\ln\frac{C_{k-1}}{C_k}\right]^{-\beta}\phi_{k-1,\varepsilon}(\theta)^\beta$$

for any $\beta > 0$. Using Lemma 5, we deduce:

$$\begin{aligned}
\int_{T_{k-1}}^{T}\int_{\Omega} \mathbf{1}_{\{\theta+\varepsilon\le C_k\}}\rho^2\,dx\,dt &\le \left[\ln\frac{C_{k-1}}{C_k}\right]^{-\beta}\int_{T_{k-1}}^{T}\int_{\Omega}\rho^{2-\alpha}\rho^{\alpha}\phi_{k-1,\varepsilon}(\theta)^{\beta}\,dx\,dt\\
&\le \left[\ln\frac{C_{k-1}}{C_k}\right]^{-\beta}\|\rho^{2-\alpha}\|_{L^{p_1'}L^{q_1'}}\|\rho^{\alpha}\phi_{k-1,\varepsilon}(\theta)^{\beta}\|_{L^{p_1}(L^{q_1})}\\
&\le \left[\ln\frac{C_{k-1}}{C_k}\right]^{-\beta}\|\rho^{2-\alpha}\|_{L^{p_1'}L^{q_1'}}\left(U_{k-1,\varepsilon}^{\frac{\alpha+\beta}{2}}+U_{k-1,\varepsilon}^{\beta}\right),
\end{aligned}$$

with α, β, p_1 and q_1 satisfying (22) and p_1', q_1' the conjugate exponents.

Lemma 6 follows if we can show that we can choose α, β, p_1 and q_1 satisfying (22) such that

$$\gamma = \min(\frac{\alpha+\beta}{2},\beta) > 1 \quad \text{and} \quad \|\rho^{2-\alpha}\|_{L^{p_1'}L^{q_1'}} \le C.$$

First, we note that if $(\alpha+\beta)/2 > 1$, then (22) implies:

$$\begin{aligned}
p_1 &< \frac{1}{1-\alpha},\\
q_1 &< \frac{3}{2\alpha+1}
\end{aligned}$$

(which requires $\alpha < 1$ and $\beta > 1$). The corresponding conditions for the conjugate exponents read:

$$\begin{aligned}
p_1' &> \frac{1}{\alpha}\\
q_1' &> \frac{3}{2(1-\alpha)}.
\end{aligned}$$

Therefore we need to have

$$\|\rho^{2-\alpha}\|_{L^{p_1'}L^{q_1'}} = \left[\int_{T_{k-1}}^{T}\left(\int_{\Omega}\rho^{\frac{3(2-\alpha)}{2(1-\alpha)}}\,dx\right)^{\frac{2(1-\alpha)}{3}\frac{1}{\alpha}}dt\right]^{\alpha} < C,$$

for some $\alpha \in (0,1)$. This is satisfied for some α close enough to 0 since $\rho \in L^{\infty}(0,T;L^{p}(\Omega))$ for some $p > 3$. $\Box$

3.3 Proof of Theorem 1

We are now ready to complete the proof of Theorem 1. Lemmas 3 and 6 yield:

$$U_{k,\varepsilon} \le C\left[\ln\frac{C_{k-1}}{C_k}\right]^{-\beta}U_{k-1,\varepsilon}^{\gamma} + \int_{\Omega}\rho_0\phi_{k,\varepsilon}(\theta_0)\,dx, \qquad \forall k \ge 2 \tag{28}$$

if $T_k = 0$ for all k, and

$$U_{k,\varepsilon} \leq C \left[\ln \frac{C_{k-1}}{C_k}\right]^{-\beta} U_{k-1}^{\gamma} + \frac{1}{T_k - T_{k-1}} \int_{T_{k-1}}^{T_k} \int_{\Omega} \rho \phi_{k,\varepsilon}(\theta)\, dx\, dt \qquad \forall k \geq 2 \quad (29)$$

if $(T_k)_{k\in\mathbb{N}}$ is an increasing sequence of positive numbers.

To complete the proof, we will use (28) in the case when the initial datum is bounded away from zero and (29) in the general case.

