Global weak solutions for a Vlasov-Fokker-Planck/Navier-Stokes system of equations

A.Mellet*† and A.Vasseur‡

Department of Mathematics
University of Texas at Austin

Abstract

We establish the existence of a weak solutions for a coupled system of kinetic and fluid equations. More precisely, we consider a Vlasov-Fokker-Planck equation coupled to compressible Navier-Stokes equation via a drag force. The fluid is assumed to be barotropic with γ-pressure law ($\gamma > 3/2$). The existence of weak solutions is proved in a bounded domain of $\mathbb{R}^3$ with homogeneous Dirichlet conditions on the fluid velocity field and Dirichlet or reflection boundary conditions on the kinetic distribution function.

1 Introduction

In this paper, we establish the existence of weak solutions for a system of a kinetic equation coupled with compressible isentropic (or barotropic) Navier-Stokes equations. Such a system models, for example, the evolution of dispersed particles in a fluid: The cloud of particles is described by its distribution function $f(x, v, t)$, solution to a Vlasov-Fokker-Planck equation:

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v(F_d f - \nabla_v f) = 0. \tag{1}$$

The fluid, on the other hand, is modeled by macroscopic quantities: Its density $\rho(x, t) \geq 0$ and its velocity field $u(x, t) \in \mathbb{R}^3$. Assuming that the fluid is viscous, compressible and barotropic, we are led to consider the following Navier-Stokes system of equations:

$$\left\{ \begin{array}{l} \partial_t \rho + \mathrm{div}_x(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) + \nabla_x p - \mu \Delta u - \lambda \nabla \mathrm{div} u = F_f \end{array} \right. \tag{2}$$

*mellet@math.utexas.edu

†A. Mellet was partially supported by NSF grant DMS-0456647

‡vasseur@math.utexas.edu

with $\mu > 0$ and $\mu + \lambda = \nu > 0$.

In this system, the fluid-particles interactions are taken into account via a friction (or drag) force F_d exerted by the fluid onto the particles. This force typically depends on the relative velocity $u(x,t) - v$ and on the density ρ of the fluid. In this paper, we will consider a very simple model, in which F_d is independent on ρ and proportional to the relative velocity of the fluid and the particles:

$$F_d = F_0(u(x,t) - v)$$

where the coefficient F_0 is a constant (we will take $F_0 = 1$). Note that the right hand-side of the moment equation in the Navier-Stokes system has to take into account the action of the cloud of particles on the fluid, and so

$$F_f = -\int F_d f dv = F_0 \int (v - u(x,t)) f(x,v,t)\, dv.$$

Finally, we will assume that the pressure follows a γ-law:

$$p(\rho) = \rho^\gamma,$$

with $\gamma > 3/2$.

This kind of system arises in a lot of industrial applications. One example is the analysis of sedimentation phenomenon, with applications in medicine, chemical engineering or waste water treatment (see Berres, Bürger, Karlsen, and Tory [2], Gidaspow [9], Sartory [17], Spannenberg and Galvin [18]). Such systems are also used in the modeling of aerosols and sprays with applications, for instance, in the study of Diesel engines (see Williams [20], [19]).

In a related paper, [14], we investigate the asymptotic regime corresponding to a strong drag force and strong Brownian motion. More precisely, we consider the following system of singular equations:

$$\begin{aligned}
&\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon}\mathrm{div}_v(f_\varepsilon(u - v) - \nabla_v f_\varepsilon) = 0\\
&\partial_t \rho_\varepsilon + \mathrm{div}_x(\rho_\varepsilon u_\varepsilon) = 0\\
&\partial_t(\rho_\varepsilon u_\varepsilon) + \mathrm{div}_x(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla_x \rho_\varepsilon^\gamma - \mu\Delta u - \lambda\nabla \mathrm{div} u = \frac{1}{\varepsilon}(j_\varepsilon - n_\varepsilon u_\varepsilon)
\end{aligned}$$

where $n_\varepsilon = \displaystyle\int f_\varepsilon(x,v,t)\, dv$ and $j_\varepsilon = \displaystyle\int v f_\varepsilon(x,v,t)\, dv$, and we prove that weak solutions $(f_\varepsilon, \rho_\varepsilon, u_\varepsilon)$ that satisfies a natural entropy inequality converges, as ε goes to zero, to $(M_{n,u}, \rho, u)$, where $M_{n,u}$ denotes the Maxwellian distribution with density $n(x,t)$ and velocity $u(x,t)$. Moreover, (n, ρ, u) solve the following system of hydrodynamic equations:

$$\begin{cases}
\partial_t n + \mathrm{div}_x(nu) = 0\\
\partial_t \rho + \mathrm{div}_x(\rho u) = 0\\
\partial_t((\rho + n)u) + \mathrm{div}_x((\rho + n)u \otimes u) + \nabla_x(n + \rho^\gamma) - \mu\Delta u - \lambda\nabla \mathrm{div} u = 0
\end{cases}$$

This asymptotic analysis was first performed (formally) by J. Carrillo and T. Goudon in [5]. The rigorous approach is developed in [14] when $\nu \geq 0$ and $\gamma \in (1,2)$. We also refer to [5] for further considerations on various modelling issues and stability properties of this system of equations.

The purpose of this paper is thus to prove that for fixed $\varepsilon > 0$, there exists a solution of the coupled system of equations (1)-(2) that satisfies all the hypothesis necessary to apply our convergence result of [14]. We work under slightly different hypothesis (namely $\nu > 0$ and $\gamma > 3/2$), but we note that when $\nu > 0$ and $\gamma \in (3/2, 2)$, both the existence result and the asymptotic result hold.

As in [14], we assume that (1)-(2) is set in a bounded subset Ω of $\mathbb{R}^3$. The system has to be supplemented with initial and boundary conditions. We will only make relatively minimal assumptions on the initial data

$$f(x,v,0) = f_0(x,v), \quad \rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x),$$

and on $\partial\Omega$, we will consider homogeneous Dirichlet boundary conditions for the fluid equations and Dirichlet or reflection conditions for the distribution function f. Naturally, the case of periodic boundary conditions could be treated similarly.

The existence of solutions for kinetic equations coupled with hydrodyamic equations has been studied before. In particular, the coupling of Vlasov-Fokker-Planck equation with Poisson equation (in that case u is an electric field) was investigated by J. Carrillo et al. [6], [3] in the case of Dirichlet and reflection boundary conditions, and by F. Bouchut [4] when the equation is set in the whole space $\Omega = \mathbb{R}^N$. Global existence results for the coupling of kinetic equations with incompressible Navier-Stokes equations was proved by Hamdache in [11]. The existence of solutions for short time in the case of the hyperbolic system (i.e. no viscosity in the Navier-Stokes equation ($\nu = 0$) and no Brownian effect in the kinetic equation) is proved by Baranger and Desvillettes in [1].

On the other hand, the main existence result for compressible Navier-Stokes equation is due to P.-L. Lions in [12]. It shows the existence of weak solutions under only physical assumptions for the initial data (finite mass and energy), and when $\gamma > 9/5$ (in dimension 3). This result was later improved by E. Feireisl [8] to include the case of all $\gamma > 3/2$. In our study, we will use many of the idea of P.-L. Lions (in particular to regularize the system), and one of the key point will be the compactness results for the density (in its improved form establised by E. Feireisl) which will allow us to pass to the limit in the regularized problem.

In the case of reflection boundary condition for the particles, one of the difficulty will be to deal with the lack of regularity of the trace of f along the boundary. This is a very classical problem, which has been addressed, in particular by Hamdache [10] and Cercignani et al. [7] for the Botlzmann equation, and by J. Carrillo [6] for the Vlasov-Poisson-Fokker-Planck system of equations. Our main reference on those issue will be the more recent work of S. Mischler (see [16] and [15]).

In the next section, we describe our result more precisely. In particular we specify the boundary conditions, make precise the notion of weak solutions, and state our result. The proof of the existence of weak solutions with Dirichlet boundary condition on the kinetic variable f is then detailed in Section 3. The last section deals with the case of reflection conditions.

2 Main results

2.1 Notion of weak solutions

The system under consideration is the following:

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v((u-v)f - \nabla_v f) = 0 \tag{3}$$

$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0 \tag{4}$$

$$\partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) + \nabla_x \rho^\gamma - \mu \Delta u - \lambda \nabla \mathrm{div} u = (j - nu) \tag{5}$$

where

$$n(x,t) = \int_{\mathbb{R}^3} f(x,v,t)\,dv\,, \qquad j(x,t) = \int_{\mathbb{R}^3} v\,f(x,v,t)\,dv.$$

The kinetic variable v (the velocity) lies in $\mathbb{R}^3$, while the space variable x lies in a bounded subset Ω of $\mathbb{R}^3$. We assume that the boundary $\partial\Omega$ is a smooth hypersurface and we consider homogeneous Dirichlet condition for the velocity field u of the fluid:

$$u(x,t) = 0 \qquad\qquad \forall x \in \partial\Omega,\ \forall t > 0\,.$$

To make precise the boundary conditions on the kinetic distribution function, we denote the traces of f by $\gamma^\pm f(x,v,t) = f|_{\Sigma^\pm}$, where

$$\Sigma^\pm = \{(x,v) \in \partial\Omega \times \mathbb{R}^3 \mid \pm v \cdot r(x) > 0\}.$$

We also introduce

$$L^p(\Sigma^\pm) = \left\{ g(x,v)\,;\ \left(\int_{\Sigma^\pm} |g(x,v)|^p |v \cdot r(x)|\, d\sigma(x)\, dv \right)^{1/p} < \infty \right\}$$

where $d\sigma(x)$ denotes the euclidean metric on $\partial\Omega$. We consider two types of boundary conditions: the case of Dirichlet boundary conditions, which read

$$\gamma^- f(x,v,t) = g(x,v) \qquad \forall (x,v) \in \Sigma^-, \tag{6}$$

and the case of reflection conditions, which can be written as

$$\gamma^- f(x,v,t) = \mathcal{B}(\gamma^+ f) \qquad \forall (x,v) \in \Sigma^-, \tag{7}$$

(the hypothesis on the boundary operator $\mathcal{B}$ will be discussed in the next section).

