Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations

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Abstract

We consider Navier-Stokes equations for compressible viscous fluids in one dimension. It is a well known fact that if the initial datum are smooth and the initial density is bounded by below by a positive constant, then a strong solution exists locally in time. In this paper, we show that under the same hypothesis, the density remains bounded by below by a positive constant uniformly in time, and that strong solutions therefore exist globally in time. Moreover, while most existence results are obtained for positive viscosity coefficients, the present result holds even if the viscosity coefficient vanishes with the density. Finally, we prove that the solution is unique in the class of weak solutions satisfying the usual entropy inequality. The key point of the paper is a new entropy-like inequality introduced by Bresch and Desjardins for the shallow water system of equations. This inequality gives additional regularity for the density (provided such regularity exists at initial time).

1 Introduction

This paper is devoted to the existence of global strong solutions of the following Navier-Stokes equations for compressible isentropic flow:

$$\rho_t + (\rho u)_x = 0 \tag{1}$$
$$(\rho u)_t + (\rho u^2)_x + p(\rho)_x = (\mu(\rho)\, u_x)_x, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{2}$$

with possibly degenerate viscosity coefficient.

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Throughout the paper, we will assume that the pressure $p(\rho)$ obeys a gamma type law

$$p(\rho) = \rho^\gamma, \qquad \gamma > 1, \tag{3}$$

(though more general pressure laws could be taken into account).

The viscosity coefficient $\mu(\rho)$ is often assumed to be a positive constant. However, it is well known that the viscosity of a gas depends on the temperature, and thus on the density (in the isentropic case). For example, the Chapman-Enskog viscosity law for hard sphere molecules predicts that $\mu(\rho)$ is proportional to the square root of the temperature (see [CC70]). In the case of monoatomic gas ($\gamma = 5/3$), this leads to $\mu(\rho) = \rho^{1/3}$. More generally, $\mu(\rho)$ is expected to vanish as a power of the ρ on the vacuum. In this paper, we consider degenerate viscosity coefficients that vanish for $\rho = 0$ at most like ρ^α for some $\alpha < 1/2$. In particular, the cases $\mu(\rho) = \nu$ and $\mu(\rho) = \nu\rho^{1/3}$ (with ν positive constant) are included in our result (see conditions (6)-(7) for details).

One-dimensional Navier-Stokes equations have been studied by many authors when the viscosity coefficient μ is a positive constant. The existence of weak solutions was first established by A. Kazhikhov and V. Shelukhin [KS77] for smooth enough data close to the equilibrium (bounded away from zero). The case of discontinuous data (still bounded away from zero) was addressed by V. Shelukhin [She82, She83, She84] and then by D. Serre [Ser86a, Ser86b] and D. Hoff [Hof87]. First results concerning vanishing intial density were also obtained by V. Shelukhin [She86]. In [Hof98], D. Hoff proved the existence of global weak solutions with large discontinuous initial data, possibly having different limits at $x = \pm\infty$. He proved moreover that the constructed solutions have strictly positive densities (vacuum states cannot form in finite time). In dimension greater than two, similar results were obtained by A. Matsumura and T. Nishida [MN79] for smooth data and D. Hoff [Hof95] for discontinuous data close to the equilibrium. The first global existence result for initial density that are allowed to vanish was due to P.-L. Lions (see [Lio98]). The result was later improved by E. Feireisl et al. ([FNP01] and [Fei04]).

Another question is that of the regularity and uniqueness of the solutions. This problem was first analyzed by V. Solonnikov [Sol76] for smooth initial data and for small time. However, the regularity may blow-up as the solution gets close to vacuum. This leads to another interesting question of whether vacuum may arise in finite time. D. Hoff and J. Smoller ([HS01]) show that any weak solution of the Navier-Stokes equations in one space dimension do not exhibit vacuum states, provided that no vacuum states are present initially. More precisely, they showed that if the initial data satisfies

$$\int_E \rho_0(x)\, dx > 0$$

for all open subsets $E \subset \mathbb{R}$, then

$$\int_E \rho(x,t)\, dx > 0$$

for every open subset $E \subset \mathbb{R}$ and for every $t \in [0, T]$.

The main theorem of this paper states that the strong solutions constructed by V. Solonnikov in [Sol76] remain bounded away from zero uniformly in time (i.e. vacuum never arises) and are thus defined globally in time. This result can be seen as the equivalent of the result of D. Hoff in [Hof95] for strong solutions instead of weak solutions. Another interest of this paper is the fact that unlike all the references mentioned above, the result presented here is valid with degenerate viscosity coefficients.

Note that compressible Navier-Stokes equations with degenerate viscosity coefficients have been studied before (see for example [LXY98], [OMNM02], [YYZ01] and [YZ02]). All those papers, however, are devoted to the case of compactly supported initial data and to the description of the evolution of the free boundary. We are interested here in the opposite situation in which vacuum never arises.