3.3.1 Case without initial layer

In this subsection we assume that the initial datum verifies:

$$\theta_0(x) \geq \delta_0 > 0.$$

In that case, we take $T_k = 0$ for every k and we choose M such that $e^{-M/2} < \delta_0$. This implies in particular that

$$\phi_{k,\varepsilon}(\theta_0) = 0 \qquad \forall k \in \mathbb{N},$$

for any $\varepsilon > 0$. Inequality (28) thus becomes

$$U_{k,\varepsilon} \leq C \left[\ln \frac{C_{k-1}}{C_k}\right]^{-\beta} U_{k-1,\varepsilon}^{\gamma},$$

for some $\gamma > 1$. Moreover our choice of constants C_k yield:

$$\frac{C_{k-1}}{C_k} = e^{M 2^{-k}},$$

and thus

$$\left[\ln \frac{C_{k-1}}{C_k}\right]^{-\beta} = \frac{2^{k\beta}}{M^{\beta}}.$$

We deduce:

$$U_{k,\varepsilon} \leq C \frac{2^{k\beta}}{M^{\beta}} U_{k-1,\varepsilon}^{\gamma} \qquad \forall k \in \mathbb{N}.$$

Since $\gamma > 1$, it is a classical result that for M large enough (depending only on U_0 and thus independent on ε), we have:

$$\lim_{k\to\infty} U_{k,\varepsilon} = 0.$$

In particular, this yields

$$\int_0^T \int_{\Omega} \kappa(\theta) \left| \nabla \left[\ln \frac{e^{-M}}{\theta + \varepsilon}\right]_+ \right|^2 dx\, dt = 0,$$

and

$$\int_\Omega \rho \left[\ln \frac{e^{-M}}{\theta+\varepsilon}\right]_+ dx = 0.$$

The first equality gives (using (11)) that $\left[\ln \frac{e^{-M}}{\theta+\varepsilon}\right]_+$ is constant in Ω for all t, and the second equality implies that this constant is 0 (unless $\rho(t,\cdot)=0$ which contradicts the conservation of mass). We deduce that for almost every $t\in[0,T]$ and $x\in\mathbb{R}^3$ we have

$$\theta(t,x)+\varepsilon \geq e^{-M}.$$

Since this inequality holds for any $\varepsilon>0$, the theorem follows. □

3.3.2 Case with the time layer

We now remove the assumption that the initial temperature is bounded away from zero and only assume that the initial entropy is finite, i.e.

$$\int_{\mathbb{R}^3} \rho[\ln(1/\theta_0)]_+ \, dx < +\infty,$$

which is enough to guarantee (see (21)) that

$$U_{0,\varepsilon} \leq C < +\infty$$

with C independent on ε. Let t_0 be a fixed positive number and set $T_k = t_0(1-2^{-k})$.

The only thing left to do is to show that we can control the second term in the right hand side of (29). For that, we note that we have (using (26)):

$$\begin{aligned}
\int_{T_{k-1}}^{T_k}\int_\Omega \rho\phi_{k,\varepsilon}(\theta)\,dx\,dt &\leq \int_{T_{k-1}}^{T}\left[\int_\Omega \phi_{k,\varepsilon}(\theta)^6\,dx\right]^{1/6}\left[\int_\Omega \rho^{6/5}\mathbf{1}_{\{\theta+\varepsilon\leq C_k\}}\right]^{5/6} dt\\
&\leq ||\phi_{k,\varepsilon}(\theta)||_{L^2L^6}\left[\int_{T_{k-1}}^{T}\left[\int_\Omega \rho^{6/5}\mathbf{1}_{\{\theta+\varepsilon\leq C_k\}}\right]^{5/3} dt\right]^{1/2}\\
&\leq \left(U_{k,\varepsilon}^{1/2}+U_{k,\varepsilon}\right)\left[\int_{T_{k-1}}^{T}\left[\int_\Omega \rho^{6/5}\mathbf{1}_{\{\theta+\varepsilon\leq C_k\}}\right]^{5/3} dt\right]^{1/2}.
\end{aligned}$$