In this framework, we say that (f, ρ, u) is a weak solution of (3)-(5) on $[0, T]$ if

$$
\begin{aligned}
& f(x,v,t) \geq 0 \qquad \forall (x,v,t) \in \Omega \times \mathbb{R}^3 \times (0,T) \\
& f \in C([0,T]; L^1(\Omega \times \mathbb{R}^3)) \cap L^\infty(0,T; L^1 \cap L^\infty(\Omega \times \mathbb{R}^3)), \\
& |v|^2 f \in L^\infty(0,T; L^1(\Omega \times \mathbb{R}^3))
\end{aligned}
$$

and

$$
\begin{aligned}
& \rho(x,t) \geq 0 \qquad \forall (x,t) \in \Omega \times (0,T) \\
& \rho \in L^\infty(0,T; L^\gamma(\Omega)) \cap C([0,T]; L^1(\Omega)) \\
& u \in L^2(0,T; H^1_0(\Omega)), \quad \rho\, |u|^2 \in L^\infty(0,T; L^1(\Omega)) \\
& \rho\, u \in C([0,T]; L^{2\gamma/(\gamma+1)}(\Omega) - w)
\end{aligned}
$$

where (4)-(5) holds in the sense of distribution. In particular, we will see that the conditions on f yield $n(x,t) \in L^\infty(0,T; L^{6/5}(\Omega))$ which is enough to give a meaning to the product $n\, u$ in $L^1((0,T) \times \Omega)$.

In the case of Dirichlet boundary conditions (6), we also ask that

$$
\gamma^\pm f \in L^1(0,T; L^1(\Sigma^\pm)), \qquad \text{and} \qquad \gamma^-(f) = g
$$

and (3) hods in the sense of distribution, that is for any $\varphi \in \mathcal{C}^\infty(\overline{\Omega} \times \mathbb{R}^3 \times [0,T])$ we have

$$
\begin{aligned}
\int_0^T \int_{\Omega \times \mathbb{R}^N} f \big[\partial_t \varphi + v \cdot \nabla_x \varphi + (u - v) \cdot \nabla_v \varphi + \Delta_v \varphi \big] \, dx\, dv\, dt \\
+ \int_{\Omega \times \mathbb{R}^N} f_0 \varphi(x,v,0) \, dx\, dv + \int_0^T \int_\Sigma (v \cdot r(x)) \gamma f\, \varphi d\sigma(x) \, dv\, dt = 0. \quad (8)
\end{aligned}
$$

In the case of reflection boundary conditions, it is a well known fact that grazing collisions with the boundary of the domain are responsible for a loss of regularity of the traces of f. In general, we cannot expect to have $\gamma^\pm f$ in $L^1(0,T; L^1(\Sigma^\pm))$, and we only ask that (3) be satisfied in the following sense:

$$
\begin{aligned}
\int_0^T \int_{\Omega \times \mathbb{R}^N} f \big[\partial_t \varphi + v \cdot \nabla_x \varphi + (u - v) \cdot \nabla_v \varphi + \Delta_v \varphi \big] \, dx\, dv\, dt \\
+ \int_{\Omega \times \mathbb{R}^N} f_0 \varphi(x,v,0) \, dx\, dv = 0 \qquad (9)
\end{aligned}
$$

for any $\varphi \in \mathcal{C}^\infty(\overline{\Omega} \times \mathbb{R}^3 \times [0,T])$ such that $\varphi(\cdot, T) = 0$ and $\gamma^+ \varphi = \mathcal{B}^* \gamma^- \varphi$ on $\Sigma^+ \times [0,T]$.

Remark 2.1 *The weak formulation (9), together with the entropy inequality is all that is needed for the asymptotic result of [14] to apply.*

2.2 Reflection boundary conditions

We recall that Ω is a bounded subset of $\mathbb{R}^3$. Moreover, we will assume that $\partial\Omega$ is a smooth hypersurface. More precisely, we require that he following holds:

> $\partial\Omega$ is a C^1 manifold. Moreover, there exists a $W^{1,\infty}$ vector field $r(x)$ defined in $\overline{\Omega}$ which is equal to the outward unit normal vector to $\partial\Omega$ for all $x \in \partial\Omega$.

In its most general form, the reflection operator $\mathcal{B}$ can be written as

$$\mathcal{B}f(v) = \int_{v'\cdot n>0} B(t,x,v,v')f(v')\,|v'\cdot r(x)|dv'$$

The usual assumptions on the kernel B are the following:

(i) The operator $\mathcal{B}$ is non-negative (i.e. $B \geq 0$).

(ii) For every $v' \in \mathbb{R}^3$ such that $v'\cdot n > 0$, we have

$$\int_{v\cdot n<0} B(t,x,v,v')|v\cdot r(x)|\,dv = 1$$

(iii) If $M(v) = (2\pi)^{-3/2}\exp(-|v|^2/2)$ denote the Maxwellian distribution, we have

$$\int_{v'\cdot n>0} B(t,x,v,v')M(v')|v'\cdot r(x)|\,dv' = M(v).$$

Those three conditions are very classical in kinetic theory, and it is readily seen that (i)-(iii) implies that $\mathcal{B}$ is a bounded operator from $L^1(\Sigma^+)$ into $L^1(\Sigma^-)$, with

$$\|\mathcal{B}\|_{\mathcal{L}(L^1(\Sigma^+),L^1(\Sigma^-))} \leq 1.$$

However, we will also need to control the trace of f in $L^p(\Sigma)$ for $p > 1$, and to that purpose we need to assume the following condition:

(iv) The operator $\mathcal{B}$ is a bounded operator from $L^p(\Sigma^+)$ into $L^p(\Sigma^-)$ for all $p \in [1,\infty]$, with

$$\|\mathcal{B}\|_{\mathcal{L}(L^p(\Sigma^+),L^p(\Sigma^-))} \leq 1.$$

This additional condition is satisfied, for example, if the reflections are elastic, i.e. if we have

$$B(x,t,v,v') = b(x,t,v,v')\delta(|v|^2 - |v'|^2).$$

For example, we can take

$$\mathcal{B}(g) = \alpha J(g) + (1-\alpha)K(g)$$

with a local reflection operator J defined by

$$J(g)(x,v) = g(x,R_x v)$$

with $R_x v = -v$ (inverse reflection) or $R_x v = v - 2(v \cdot r(x))\, r(x)$ (specular reflexion), and the elactic diffusive operator given by

$$K(g)(x,v) = \frac{1}{4\pi|v|^2} \int_{S^+(x,v)} g(x,v') v' \cdot r \, dv' \, .$$

where $S^+(x,v) = \{v' \in \mathbb{R}^3\, ;\, v' \cdot r(x) > 0, |v'|^2 = |v|^2\}$.

Note that condition (ii) on B yields that if $\gamma\varphi(x,v) = g(x)$ is independent on v then $\gamma^+\varphi = \mathcal{B}^*\gamma^-\varphi$. So the weak formulation (9) holds in particular for test function φ indepedent of v.

Finally, hypothesis (i)-(iii) yield the following classical lemma which will be crucial in the sequel to make use of the entropy inequality:

Lemma 2.1 (Cercignani et al. [7]) *Let $\gamma f \geq 0$ satisfy $\gamma^- f = \mathcal{B}\gamma^+ f$ and assume that $(1 + |v|^2 + |\log \gamma f|)\gamma f \in L^1(\Sigma^\pm)$. Then we have*

$$\int_{\mathbb{R}^3} (v \cdot r(x)) \gamma f \, dv = 0$$

and

$$\int_{\mathbb{R}^3} (v \cdot r(x)) \left(\frac{|v|^2}{2} + \log(\gamma f) \right) \gamma f \, dv \geq 0$$

2.3 Entropy inequality

Before stating our main results, we briefly review the classical energy/entropy inequalities satisfied by smooth solutions of (3) and (4)-(5).