The new tool that allows us to obtain those results is an entropy inequality that was derived by D. Bresch and B. Desjardins in [BD02] for the multi-dimensional Korteweg system of equations (which corresponds to the case $\mu(\rho) = \rho$ and with an additional capillary term) and later generalized by the same authors (see [BD04]) to include other density-dependent viscosity coefficients. In the one dimensional case, a similar inequality was introduced earlier by V. A. Vaĭgant [Vaĭ90] for flows with constant viscosity (see also V. V. Shelukhin [She98]).

The main interest of this inequality is to provide further regularity for the density. When $\mu(\rho) = \rho$, for instance, it implies that the gradient of $\sqrt{\rho}$ remains bounded for all time provided it was bounded at time $t = 0$. This has very interesting consequences for many hydrodynamic equations. In [She98], V. V. Shelukhin establishes the existence of a unique weak solution for one dimensional flows with constant viscosity coefficient. In higher dimension, D. Bresch, B. Desjardin and C.K. Lin use this inequality to establish the stability of weak solutions for the Korteweg system of equations in [BDL03] and for the compressible Navier-Stokes equations with an additional quadratic friction term in [BD03]. In [MV06], we establish the stability of weak solutions for the compressible isentropic Navier-Stokes equations in dimension 2 and 3 (without any additional terms). We also refer to [BD05] for recent developments concerning the full system of compressible Navier-Stokes equations (for heat conducting fluids).

At this point, we want to stress out the fact that in dimension 2 and higher, this inequality holds only when the two viscosity coefficients satisfy a relation that considerably restricts the range of admissible coefficients (and in particular implies that one must have $\mu(0) = 0$). This necessary condition disappear in dimension 1, as the two viscosity coefficients become one (the derivation of the inequality is also much simpler in one dimension).

Another particularity of the dimension 1, is that the inequality gives control on some negative powers of the density (this is not true in higher dimensions).

This will allow us to show that vacuum cannot arise if it was not present at time $t = 0$.

Finally, we point out that the present result is very different from that of [MV06] where the density was allowed to vanish (and the difficulty was to control the velocity u on the vacuum). Naturally, a result similar to that of [MV06] holds in dimension one, though it is not the topic of this paper.

Our main result is made precise in the next section. Section 3 is devoted to the derivation of the fundamental entropy inequalities and a priori estimates. The existence part of Theorem 2.1 is proved in Section 4. The uniqueness is addressed in Section 5.

2 The result

Following D. Hoff in [Hof98], we work with positive initial data having (possibly different) positive limits at $x = \pm\infty$: We fix constant velocities u_+ and u_- and constant positive density $\rho_+ > 0$ and $\rho_- > 0$, and we let $\overline{u}(x)$ and $\overline{\rho}(x)$ be two smooth monotone functions satisfying

$$\overline{\rho}(x) = \rho_\pm \quad \text{when } \pm x \geq 1\,, \quad \overline{\rho}(x) > 0 \quad \forall x \in \mathbb{R}, \tag{4}$$

and

$$\overline{u}(x) = u_\pm \quad \text{when } \pm x \geq 1. \tag{5}$$

We recall that the pressure satisfies $p(\rho) = \rho^\gamma$ for some $\gamma > 1$ and we assume that there exists a constant $\nu > 0$ such that the viscosity coefficient $\mu(\rho)$ satisfies

$$\begin{array}{lll} \mu(\rho) \geq \nu\rho^\alpha & \forall \rho \leq 1 & \text{for some } \alpha \in [0, 1/2), \\ \mu(\rho) \geq \nu & \forall \rho \geq 1, & \end{array} \tag{6}$$

and

$$\mu(\rho) \leq C + Cp(\rho) \qquad \forall \rho \geq 0. \tag{7}$$

Note that (7) is only a restriction on the growth of μ for large ρ. Examples of admissible viscosity coefficients include $\mu(\rho) = \nu$ and $\mu(\rho) = \rho^{1/3}$.

Our main theorem is the following:

Theorem 2.1 *Assume that the initial data $\rho_0(x)$ and $u_0(x)$ satisfy*

$$\begin{array}{c} 0 < \underline{\kappa}_0 \leq \rho_0(x) \leq \overline{\kappa}_0 < \infty, \\ \rho_0 - \overline{\rho} \in H^1(\mathbb{R}), \\ u_0 - \overline{u} \in H^1(\mathbb{R}), \end{array} \tag{8}$$

for some constants $\underline{\kappa}_0$ and $\overline{\kappa}_0$. Assume also that $\mu(\rho)$ verifies (6) and (7). Then there exists a global strong solution (ρ, u) of (1)-(2)-(3) on $\mathbb{R}^+ \times \mathbb{R}$ such that for every $T > 0$:

$$\begin{array}{l} \rho - \overline{\rho} \in L^\infty(0, T; H^1(\mathbb{R})), \\ u - \overline{u} \in L^\infty(0, T; H^1(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R})). \end{array}$$

Moreover, for every $T > 0$, there exist constants $\underline{\kappa}(T)$ and $\overline{\kappa}(T)$ such that

$$0 < \underline{\kappa}(T) \le \rho(x,t) \le \overline{\kappa}(T) < \infty \qquad \forall (t,x) \in (0,T) \times \mathbb{R}.$$

Finally, if $\mu(\rho) \ge \nu > 0$ for all $\rho \ge 0$, if μ is uniformly Lipschitz and if $\gamma \ge 2$ then this solution is unique in the class of weak solutions satisfying the usual entropy inequality (16).