Moreover, proceeding as in the proof of Lemma 6, we check that when $\theta+\varepsilon\leq C_k$, we have $\phi_{k-1,\varepsilon}(\theta)\geq \ln\frac{C_{k-1}}{C_k}$ and so

$$\mathbf{1}_{\{\theta+\varepsilon\leq C_k\}} \leq \left[\ln\frac{C_{k-1}}{C_k}\right]^{-\alpha}\phi_{k-1,\varepsilon}(\theta)^\alpha,$$

for any $\alpha>0$. We deduce

$$\begin{aligned}
&\int_{T_{k-1}}^{T_k}\int_\Omega \rho\phi_{k,\varepsilon}(\theta)\,dx\,dt\\
&\qquad\leq \left(U_{k,\varepsilon}^{1/2}+U_{k,\varepsilon}\right)\left[\ln\frac{C_{k-1}}{C_k}\right]^{-\frac{5}{6}\alpha}\left[\int_{T_{k-1}}^{T}\left[\int_\Omega \rho^{6/5}\phi_{k-1,\varepsilon}(\theta)^\alpha\,dx\right]^{5/3} dt\right]^{1/2}.
\end{aligned}$$

The last term can be bounded as follows:

$$\left(\int_\Omega \rho^{6/5}\phi_{k-1,\varepsilon}(\theta)^\alpha\,dx\right)^{5/3} \le \left(\int_\Omega \rho^{\frac{6/5-\alpha}{1-\alpha}}\,dx\right)^{\frac{5}{3}(1-\alpha)}\left(\int_\Omega \rho\phi_{k-1,\varepsilon}(\theta)\,dx\right)^{5\alpha/3}.$$

If we take $\alpha > 3/5$, we deduce that there exists positive numbers β_1, β_2, and $\gamma' > 1$ such that:

$$\frac{1}{T_k - T_{k-1}}\int_{T_{k-1}}^{T_k}\int_\Omega \rho\phi_{k,\varepsilon}(\theta)\,dx\,dt \le \frac{1}{T_k - T_{k-1}}\left[\ln\frac{C_{k-1}}{C_k}\right]^{-\beta_1}\|\rho\|_{L^\infty L^p}^{\beta_2}U_{k-1,\varepsilon}^{\gamma'}. \tag{30}$$

with $p = \frac{6/5-\alpha}{1-\alpha}$. Note that for α close to $3/5$, we have p close to $3/2$ and so $\|\rho\|_{L^\infty L^p} < C$

Finally, recalling that $C_k = e^{-M[1-2^{-k}]}$ and $T_k = t_0(1-2^{-k})$, we get:

$$\left[\ln\frac{C_{k-1}}{C_k}\right]^{-1} = \frac{2^k}{M} \qquad \text{and} \qquad \frac{1}{T_k - T_{k-1}} = \frac{2^k}{t_0}$$

and so (29) and (30) yield

$$U_{k,\varepsilon} \le C\frac{2^{\beta k}}{M^\beta}U_{k-1,\varepsilon}^{\gamma} + C\frac{2^{\beta_2' k}}{t_0\,M^{\beta_1}}U_{k-1,\varepsilon}^{\gamma'}.$$

The proof can now be completed as in the case without time initial layer. □

4 Proof of Corollary 2

In this last section, we show how to deal with the temperature damping term which is necessary in Feireisl's result to prove the existence of admissible solutions. The important point is to get estimates that are independent of δ, so that the result holds when passing to the limit $\delta \to 0$.

We only treat the case without initial layer (the general case is left to the reader). We thus have $\phi_{k,\varepsilon}(\theta_0) = 0$ and we can take $T_k = 0$ for all $k \in \mathbb{N}$. Let $(\rho_\delta, u_\delta, \theta_\delta)$ be the solution of (14), (15) satisfying (16). We define:

$$\begin{aligned} U_{k,\varepsilon,\delta} := \sup_{0\le t\le T}\left(\int_\Omega (\rho_\delta+\delta)\phi_{k,\varepsilon}(\theta_\delta)\,dx\right) + \int_0^T\int_\Omega \frac{\nu(\theta_\delta)}{\theta_\delta+\varepsilon}\mathbf{1}_{\{\theta_\delta+\varepsilon\le C_k\}}|D(u_\delta)|^2\,dx\,dt \\ + \int_{T_k}^T\int_\Omega \frac{\kappa(\theta_\delta)}{(\theta_\delta+\varepsilon)^2}\mathbf{1}_{\{\theta_\delta+\varepsilon\le C_k\}}|\nabla\theta_\delta|^2\,dx\,dt. \end{aligned}$$