First of all, setting

$$\mathscr{E}_1(f) = \int_{\mathbb{R}^N} \left(\frac{|v|^2}{2} f + f \log f \right) \, dv,$$

it is readily seen that (multipying (3) by $\frac{|v|^2}{2} + \log f + 1$ and integrating with respect to x, v) smooth solutions of (3) satisfy:

$$\begin{aligned} \frac{d}{dt} \int_\Omega \mathscr{E}_1(f) \, dx &+ \int_\Omega \int_{\mathbb{R}^N} |(u - v) f - \nabla_v f|^2 \frac{1}{f} \, dv \, dx \\ &+ \int_{\partial\Omega \times \mathbb{R}^3} (v \cdot r(x)) \left(\frac{|v|^2}{2} + \log \gamma f + 1 \right) \gamma f \, d\sigma(x) \, dv \\ &= \int_\Omega \int_{\mathbb{R}^N} u(u - v) f \, dv \, dx. \end{aligned}$$

Next, defining

$$\mathscr{E}_2(\rho, u) = \rho \frac{|u|^2}{2} + \frac{1}{\gamma - 1} \rho^\gamma \, ,$$

it is a well-konwn fact that smooth solutions of (4)-(5) satisfy:

$$\frac{d}{dt}\int_\Omega \mathscr{E}_2(\rho,u)\,dx + \nu\int_\Omega |\nabla_x u|^2\,dx = \int_\Omega\int_{\mathbb{R}^N} u(u-v)f\,dv\,dx.$$

We deduce the following proposition:

Proposition 2.1 *The function*

$$\mathscr{E}(f,\rho,u) = \int_{\mathbb{R}^N}\left[\frac{|v|^2}{2}f + f\log f\right]\,dv + \rho\frac{|u|^2}{2} + \frac{1}{\gamma-1}\rho^\gamma$$

is an entropy for the system (3)-(5), with dissipation

$$D(f,u) = \int_\Omega\int_{\mathbb{R}^N}\left|(u-v)f - \nabla_v f\right|^2\frac{1}{f}\,dv\,dx + \nu\int_\Omega |\nabla_x u|^2\,dx.$$

More precisely, if (f,ρ,u) is a smooth solution of (3)-(5), then the following energy equality holds:

$$\begin{aligned}\int_\Omega\left[\mathscr{E}(f,\rho,u)(t)\right]dx + \int_0^t\int_{\Omega\times\mathbb{R}^N}|(u-v)f - \nabla_v f|^2\frac{1}{f}\,dv\,dx\,ds \\ +\nu\int_0^t\int_\Omega|\nabla_x u|^2\,dx\,ds = \int_\Omega \mathscr{E}(f_0,\rho_0,u_0)\,dx \\ -\int_0^t\int_{\partial\Omega\times\mathbb{R}^3}(v\cdot r(x))\left(\frac{|v|^2}{2} + \log\gamma f + 1\right)\gamma f\,d\sigma(x)\,dv\,ds.\end{aligned}$$

The entropy inequality will be crucial in deriving a priori estimates on the solutions (f,ρ,u) of the approximated system of equations. Note however that the entropy $\mathscr{E}_1$ may be negative (when $f\le 1$). We will thus make extensive use of the following classical Lemma [7]:

Lemma 2.2 *Assume that $f\ge 0$ and $|v|^2 f\in L^1(\Omega\times\mathbb{R}^3)$. Then*

$$\int_{\Omega\times\mathbb{R}^3} f\log^- f\,dx\,dv \le C(\varepsilon,\Omega) + \varepsilon\int |v|^2 f\,dx\,dv$$

for any $\varepsilon>0$. Similarly, if $|v|^2\gamma f\in L^1((0,T)\times\Sigma^\pm)$, then $\gamma f\log^-(\gamma f)\in L^1((0,T)\times\Sigma^\pm)$ and a similar estimate holds.

2.4 Main results

We can now state our main results. Throughout the paper, we will assume that the intial data have finite mass and energy. More precisely, we assume:

$$f_0(x,v)\ge 0 \quad \forall (x,v)\in\Omega\times\mathbb{R}^N, \qquad f_0\in L^1\cap L^\infty(\Omega\times\mathbb{R}^3) \tag{10}$$

and

$$\rho_0(x) \geq 0 \qquad \forall x \in \Omega \qquad\qquad \rho_0 \in L^1(\Omega) \tag{11}$$

$$\int_\Omega \mathscr{E}(f_0, \rho_0, u_0)\; dx < +\infty \tag{12}$$

In the case of Dirichlet boundary condition, we also assume that

$$\begin{cases} g(x,v,t) \geq 0 \qquad g \in L^1 \cap L^\infty(\Sigma^- \times (0,T)) \\ \displaystyle\int_0^T \int_{\Sigma^-} |v|^2\, g(x,v)\, |v \cdot r(x)|\, d\sigma(x)\, dv\, dt < \infty \end{cases} \tag{13}$$

Theorem 2.1 (Dirichlet boundary conditions) *Let f_0, ρ_0 and u_0 satisfy (10)-(12) and let g satisfies (13) for every $T > 0$. Assume that $\gamma > 3/2$ and $\nu > 0$. Then there exists a weak solution (f, ρ, u) of (3)-(5) satisfying (6) defined globally in time. Moreover, f satisfies the additional bounds for every $T > 0$:*

(i) $\gamma^+ f \in L^p(0,T; L^p(\Sigma^+))$ *for all* $p \in [0, \infty]$

(ii) $|v|^2 \gamma^+ f \in L^1(0,T; L^1(\Sigma^+))$.

and the following entropy inequality holds:

$$\int_\Omega \mathscr{E}(f(t), \rho(t), u(t))\, dx + \int_0^t D(f,u)\, ds + \nu \int_0^t \int_\Omega |\nabla u|^2\, dx\, ds$$
$$+ \int_0^t\int_{\partial\Omega\times\mathbb{R}^3} (v \cdot r(x)) \left(\frac{|v|^2}{2} + \log \gamma f + 1\right) \gamma f\, d\sigma(x) dv ds \leq \int_\Omega \mathscr{E}(f_0, \rho_0, u_0)\, dx \tag{14}$$

The proof of this result will be developed in Section 3. It relies on the introduction of a regularized system of equation (for which the existence of a solutions is given by a fixed point argument), and it makes use of a compactness result of P.L. Lions [12] and E. Feireisl [8] for weak solutions of compressible Navier-Stokes equation.

In the case of reflection boundary condition, we have the following result:

Theorem 2.2 (Reflection boundary conditions) *Let f_0, ρ_0 and u_0 satisfy (10)-(12) and assume that $\gamma > 3/2$. Then there exists a weak solution (f, ρ, u) of (3)-(5) satisfying (7) defined globally in time. Moreover the following entropy inequality holds:*

$$\int_\Omega \mathscr{E}(f(t), \rho(t), u(t))\, dx + \int_0^t D(f,u)\, ds + \nu \int_0^t \int_\Omega |\nabla u|^2\, dx\, ds$$
$$\leq \int_\Omega \mathscr{E}(f_0, \rho_0, u_0)\, dx \tag{15}$$

We recall that in this case, (3) holds in the sense of (9). In particular, we stress out the fact that we do not have γf in $L^1(0,T; L^1(\Sigma))$, and that we cannot write the weak formulation (8).

The proof of this second theorem relies on the first result and a fixed point argument on the trace. It will be the object of Section 4.

3 Proof of theorem 2.1: Dirichlet boundary conditions

The proof of Theorem 2.1 is divided in three steps. First, we introduce a regularized system of equations for which we prove the existence of solutions by a fixed point argument on the velocity field u. Then we investigate the properties of these solutions and show, in particular, that they satisfy an approximated entropy inequality. Finally, we pass to the limit on the regularization parameters and show that we obtain weak solutions of (3)-(5) satisfying the entropy inequality.

3.1 Regularization

In this section, we regularized the system (3)-(5) and construct a solution of the regularized system of equations. The main difficulty is to control the right hand side in (5), i.e. to control some L^p norm of the kinetic density $n(x,t)$ and current $j(x,t)$. To that purpsoe, we modify the kinetic equation by truncating the velocity field u: We consider

$$\partial_t f + v\cdot\nabla_x f + \mathrm{div}_v((\chi_\lambda(u) - v)f - \nabla_v f) = 0 \tag{16}$$

where

$$\chi_\lambda(u) = u\,1_{\{|u|\leq\lambda\}}.$$

We need to modify Navier-Stokes system of equations accordingly, in order to preserve the entropy inequality. This can be done by replacing the right hand side in (5) by $(j - n\,u)1_{\{|u|\leq\lambda\}}$. Finally, following P.-L. Lions, we regularize the transport term $\rho\partial_t + \rho u\cdot\nabla$ in the moment equation, leading to the following system of equations:

$$\partial_t\rho + \mathrm{div}_x(\rho u) = 0 \tag{17}$$

$$\partial_t(\rho_k u) + \mathrm{div}_x((\rho u)_k\otimes u) + \nabla_x\rho^\gamma - \mu\Delta u - \lambda\nabla\mathrm{div}u = (j - n\,u)1_{\{|u|\leq\lambda\}} \tag{18}$$

where the subscript k in (18) denotes the convolution with a mollifier $h_k(x)$ with respect to x:

$$\rho_k = \rho\star h_k \qquad (\rho u)_k = (\rho u)\star h_k,$$

Finally, we need to regularize the initial data (f_0,ρ_0,u_0) and boundary condition g. Here also, the main issue is to gain control on some norms of $n(x,t)$ and $j(x,t)$, and this is achieved by taking f_0 and g with finite moments of order higher than those given by the energy. More precisely, we assume that (f_0,ρ_0,u_0) satisfy (10)-(12), and

$$\int_{\Omega\times\mathbb{R}^3} |v|^m f_0(x,v)\,dx\,dv < +\infty \qquad \forall m\in[0,m_0],\ \text{with } m_0 > 5, \tag{19}$$

We also assume that g satisfies (13) and

$$\int_0^t \int_{\Sigma^-} |v|^m g\, |v\cdot r(x)|\, d\sigma(x)\, dv\, ds < \infty \qquad \forall m \in [0, m_0], \text{ with } m_0 > 5. \quad (20)$$

The main result of this section is the following proposition:

Proposition 3.1 *Let f_0, ρ_0 and u_0 satisfy (10)-(12) and (19). and let g satisfy (13) and (20). Then, for all k and $\lambda > 0$ and for all $T > 0$, the system of equations (16)-(18) has a weak solution (f, ρ, u) defined on $[0, T]$ with initial condition*

$$f|_{t=0} = f_0, \quad \rho|_{t=0} = \rho_0, \quad u|_{t=0} = u_0.$$

Moreover, the following inequalities hold

$$\sup_{t\in[0,T]} ||f(t)||_{L^p(\mathbb{R}^3\times\Omega)} \leq e^{\frac{3}{p'}T} \left(||f_0||_{L^p(\mathbb{R}^3\times\Omega)} + \|g\|_{L^p((0,T)\times\Sigma^-)}\right) \quad (21)$$

$$||\gamma^+ f||_{L^p((0,T)\times\Sigma^+)} \leq ||f_0||_{L^p(\mathbb{R}^3\times\Omega)} + \|g\|_{L^p((0,T)\times\Sigma^-)} \quad (22)$$

(note that those bounds are independent of λ).

The notion of weak solutions that we use for (16)-(18) is similar to that introduced in Section 2.1 for (3)-(5). In particular, we request that the weak formulation (8) holds (with $\chi_\lambda(u)$ instead of u) and that

$$\gamma^- f(x, v, t) = g(x, v, t) \qquad \forall (x, v, t) \in \Sigma^- \times (0, T).$$

The rest of this section is devoted to the proof of Proposition 3.1. It follows from a fixed point argument on the velocity field: For a given $\tilde{u}$, there exists a solution f to the Vlasov-Fokker-Planck equations (16). Setting $n = \int f\, dv$ and $j = \int v f\, dv$, we can then define (ρ, u) solution of (17)-(18) with the force term in (18) given by

$$(j - n\,\tilde{u})1_{\{|\tilde{u}|\leq\lambda\}}$$

The key point is thus to have enough regularity on n and j in order to make use of the result of P.-L. Lions [12] on compressible Navier-Stokes equations and get the existence of such a (ρ, u). A simple fixed point argument then yields the existence of a $\tilde{u}$ such that $\tilde{u} = u$ which in turn gives Proposition 3.1.

We thus start by taking

$$\tilde{u} \in L^2(0, T, L^2(\Omega)),$$

and we consider the following boundary value problem:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + \mathrm{div}_v((\chi_\lambda(\tilde{u}) - v)f - \nabla_v f) = 0 & \text{in } \Omega\times\mathbb{R}^3\times(0,T) \\ \gamma^- f(x, v, t) = g(x, v, t) & \text{in } \Sigma^-\times(0,T) \\ f(x, v, 0) = f_0(x, v) & \text{in } \Omega\times\mathbb{R}^3 \end{cases} \quad (23)$$

Thanks to the truncation, we have

$$\chi_\lambda(\tilde{u}) \in L^\infty(\Omega \times (0,T)).$$

It is thus a classical result (see J. Carrillo [6]) that the V-F-P equation (23) has a weak solution $f \geq 0$ as soon as f_0 satisfies (10) and (12) and g satisfies (13).

This solution satisfies, for all $T > 0$,

$$\begin{aligned}
&f \in L^\infty(0,T; L^1 \cap L^\infty(\Omega \times \mathbb{R}^3)),\\
&\nabla_v f \in L^\infty(0,T; L^2(\Omega \times \mathbb{R}^3))\\
&|v|^2 f \in L^\infty(0,T; L^1(\Omega \times \mathbb{R}^3))\\
&\gamma^+ f \in L^2((0,T) \times \Sigma^+).
\end{aligned}$$

and (23) holds in the sense of (8) with $\chi_\lambda(\tilde{u})$ instead of u. Moreover, we have the following bounds (see [6]):

Lemma 3.1 *For any f_0 satisfying (10), g satisfying (13), and for any $\tilde{u} \in L^2(0,T,L^2(\Omega))$, the solution $f(x,v,t)$ of the Vlasov-Fokker-Planck equation (23) satisfies inequalities (21) and (22).*

Moreover, for any positive m, there exists a constant C depending on λ, m such that

$$\begin{aligned}
\int_{\Omega\times\mathbb{R}^3} |v|^m f(x,v,t)\,dx\,dv \;\leq\; & C(\lambda,m)\int_{\Omega\times\mathbb{R}^3} |v|^m f_0(x,v)\,dx\,dv\\
&+C(\lambda,m)\int_0^t\int_{\Sigma^-} |v\cdot n||v|^m g(x,v,t)\,d\sigma(x)\,dv. \quad (24)
\end{aligned}$$

and similar bounds hold for $\int_0^t\int_{\Sigma^+} |v\cdot n||v|^m \gamma^+ f\,d\sigma(x)\,dv$.

We recall, for further references, that the L^p bounds are obtained by multiplying (16) by pf^{p-1}, which yields

$$\begin{aligned}
\frac{d}{dt}\int f^p\,dx\,dv + \int_\Sigma (v\cdot r(x)) f^p d\sigma(x)\,dv&\\
-\int 3(p-1) f^p\,dx\,dv + 4\int \frac{p-1}{p}|\nabla_v f^{p/2}|^2\,dx\,dv = 0& \qquad (25)
\end{aligned}$$

Thanks to (24) and Hypothesis (19) and (20), we are now able to control some L^p norms of the moments $n(x,t)$ and $j(x,t)$, by mean of the following classical lemma:

Lemma 3.2 *Assume that f satisfies*

$$\|f\|_{L^\infty([0,T]\times\mathbb{R}^3\times\Omega)} < M$$

and

$$\int\int |v|^m f(x,v,t)\,dx\,dv \leq M \qquad \forall t \in (0,T), \qquad \forall m \in [0,m_0].$$

Then there exists a constant $C(M)$ such that

$$\begin{array}{ll} ||n(t)||_{L^p(\Omega)} \le C(M), & \textit{for every } p \in [1, (m_0+3)/3) \\ ||j(t)||_{L^p(\Omega)} \le C(M), & \textit{for every } p \in [1, (m_0+3)/4) \end{array} \tag{26}$$

for all $t \in [0, T]$.

In particular, under Hypothesis (19)-(20), the kinetic density and current $n(x,t)$ and $j(x,t)$ are bounded in $L^\infty(0,T; L^2(\Omega))$ (by a constant depending on λ).

Proof. Let $p \in (1, \infty)$ and let q be such that $1/p + 1/q = 1$. Then, for all r, we have:

$$\begin{aligned} n(x,t) &= \int (1+|v|)^{r/p} f^{1/p}(v) \frac{f^{1/q}}{(1+|v|)^{r/p}}\, dv \\ &\le \left(\int (1+|v|)^r f(v)\, dv \right)^{1/p} \left(\int \frac{f(v)}{(1+|v|)^{rq/p}}\, dv \right)^{1/q} \end{aligned}$$

In particular, if $rq/p > 3$, we deduce

$$n(x,t) \le C ||f(t)||_{L^\infty}^{1/q} \left(\int (1+|v|)^r f(v)\, dv \right)^{1/p}$$

and so

$$||n(t)||_{L^p}^p = \int n(x,t)^p\, dx \le C \int \int (1+|v|)^r f(v)\, dv\, dx.$$

Finally, note that the condition $rq/p > 3$ is equivalent to

$$p < \frac{r+3}{3},$$

A similar argument holds for the current:

$$\begin{aligned} j(x,t) &\le \int (1+|v|)^{r/p} f^{1/p}(v) \frac{f^{1/q}}{(1+|v|)^{r/p-1}}\, dv \\ &\le \left(\int (1+|v|)^r f(v)\, dv \right)^{1/p} \left(\int \frac{f(v)}{(1+|v|)^{rq/p-q}}\, dv \right)^{1/q} \end{aligned}$$

In particular, if $rq/p - q > 3$, we deduce

$$j(x,t) \le C ||f(t)||_{L^\infty}^{1/q} \left(\int (1+|v|)^r f(v)\, dv \right)^{1/p}$$

and so

$$||j(t)||_{L^p}^p = \int n(x,t)^p\, dx \le C \int \int (1+|v|)^r f(v)\, dv\, dx,$$

and the condition $rq/p - q > 3$ is equivalent to

$$p < \frac{r+3}{4}.$$

$\square$

Next, we consider the regularized Navier-Stokes system of equations with force term $(j - n\tilde{u})1_{\{|\tilde{u}|\leq\lambda\}}$:

$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0 \tag{27}$$

$$\partial_t(\rho_k u) + \mathrm{div}_x((\rho u)_k \otimes u) + \nabla_x \rho^\gamma - \mu\Delta u - \lambda\nabla\mathrm{div} u = (j - n\tilde{u})1_{\{|\tilde{u}|\leq\lambda\}} \tag{28}$$

with the initial condition

$$\rho(x,0) = \rho_0(x) \qquad u(x,0) = u_0(x).$$

We note that the right hand side is bounded in $L^\infty(0,T;L^2(\Omega))$:

$$||(j - n\tilde{u})1_{\{|\tilde{u}|\leq\lambda\}}||_{L^2(\Omega)} \leq ||j||_{L^2(\Omega)} + \lambda||n||_{L^2(\Omega)},$$

so the existence of a weak solution (ρ, u) of the system (27)-(28) can be proved as in P.-L. Lions [12]. More precisely, we can prove:

Lemma 3.3 *The system of equations (27)-(28) has a weak solution (ρ, u) satisfying the following entropy inequality:*

$$\frac{d}{dt}\int \rho_k \frac{u^2}{2} + \frac{1}{1-\gamma}\rho^\gamma \, dx + \nu \int |\nabla u|^2 = \int (j - n\tilde{u})1_{\{|\tilde{u}|\leq\lambda\}} u dx. \tag{29}$$

We can therefore introduce the operator

$$\begin{array}{rcl} \mathcal{T} : L^2((0,T)\times\Omega) & \to & L^2((0,T)\times\Omega) \\ \tilde{u} & \mapsto & u \end{array}$$

and Proposition 3.1 follows if we can prove the existence of a fixed point for $\mathcal{T}$. This will be a consequence of the following lemma:

Lemma 3.4 *There exists a constant $C(k, \lambda, T)$ such that*

$$||\mathcal{T}\tilde{u}||_{L^2((0,T),H^1(\Omega))} \leq C$$

and

$$||\partial_t \mathcal{T}\tilde{u}||_{L^2((0,T),H^{-1}(\Omega))} \leq C$$

From this lemma, it follows that the operator $\mathcal{T}$ is compact in $L^2((0,T)\times\Omega)$ and that the image of $\mathcal{T}$ is bounded in $L^2((0,T)\times\Omega)$. By Schauder's fixed point theorem, we deduce that the operator $\mathcal{T}$ has a fixed point in $L^2((0,T)\times\Omega)$. The corresponding (f, ρ, u) furnishes a solution of (16)-(18), which concludes the proof of Proposition 3.1.

Proof of Lemma 3.4. Let $u = \mathcal{T}\tilde{u}$. Using (26) and Sobolev's inequalities we have:

$$\begin{aligned} \left|\int (j - n\tilde{u})1_{\{|\tilde{u}|\leq\lambda\}} u \, dx\right| &\leq ||j||_{L^2}||u||_{L^2} + \lambda||n||_{L^2}||u||_{L^2} \\ &\leq (||j||_{L^2} + \lambda||n||_{L^2})^2 + ||u||^2_{L^2} \end{aligned}$$

Hence (29) gives

$$\frac{d}{dt}\int \rho_k \frac{u^2}{2} + \frac{1}{1-\gamma}\rho^\gamma\,dx + \nu \int |\nabla u|^2\,dx$$
$$\leq ||u||_{L^2}^2 + C. \tag{30}$$

Next, we note that since Ω is bounded and $\int_\Omega \rho(t,x)\,dx = \int_\Omega \rho_o(x)\,dx > 0$, we have $\rho_k(t,x) = \rho \star h_k \geq c_k > 0$ for all x, t. In particular, it follows that

$$||u||_{L^2}^2 \leq C_k \int \rho_k \frac{u^2}{2}\,dx.$$

Inequality (30) thus becomes

$$\frac{d}{dt}\int \rho_k \frac{u^2}{2} + \frac{1}{1-\gamma}\rho^\gamma\,dx + \nu \int |\nabla u|^2\,dx$$
$$\leq C(k)\int \rho_k \frac{u^2}{2}\,dx + C,$$

and Gronwall lemma gives $||u||^2_{L^\infty(0,T;L^2(\Omega))} \leq C(k)$, which gives the first estimate in Lemma 3.4.

To deduce the second estimate, we note that (27) yields

$$\partial_t \rho_k + \mathrm{div}_x(\rho u)_k = 0$$

so that (28) can be rewritten as

$$\rho_k \partial_t u + (\rho u)_k \cdot \nabla u + \nabla_x \rho^\gamma - \nu \Delta u = \frac{1}{\varepsilon}(j - n\tilde{u})1_{\{|\tilde{u}|\leq\lambda\}}.$$

The result follows using once again the fact that $\rho_k \geq c_k > 0$. $\square$

3.2 Approximated entropy inequality

In order to obtain further estimates on the solution (f,ρ,u) of the approximated system (16)-(18), we prove that it satisfies an approximated entropy inequality.

We will use the following notations for the approximated entropy functions:

$$\mathscr{E}_k = \mathscr{E}_1 + \mathscr{E}_{2,k}$$

with

$$\mathscr{E}_1(f) = \int \left[\frac{|v|^2}{2} f + f\log f\right] dv$$

and

$$\mathscr{E}_{2,k}(\rho,u) = \rho_k \frac{|u|^2}{2} + \frac{1}{\gamma-1}\rho^\gamma.$$

We also introduce the approximated dissipation

$$D_\lambda(f,u) = \int\int |(\chi_\lambda(u) - v)f - \nabla f|^2 \frac{1}{f}\, dx\, dv.$$

Then, we have the following result

Proposition 3.2 *The weak solution (f, ρ, u) of (16)-(18) given by Proposition 3.1 satisfies the following entropy inequality:*

$$\int \mathscr{E}_k(f(t), \rho(t), u(t))\, dx + \int_0^t D_\lambda(f(s), u(s))\, ds + \nu \int_0^t \int_\Omega |\nabla u|^2\, dx\, ds \\ \leq \int \mathscr{E}_k(f_0, \rho_0, u_0)\, dx - \int_0^t \int_\Sigma (v \cdot r) \left(\frac{|v|^2}{2} + \log \gamma f + 1 \right) \gamma f\, d\sigma(x)\, dv\, ds. \quad (31)$$

Proof. J. Carrillo proves in [6] that (since $\chi_\lambda(u) \in L^\infty((0,T) \times \mathbb{R}^3)$) the weak solution f of (23) given by Lemma 3.1 satisfies:

$$\frac{d}{dt} \int \left[\frac{|v|^2}{2} f + f \log f \right] dv\, dx + \int\int |(\chi_\lambda(u) - v)f - \nabla f|^2 \frac{1}{f}\, dx\, dv \\ = \int\int \chi_\lambda(u)(\chi_\lambda(u) - v) f\, dx\, dv \\ - \int_0^t \int_\Sigma (v \cdot r) \left(\frac{|v|^2}{2} + \log \gamma f + 1 \right) \gamma f\, d\sigma(x)\, dv\, ds.$$

(formally, this equality is obtained by multiplying (23) by $|v|^2/2 + \log f$).

Thus, using Lemma 3.3 together with the fact that

$$\begin{aligned} \int (j - nu) 1_{\{|u| \leq \lambda\}} u\, dx &= \int\int (v - u) f 1_{\{|u| \leq \lambda\}} u\, dx \\ &= \int\int \chi_\lambda(u)(\chi_\lambda(u) - v) f\, dx\, dv. \end{aligned}$$

we easily deduce (31). □

3.3 Proof of Theorem 2.1

We now use the results of the previous sections to construct a weak solutions of (3)-(5). We assume that the initial data (f_0, ρ_0, u_0) satisfy (10)-(12) and that the boundary data g satisfy (13). In order to use the result of the previous section, we thus consider approximating sequences (f_0^n, ρ_0^n, u_0^n) and g^n satisfying (10)-(13) uniformly with respect to n and such that for every n, (19)-(20) hold.

For every λ, k, n, we denote by (f_n, ρ_n, u_n) the solutions of (16)-(18) with initial data

$$f|_{t=0} = f_0^n, \quad \rho|_{t=0} = \rho_0^n, \quad u|_{t=0} = u_0^n$$

(we do not keep track of the subscripts k and λ for the sake of clarity).

A priori estimates.
Lemma 3.1 yields the existence of a constant C independent on λ, k and n such that

$$\begin{aligned}\|f_n\|_{L^\infty(0,T;L^p(\mathbb{R}^3\times\Omega)} &\le C\\ \|\gamma f_n\|_{L^\infty(0,T;L^p(\Sigma)} &\le C\end{aligned}$$

for all $p\in[1,\infty]$.

Moreover, using Lemma 2.2, we have

$$\begin{aligned}
&-\int_0^t\int_\Sigma (v\cdot r(x))\left(\frac{|v|^2}{2}+\log\gamma f+1\right)\gamma f\,d\sigma(x)\,dv\,ds\\
&\quad\le -\int_0^t\int_{\Sigma^+}|v\cdot r(x)|\left(\frac{|v|^2}{2}+\log^+(\gamma^+f)+1\right)\gamma^+f\,d\sigma(x)\,dv\,ds\\
&\qquad+\int_0^t\int_{\Sigma^+}|v\cdot r(x)|\log^-(\gamma^+f)\gamma^+f\,d\sigma(x)\,dv\,ds\\
&\qquad+\int_0^t\int_{\Sigma^-}|v\cdot r(x)|\left(\frac{|v|^2}{2}+\log g+1\right)g\,d\sigma(x)\,dv\,ds\\
&\quad\le -\frac12\int_0^t\int_{\Sigma^+}|v\cdot r(x)|\left(\frac{|v|^2}{2}+\log^+(\gamma^+f)+1\right)\gamma^+f\,d\sigma(x)\,dv\,ds\\
&\qquad+\int_0^t\int_{\Sigma^-}|v\cdot r(x)|\left(\frac{|v|^2}{2}+\log g+1\right)g\,d\sigma(x)\,dv\,ds+C
\end{aligned}$$

and proceeding similarly with $\mathscr{E}(f,\rho,u)$, Proposition 3.2 gives

$$\begin{aligned}
&\int\left(\frac{|v|^2}{4}+|\log f|\right)f\,dv\,dx+\int\rho_k\frac{|u|^2}{2}+\frac{1}{\gamma-1}\rho^\gamma\,dx\\
&+\frac12\int_0^t\int_{\Sigma^+}|v\cdot r(x)|\left(\frac{|v|^2}{2}+\log^+(\gamma^+f)+1\right)\gamma^+f\,d\sigma(x)\,dv\,ds\\
&+\int_0^t D_\lambda(f,u)\,ds+\nu\int_0^t\int_\Omega|\nabla u|^2\,dx\,ds\\
&\qquad\le\int\mathscr{E}_k(f_0,\rho_0,u_0)\,dx\\
&\qquad\qquad+\int_0^t\int_{\Sigma^-}|v\cdot r(x)|\left(\frac{|v|^2}{2}+\log g+1\right)g\,d\sigma(x)\,dv\,ds+C.
\end{aligned}\tag{32}$$