Using (18), it is readily seen that Lemma 3 with the additional damping term becomes:

$$U_{k,\varepsilon,\delta} \le R\int_0^T\!\!\int_\Omega \rho_\delta\frac{\theta_\delta}{\theta_\delta+\varepsilon}\mathbf{1}_{\{\theta_\delta+\varepsilon\le C_k\}}|\mathrm{div}u_\delta|\,dx\,dt + \delta\int_0^T\!\!\int_\Omega \frac{\theta_\delta^{\alpha'+1}}{\theta_\delta+\varepsilon}\mathbf{1}_{\{\theta_\delta+\varepsilon\le C_k\}}\,dx\,dt.$$

Together with Lemma 6, this implies:

$$U_{k,\varepsilon,\delta} \;\leq\; C\frac{2^{k\beta}}{M^\beta}U^{\gamma}_{k-1,\varepsilon,\delta} + \delta\int_0^T\int_\Omega \frac{\theta_\delta^{\alpha'+1}}{\theta_\delta+\varepsilon}\mathbf{1}_{\{\theta_\delta+\varepsilon\leq C_k\}}\,dx\,dt. \tag{31}$$

In order to control the last term in the right hand side, we note that when $\theta_\delta+\varepsilon\leq C_k$, we have (provided that M is large enough, so that $C_0\leq 1$):

$$\frac{\theta_\delta^{\alpha'+1}}{\theta_\delta+\varepsilon}\leq 1$$

for all $\varepsilon>0$. Next, we note that when $\theta_\delta+\varepsilon\leq C_k$, we have $\phi_{k-1,\varepsilon}(\theta_\delta)\geq \ln\frac{C_{k-1}}{C_k}$. It follows that

$$\mathbf{1}_{\{\theta_\delta+\varepsilon\leq C_k\}}\leq\left[\ln\frac{C_{k-1}}{C_k}\right]^{-\beta}\phi_{k-1,\varepsilon}(\theta_\delta)^\beta$$

for any $\beta>0$. Using Lemma 5 with ρ replaced by $\rho+\delta$, we deduce:

$$\begin{aligned}
\delta\int_0^T\int_\Omega \mathbf{1}_{\{\theta_\delta+\varepsilon\leq C_k\}}\,dx\,dt \;&\leq\; \delta^{1-\alpha}\left[\ln\frac{C_{k-1}}{C_k}\right]^{-\beta}\int_0^T\int_\Omega(\rho+\delta)^\alpha\phi_{k-1,\varepsilon}(\theta_\delta)^\beta\,dx\,dt\\
&\leq\; \delta^{1-\alpha}\left[\ln\frac{C_{k-1}}{C_k}\right]^{-\beta}T^{1/p_1'}|\Omega|^{1/q_1'}\left\|(\rho+\delta)^\alpha\phi_{k-1,\varepsilon}(\theta_\delta)^\beta\right\|_{L^{p_1}L^{q_1}}\\
&\leq\; C\delta^{1-\alpha}\left[\ln\frac{C_{k-1}}{C_k}\right]^{-\beta}\left(U^{\frac{\alpha+\beta}{2}}_{k-1,\varepsilon,\delta}+U^{\beta}_{k-1,\varepsilon,\delta}\right),
\end{aligned}$$

with α, β, p_1 and q_1 satisfying (22) and p_1', q_1' the conjugate exponents, with $p_1>1$ and $q_1>1$ close to 1. As in Section 3.2, we can choose $\alpha<1$ and $\beta>1$ such that

$$\gamma=\min(\frac{\alpha+\beta}{2},\beta)>1$$

if we take p_1 and q_1 close enough to 1. Since $\delta^{1-\alpha}\leq 1$, we deduce

$$U_{k,\varepsilon,\delta}\;\leq\; C\frac{2^{k\beta}}{M^\beta}U^\gamma_{k-1,\varepsilon,\delta}$$

where all the constants are independent of δ (and ε). As in the previous section, this implies that there exists a constant $\eta_T>0$ (independent on δ) such that:

$$\theta_\delta(t,x)\geq\eta_T,\qquad 0\leq t\leq T,\ x\in\Omega.$$

Passing to the limit with respect to δ, this shows that Feireisl's solution to (1), (2), (4) satisfies $\theta\geq\eta_T>0$ in $\Omega\times[0,T]$. $\square$

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