We deduce that there exists a constant C independent on λ, k and n such that

$$\begin{aligned}
\int_{\Omega\times\mathbb{R}^3}(1+|v|^2)f_n(x,v,t)\,dx\,dv &\le C\quad\forall t\in[0,T]\\
\int_0^T\int_{\Sigma^+}(1+|v|^2)\gamma^+f_n(x,v,t)\,|v\cdot r(x)|\,d\sigma(x)\,dv\,ds &\le C,
\end{aligned}$$

and

$$\begin{aligned}
&\|\rho_n\|_{L^\infty(0,T;L^1\cap L^\gamma(\Omega))} \leq C\\
&\|\sqrt{\rho_n}u_n\|_{L^\infty(0,T;L^2(\Omega))} \leq C\\
&||\nabla u_n||_{L^2((0,T)\times\Omega)} \leq C.
\end{aligned}$$

This implies in particular the following result (using Lemma 3.2 with $m_0 = 2$):

Lemma 3.5 *There exists a constant C independent on λ, k and n such that*

$$||n_n(t)||_{L^p} \leq K \quad \forall p < \frac{5}{3}$$

and

$$||j_n(t)||_{L^p} \leq K \quad \forall p < \frac{5}{4}$$

Limits.
We now explain how to take the limit as λ, k and n go to infinity. In order to keep things simple, we will keep the notation (f_n, ρ_n, u_n) for the solutions constructed in the previous section, being understood that the limit with respect to λ and k are treated in a similar way.

First of all, the a priori estimates gives the existence of a function f such that

$$f_n \rightharpoonup f \qquad L^\infty(0,T : L^p(\Omega\times\mathbb{R}^3)) - \text{weak}$$

for all $p \in (1,\infty)$. Moreover, for any $\varphi(x)$ smooth, compactly supported test function, we have

$$\begin{aligned}
&\int\left(j_n - \int vf\,dv\right)\varphi(x)\,dx\\
&\leq \left(\iint (f_n - f)(1+|v|^2)\varphi(x)dxdv\right)^{\frac{2}{3}}\left(\iint (f_n - f)\frac{\varphi(x)}{1+|v|}dxdv\right)^{\frac{1}{3}}\\
&\leq C\left(\iint (f_n - f)\frac{\varphi(x)}{1+|v|}\,dx\,dv\right)^{1/3}
\end{aligned}$$

which goes to zero as n goes to infinity since $\frac{\varphi(x)}{1+|v|}$ lies in $L^4(\Omega\times\mathbb{R}^3)$ and f_n converges weakly to f in L^p for all $p > 1$. It follows that

$$\begin{aligned}
&j_n \rightharpoonup j \qquad L^\infty(0,T : L^p(\Omega\times\mathbb{R}^3)) - \text{weak} \quad \forall p \in (1, 5/4)\\
&n_n \rightharpoonup n \qquad L^\infty(0,T : L^p(\Omega\times\mathbb{R}^3)) - \text{weak} \quad \forall p \in (1, 5/3)
\end{aligned}$$

with $j = \int vf\,dv$ and $n = \int f\,dv$.

Next, since Ω is bounded and u satisfies homogeneous Dirichlet boundary conditions, Poincaré inequality yields

$$||u||_{L^6(\Omega)} \leq C||u||_{H^1(\Omega)} \leq ||\nabla u||_{L^2},$$

so that u_n is bounded in $L^2(0,T;L^6(\Omega))$. We then have

$$\|n_n u_n\|_{L^2(0,T;L^{6/5}(\Omega))} \leq \|u_n\|_{L^2(0,T;L^6(\Omega))}\|n_n\|_{L^2(0,T;L^{3/2}(\Omega))}$$

where $3/2 < 5/3$. In particular, the right hand side in (18), $j_n - n_n u_n$, is bounded in $L^2(0,T;L^{6/5}(\Omega))$. This is the crucial bound that allows us to proceed as in P.-L. Lions [12] and E. Feireisl [8] to prove the stability of weak solution for compressible Navier-Stokes equations. This leads to the following convergences:

$$\begin{aligned}
&\rho_n \to \rho \text{ in } L^1((0,T)\times\Omega) \text{ and } \mathcal{C}([0,T];L^\gamma_{weak}(\Omega))\\
&u_n \to u \text{ weakly in } L^2(0,T;[W_0^{1,2}(\Omega)]^3)\\
&\rho_n u_n \to \rho u \text{ in } \mathcal{C}([0,T];L^{2\gamma/(\gamma+1)}_{weak}(\Omega))
\end{aligned}$$

Moreover, (see Lions [12]), we have

$$(\rho_n \star h_k)u_n \rightharpoonup \rho u, \quad \text{ and } \quad {\rho_n}^\gamma \rightharpoonup {\rho_n}^\gamma.$$

Thus, in order to show that (f,ρ,u) is a weak solution of (3)-(5), it only remains to show that we can pass to the limit in the coupling terms $\chi_\lambda(u_n)f_n$ in (16) and $n_n u_n 1_{\{|u_n|<\lambda\}} = n_n\chi_\lambda(u_n)$ in (18) and in the boundary conditions.

For that purpose, we write

$$n_n u_n 1_{\{|u_n|<\lambda\}} = n_n u_n - n_n u_n 1_{\{|u_n|>\lambda\}},$$

where the second term can be bounded as follows:

$$\begin{aligned}
&\|n_n u_n 1_{\{|u_n|>\lambda\}}\|_{L^1((0,T)\times\Omega)}\\
&\qquad \leq ||n_n||_{L^\infty(L^{3/2})}||u_n||_{L^2(L^6)}||1_{\{|u_n|>\lambda\}}||_{L^2(L^6)}\\
&\qquad \leq ||n_n||_{L^\infty(L^{3/2})}||u_n||_{L^2(L^6)}\left(\frac{||u_n||_{L^2(L^6)}}{\lambda}\right)\\
&\qquad \leq \frac{C}{\lambda}.
\end{aligned}$$

Thus, we only have to show that $n_n u_n$ converges in the sense of distribution to nu. We recall that n_n converges in $L^2(0,T;L^{3/2}(\Omega))$-weak and u_n converges in $L^2(0,T;L^3(\Omega))$-weak. Moreover, integrating (16) with respect to v, we find $\partial_t n_n = -\mathrm{div}_x j_n$, and so

$$\partial_t n_n \quad \text{is bounded in } L^2(0,T;W^{-1,1}(\Omega)).$$

Since $\nabla_x u_n$ is bounded in $L^2((0,T)\times\Omega)$, we can make use of a classical result (see P.-L. Lions [12] for details) to deduce

$$n_n u_n \longrightarrow nu \text{ in the sense of distribution.}$$

Therefore

$$n_n \chi_\lambda(u_n) \longrightarrow nu \text{ in the sense of distribution}$$

when λ, k and n go to infinity.

We proceed similarly with the $\chi_\lambda(u_n) f_n$: we only have to show that for any test functions $\varphi(v)$, we have

$$u_n \int f_n \varphi(v)\, dv \rightharpoonup u \int f \varphi(v)\, dv \quad \text{in the sense of distribution.}$$

We note that $\int f_n \varphi(v)\, dv$ is bounded in $L^\infty(0,T; L^p(\Omega))$ for all p. Moreover, multiplying (16) by $\varphi(v)$ and integrating with respect to v (and using the fact that $\nabla_v f_n$ is bounded in $L^2(0,T; L^2(\Omega \times \mathbb{R}^3))$), it is readily seen that

$$\int f_n \varphi(v)\, dv \quad \text{is bounded in } L^2(0,T; W^{-1,1}(\Omega)).$$

We can thus conclude as before.

Finally, we show how to handle the boundary condition for the kinetic equations: Since $\gamma^\pm f_n$ is bounded in $L^p(0,T; L^p(\Sigma^\pm))$, it converges weakly to some $h^\pm \in L^p(0,T; L^p(\Sigma^\pm))$ satisfying $h^- = g$. Passing to the limit in (8), we get:

$$\begin{aligned}\int_0^T \int_{\Omega\times\mathbb{R}^N} & f\big[\partial_t \varphi + v\cdot\nabla_x \varphi + (u-v)\cdot\nabla_v\varphi + \Delta_v \varphi\big]\, dx\, dv\, dt \\ & + \int_{\Omega\times\mathbb{R}^N} f_0 \varphi(x,v,0)\, dx\, dv + \int_0^T \int_\Sigma (v\cdot r(x)) h\, \varphi d\sigma(x)\, dv\, dt = 0. \quad (33)\end{aligned}$$

Using Carrillo [6] (see also Mischler [15]), we can now prove that f has trace γf in $L^1(0,T; L^1(\Sigma))$ and that it satisfies the Green formula. Equality (33) thus yields $h = \gamma f$ and f satisfies (8).

Entropy inequality:
So we have proved that as λ, n and k go to infinity, $(f_{\lambda,k,n}, \rho_{\lambda,k,n}, u_{\lambda,k,n})$ converge to a weak solution (f,ρ,u) of (3)-(5) with boundary conditions (6). In order to complete the proof of Theorem 2.1, it only remains to check that (f,ρ,u) satisfied the appropriate entropy inequality, but this is a direct consequence of (31). As a matter of fact, taking the limit in (31) and using the convexity of the entropy and the weak convergence of f_n, $f_n \star h_k$, ρ_n, u_n and γf_n, we deduce:

$$\begin{aligned}\int \mathscr{E}(f(t),\rho(t),u(t))\, dx &+ \int_0^t D(f(s),u(s))\, ds + \nu \int_0^t \int_\Omega |\nabla u|^2\, dx\, ds \\ &+ \int_0^t \int_{\Sigma^+} |v\cdot r| \left(\frac{|v|^2}{2} + \log \gamma^+ f + 1\right) \gamma^+ f\, d\sigma(x)\, dv\, ds \\ &\le \int \mathscr{E}(f_0,\rho_0,u_0)\, dx + \int_0^t \int_{\Sigma^-} |v\cdot r| \left(\frac{|v|^2}{2} + \log g + 1\right) g\, d\sigma(x)\, dv\, ds. (34)\end{aligned}$$

4 Proof of Theorem 2.2: Reflection Boundary conditions

In the section, we detail the proof of the existence result in the case of reflection boundary conditions. The first idea is to do a fixed point argument on the trace. Since $\|\mathcal{B}\|_{\mathcal{L}(L^1,L^1)} = 1$, we introduce an $\varepsilon \in (0,1)$, and we first construct a solution of (3)-(5) with boundary condition

$$\gamma^- f = (1-\varepsilon)\,\mathcal{B}\gamma^+ f \qquad \forall (x,v,t) \in \Sigma^- \times (0,T) \tag{35}$$

Such a solution is constructed by an iterative argument: Using Theorem 2.1, we construct a sequence (f^n, ρ^n, u^n) of solution of (3)-(5) with Dirichlet boundary condition

$$\gamma^- f^{n+1} = (1-\varepsilon)\,\mathcal{B}\gamma^+ f^n \qquad \forall (x,v,t) \in \Sigma^- \times (0,T)$$

(we can take, for example, $\gamma^+ f^0 = 0$ to initiate the sequence). Then, we note that for every n, we have (using (22)):

$$||\gamma^+ f^{n+1}||_{L^p(0,T;L^p(\Sigma^+))} \le ||f_0||_{L^p} + (1-\varepsilon)||\gamma^+ f^n||_{L^p(0,T;L^p(\Sigma^+))}$$

for all $p \in [1,\infty]$. Morever, Lemma 2.1 yields

$$\begin{aligned}
&\int_0^T \int_{\Sigma^-} |v\cdot r| \left(\frac{|v|^2}{2} + \log(\gamma^- f^{n+1}) + 1\right) \gamma^- f^{n+1}\, d\sigma(x)\, dv\, dt \\
&\qquad \le (1-\varepsilon) \int_0^T \int_{\Sigma^+} |v\cdot r| \left(\frac{|v|^2}{2} + \log(\gamma^+ f^{n}) + 1\right) \gamma^+ f^{n}\, d\sigma(x)\, dv\, dt.
\end{aligned}$$

and so, using (34), we get:

$$\begin{aligned}
&\int \mathscr{E}(f^{n+1}(t), \rho^{n+1}(t), u^{n+1}(t))\, dx \\
&\quad + \int_0^t D(f^{n+1}, u^{n+1})\, ds + \nu \int_0^t \int_\Omega |\nabla u^{n+1}|^2\, dx\, ds \\
&\quad + \int_0^t \int_{\Sigma^+} |v\cdot r| \left(\frac{|v|^2}{2} + \log(\gamma^+ f^{n+1}) + 1\right) \gamma^+ f^{n+1}\, d\sigma(x)\, dv\, ds \\
&\qquad \le \int \mathscr{E}(f_0, \rho_0, u_0)\, dx \\
&\qquad\quad + (1-\varepsilon) \int_0^t \int_{\Sigma^+} |v\cdot r| \left(\frac{|v|^2}{2} + \log(\gamma^+ f^{n}) + 1\right) \gamma^+ f^{n}\, d\sigma(x)\, dv\, ds.
\end{aligned} \tag{36}$$

Since $\varepsilon > 0$, we deduce (by iterating those estimates):

$$||\gamma^+ f^{n+1}||_{L^p(0,T;L^p(\Sigma^+))} \le \frac{1}{\varepsilon}||f_0||_{L^p} + (1-\varepsilon)^n||\gamma^+ f^1||_{L^p(0,T;L^p(\Sigma^+))}.$$

for all $p \in [1,\infty]$, and

$$\begin{aligned}
&\int_0^T\int_{\Sigma^+} |v\cdot r| \left(\frac{|v|^2}{2} + \log(\gamma^+ f^{n+1}) + 1\right) \gamma^+ f^{n+1}\, d\sigma(x)\, dv\, ds \\
&\quad\leq \frac{1}{\varepsilon}\int \mathscr{E}(f_0,\rho_0,u_0)\,dx \\
&\qquad + (1-\varepsilon)^n \int_0^T\int_{\Sigma^+} |v\cdot r| \left(\frac{|v|^2}{2} + \log(\gamma^+ f^{1}) + 1\right) \gamma^+ f^{1}\, d\sigma(x)\, dv\, ds.
\end{aligned}$$

We thus have all the necessary estimates to proceed as in the previous section and take the limit $n\to\infty$. we deduce the following proposition:

Proposition 4.1 *Assume that (f_0,ρ_0,u_0) satisfies (10)-(12), then for every $\varepsilon>0$, there exist a weak solution $(f_\varepsilon,\rho_\varepsilon,u_\varepsilon)$ of (3)-(5) with boundary value γf_ε in $L^p(0,T;L^p(\Sigma^+))$ satisfying*

$$\gamma^- f_\varepsilon = (1-\varepsilon)\,\mathcal{B}\gamma^+ f_\varepsilon \qquad \forall (x,v,t)\in\Sigma^-\times(0,T).$$

Moreover, the following entropy inequality holds (by taking the limit $n\to\infty$ is (36)):

$$\begin{aligned}
&\int \mathscr{E}(f_\varepsilon(t),\rho_\varepsilon(t),u_\varepsilon(t))\,dx + \int_0^t D(f_\varepsilon(s),u_\varepsilon(s))\,ds + \nu\int_0^t\int_\Omega |\nabla u_\varepsilon|^2\,dx\,ds \\
&+\varepsilon\int_0^t\int_{\Sigma^+} |v\cdot r| \left(\frac{|v|^2}{2} + \log(\gamma^+ f_\varepsilon) + 1\right)\gamma^+ f_\varepsilon\, d\sigma(x)\,dv\,ds \leq \int \mathscr{E}(f_0,\rho_0,u_0)\,dx.
\end{aligned}$$

Limit $\varepsilon\to 0$.
In order to prove Theorem 2.2, it only remains to show take the limit $\varepsilon\to 0$. Using (25) and (35) we see that the solution $(f_\varepsilon,\rho_\varepsilon,u_\varepsilon)$ given by Proposition 4.1 satisfies

$$\frac{d}{dt}\int f_\varepsilon^p\,dx\,dv + (1-(1-\varepsilon)^p)\int_{\Sigma^+} |v\cdot r(x)|\left|\gamma^+ f_\varepsilon\right|^p\,d\sigma(x)\,dv \leq 3(p-1)\int f_\varepsilon^p\,dx\,dv\,.$$

In particular, f_ε is bounded in L^p for all $p\in[1,\infty)$ (and $\varepsilon\|\gamma^+ f_\varepsilon\|_{L^1(0,T;L^1(\Sigma^+))}$ is bounded).

Next, we note that Lemma 2.2 yields:

$$\int_{\Sigma^+} |v\cdot r|\log^-(\gamma^+ f_\varepsilon)\gamma^+ f_\varepsilon\,d\sigma(x)\,dv\,ds \leq \int_{\Sigma^+} |v\cdot r|\frac{|v|^2}{2}\gamma^+ f_\varepsilon\,d\sigma(x)\,dv + C$$

and thus

$$\int_0^t\int_{\Sigma^+} |v\cdot r|\left(\frac{|v|^2}{2} + \log\gamma^+ f_\varepsilon + 1\right)\gamma^+ f_\varepsilon\,d\sigma(x)\,dv\,ds \geq -C$$

We deduce

$$\int \mathscr{E}(f_\varepsilon(t), \rho_\varepsilon(t), u_\varepsilon(t))\, dx + \int_0^t D(f_\varepsilon, u_\varepsilon)\, ds + \nu \int_0^t \int_\Omega |\nabla u_\varepsilon|^2\, dx\, ds$$
$$\leq \int \mathscr{E}(f_0, \rho_0, u_0)\, dx + C\varepsilon. \tag{37}$$

In particular $\int \mathscr{E}(f_\varepsilon(t), \rho_\varepsilon(t), u_\varepsilon(t))\, dx$ is bounded uniformly with respect to ε, and so $(f_\varepsilon, \rho_\varepsilon, u_\varepsilon)$ satisfies the same a priori estimate as in Section 3.3 uniformly with respect to ε, except for the estimate on the trace of f_ε. In particular, we can proceed as in Section 3.3 to take the limit in the fluid equations (4)-(5), and the entropy inequality (15) follows directly from (37), using the convexity of $\mathscr{E}$.

So, we are left with the task of passing to the limit in the weak formulation of (3) in order to get (9). For all $\varphi \in C_c^\infty((0,T) \times \Omega \times \mathbb{R}^3)$, (33) gives:

$$\int_0^T \int_{\Omega \times \mathbb{R}^N} f_\varepsilon \big[\partial_t \varphi + v \cdot \nabla_x \varphi + (u - v) \cdot \nabla_v \varphi + \Delta_v \varphi\big]\, dx\, dv\, dt$$
$$+ \int_{\Omega \times \mathbb{R}^N} f_0 \varphi(x, v, 0)\, dx\, dv + \int_0^T \int_\Sigma (v \cdot r(x)) h_\varepsilon\, \varphi d\sigma(x)\, dv\, dt = 0. \tag{38}$$

Moreover, h^ε is a non-negative function such that $h_\varepsilon^- = (1 - \varepsilon)\mathcal{B} h_\varepsilon^+$, so if we take φ such that

$$\gamma^+ \varphi = \mathcal{B}^* \gamma^- \varphi$$

we get

$$\left| \int_0^T \int_\Sigma (v \cdot r) h_\varepsilon\, \varphi\, d\sigma(x)\, dv\, dt \right| \quad = \quad \varepsilon \left| \int_0^T \int_{\Sigma^+} (v \cdot r) \gamma^+ h_\varepsilon\, \gamma^+ \varphi\, d\sigma(x)\, dv\, dt \right|.$$

If we had $\gamma^+ h_\varepsilon$ bounded in $L^1(0,T; L^1(\Sigma^+))$, it would be easy to deduce that this last term goes to zero when $\varepsilon \to 0$. However, this is not true for general reflection condition because of the lack of regularity due to grazing collisions (this is a very classical difficulty when dealing with boundary value problems for transport equations, see for instance Hamdache [10] or Cercignani et al. [7]). In general, the best we can hope for is to prove that $(v \cdot r)\gamma^\pm h$ is bounded in $L^1(0,T; L^1(\Sigma^\pm))$, which is typically not enough to pass to the limit in the weak formulation. There are several way to handle this difficulty. In the case of Maxwell reflexion, for instance, it is actually possible to show that $\gamma^+ h_\varepsilon$ is bounded in $L^1(0,T; L^1(\Sigma^\pm))$, and for more general operator, one method is to decompose $\gamma^+ h_\varepsilon$ into its projection on $\ker(I - \mathcal{B})$ (which is typically a constant times a maxwellian distribution) and on $\ker(I - \mathcal{B})^\perp$ (for which the norm of $\mathcal{B}$ is strictly less than 1). We refer to [7] for more details on this issue.

Here, we chose a different approach, which was first used in [13] for the Vlasov equation, but which applies only to the case of elastic reflection operator (such operators are the most physically relevant operator that satisfy all of the conditions listed in Section 2.2).

First, multiplying (3) by $(v \cdot r(x)) f_\varepsilon \psi(v)$ where $r(x)$ is an extension of the normal unit vector to Ω and $\psi(v)$ is a compactly supported function (supp $\varphi \in B_R$), we can show that there exists a constant $C(R, \|n\|_{W^{1,\infty}})$ such that

$$\begin{aligned}\int_0^T \int_\Sigma \psi(v) h_\varepsilon^2 (v \cdot r(x))^2 \, d\sigma(x) \, dv \, dt \\ \leq C \Big[||f_\varepsilon||^2_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}^3))} + ||\nabla_v f_\varepsilon||^2_{L^2(0,T;L^2(\Omega\times\mathbb{R}^3))} \\ + \|u\||_{L^2((0,T)\times\mathbb{R}^3)} \|f_\varepsilon\|^4_{L^\infty(0,T;L^\infty(\Omega\times\mathbb{R}^3))} \Big]\end{aligned}$$

In particular, taking $\psi(v)$ such that $\psi = 1$ on B_R, $\psi = 0$ on B_{2R}, we deduce

$$\int_0^T \int_{\partial\Omega\times B_R} h_\varepsilon^2 (v \cdot r(x))^2 \, d\sigma(x) \, dv \, dt \leq C$$

and so (using Cauchy-Schwartz inequality):

$$\int_0^T \int_{\partial\Omega\times B_R} h_\varepsilon |v \cdot r(x)| \, d\sigma(x) \, dv \, dt \leq C.$$

This is enough to pass to the limit in (38), if we take as a test function $\varphi_n(x,v,t) = \varphi(x,v,t)\psi_n(|v|^2)$ with $\psi_n(e) = 1$ on $[0,n]$ and $\phi_n(e) = 0$ for $e > n+1$. As a matter of fact, we then have

$$\left| \int_0^T \int_\Sigma (v \cdot r) h_\varepsilon \, \varphi_n \, d\sigma(x) \, dv \, dt \right| \leq \varepsilon \|\gamma^+ h_\varepsilon\|_{L^1(0,T;L^1(\partial\Omega\times B_n))} \|\gamma^+ \varphi\|_{L^\infty((0,T)\times\Sigma^+)}$$

which allows us to take the limit $\varepsilon \to 0$ for n fixed in (38). We can then let n goes to infinity to obtain (9) and conclude the proof of Theorem 2.2.

References

[1] C. Baranger and L. Desvillettes. Coupling euler and vlasov equations in the context of sprays: local smooth solutions. *To appear in JDHE*, 2006.

[2] Stefan Berres, Raimund Bürger, Kenneth H. Karlsen, and Elmer M. Tory. Strongly degenerate parabolic-hyperbolic systems modeling polydisperse sedimentation with compression. *SIAM J. Appl. Math.*, 64(1):41–80 (electronic), 2003.

[3] L. L. Bonilla, J. A. Carrillo, and J. Soler. Asymptotic behavior of an initial-boundary value problem for the Vlasov-Poisson-Fokker-Planck system. *SIAM J. Appl. Math.*, 57(5):1343–1372, 1997.

[4] François Bouchut. Existence and uniqueness of a global smooth solution for the Vlasov-Poisson-Fokker-Planck system in three dimensions. *J. Funct. Anal.*, 111(1):239–258, 1993.

[5] J. Carrillo and T. Goudon. Stability and asymptotics analysis of a fluid-particles interaction model. *C.P.D.E.*

[6] José A. Carrillo. Global weak solutions for the initial-boundary-value problems to the Vlasov-Poisson-Fokker-Planck system. *Math. Methods Appl. Sci.*, 21(10):907–938, 1998.

[7] Carlo Cercignani, Reinhard Illner, and Mario Pulvirenti. *The mathematical theory of dilute gases*, volume 106 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994.

[8] Eduard Feireisl. On compactness of solutions to the compressible isentropic Navier-Stokes equations when the density is not square integrable. *Comment. Math. Univ. Carolin.*, 42(1):83–98, 2001.

[9] Dimitri Gidaspow. *Multiphase flow and fluidization*. Academic Press Inc., Boston, MA, 1994. Continuum and kinetic theory descriptions.

[10] K. Hamdache. Initial-boundary value problems for the Boltzmann equation: global existence of weak solutions. *Arch. Rational Mech. Anal.*, 119(4):309–353, 1992.

[11] K. Hamdache. Global existence and large time behaviour of solutions for the Vlasov-Stokes equations. *Japan J. Indust. Appl. Math.*, 15(1):51–74, 1998.

[12] Pierre-Louis Lions. *Mathematical topics in fluid mechanics. Vol. 2*, volume 10 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1998. Compressible models, Oxford Science Publications.

[13] A. Mellet and S. Mischler. Uniqueness and semigroup for the Vlasov equation with elastic-diffusive reflexion boundary conditions. *Appl. Math. Lett.*, 17(7):827–832, 2004.

[14] A. Mellet and A. Vasseur. Asymptotic analysis for a Vlasov-Fokker-Planck/compressible Navier-Stokes system of equations. *preprint*, 2006.

[15] Stéphane Mischler. On the initial boundary value problem for the Vlasov-Poisson-Boltzmann system. *Comm. Math. Phys.*, 210(2):447–466, 2000.

[16] Stéphane Mischler. On the trace problem for solutions of the Vlasov equation. *Comm. Partial Differential Equations*, 25(7-8):1415–1443, 2000.

[17] W.K. Sartory. Three-component analysis of blood sedimentation by the method of characteristics. *Math. Biosci.*, 33:145–165, 1977.

[18] A. Spannenberg and K.P. Galvin. Continuous differential sedimentation of a binary suspension. *Chem. Eng. in Australia*, 21:7–11, 1996.

[19] F.A. Williams. Spray combustion and atomization. *Physics of Fluids 1*, pages 541–555, 1958.

[20] F.A. Williams. *Combustion theory.* Benjamin Cummings Publ., 2nd ed., 1985.