Note that he assumption (8) on the initial data implies, in particular that the initial entropy (or relative entropy) is finite.

When the viscosity coefficient $\mu(\rho)$ satisfies

$$\mu(\rho) \ge \nu > 0 \qquad \forall \rho \ge 0, \tag{9}$$

the existence of a smooth solution for small time is a well-known result. More precisely, we have:

Proposition 2.1 ([Sol76]) *Let (ρ_0, u_0) satisfy (8) and assume that μ satisfies (9), then there exists $T_0 > 0$ depending on $\underline{\kappa}_0$, $\overline{\kappa}_0$, $||\rho_0 - \overline{\rho}||_{H^1}$ and $||u_0 - \overline{u}||_{H^1}$ such that (1)-(2)-(3) has a unique solution (ρ, u) on $(0, T_0)$ satisfying*

$$\begin{aligned} &\rho - \overline{\rho} \in L^\infty(0, T_1, H^1(\mathbb{R}))\,, \qquad &\partial_t \rho \in L^2((0,T_1) \times \mathbb{R})\,,\\ &u - \overline{u} \in L^2(0, T_1; H^2(\mathbb{R}))\,, \qquad &\partial_t u \in L^2((0,T_1) \times \mathbb{R}) \end{aligned}$$

for all $T_1 < T_0$.
Moreover, there exist some $\underline{\kappa}(t) > 0$ and $\overline{\kappa}(t) < \infty$ such that $\underline{\kappa}(t) \le \rho(x,t) \le \overline{\kappa}(t)$ for all $t \in (0, T_0)$.

In view of this proposition, we see that if we introduce a truncated viscosity coefficient $\mu_n(\rho)$:

$$\mu_n(\rho) = \max(\mu(\rho), 1/n),$$

then there exist approximated solutions (ρ_n, u_n) defined for small time $(0, T_0)$ (T_0 possibly depending on n). To prove Theorem 2.1, we only have to show that (ρ_n, u_n) satisfies the following bounds uniformly with respect to n and T large:

$$\begin{aligned} &\underline{\kappa}(T) \le \rho_n \le \overline{\kappa}(T) \quad \forall t \in [0,T],\\ &\rho_n - \overline{\rho} \in L^\infty(0,T; H^1(\mathbb{R})),\\ &u_n - \overline{u} \in L^\infty(0,T; H^1(\mathbb{R})). \end{aligned}$$

In the next section, we derive the entropy inequalites that will be used to obtain the necessary bounds on ρ_n and u_n.

3 Entropy inequalities

In its conservative form, (1)-(2)-(3) can be written as

$$\partial_t U + \partial_x [A(U)] = \begin{bmatrix} 0 \\ (\mu(\rho) u_x)_x \end{bmatrix}$$

with the state vector

$$U = \begin{bmatrix} \rho \\ \rho\, u \end{bmatrix} = \begin{bmatrix} \rho \\ m \end{bmatrix}$$

and the flux

$$A(U) = \begin{bmatrix} \rho\, u \\ \rho\, u^2 + \rho^\gamma \end{bmatrix} = \begin{bmatrix} m \\ \dfrac{m^2}{\rho} + \rho^\gamma \end{bmatrix}.$$

It is well known that

$$\mathscr{H}(U) = \rho \frac{u^2}{2} + \frac{1}{\gamma - 1}\rho^\gamma = \frac{m^2}{2\rho} + \frac{1}{\gamma - 1}\rho^\gamma,$$

is an entropy for the system of equations (1)-(2)-(3). More precisely, if (ρ, u) is a smooth solution, then we have

$$\partial_t \mathscr{H}(U) + \partial_x \big[F(U) - \mu(\rho) u u_x\big] + \mu(\rho) u_x^2 = 0 \tag{10}$$

with

$$F(U) = \rho u \frac{u^2}{2} + \frac{\gamma}{\gamma - 1} u \rho^\gamma.$$

In particular, integrating (10) with respect to x, we immediately see that

$$\frac{d}{dt} \int_{\mathbb{R}} \left[\rho \frac{u^2}{2} + \frac{1}{\gamma - 1}\rho^\gamma\right] dx + \int_{\mathbb{R}} \mu(\rho) |u_x|^2 \, dx \leq 0. \tag{11}$$

However, since we are looking for solutions $\rho(x, t)$ and $u(x, t)$ that converges to $\rho_\pm$ and $u_\pm$ at $\pm\infty$, we do not expect the entropy to be integrable. It is thus natural to work with the relative entropy instead of the entropy.

The relative entropy is defined for any functions U and $\tilde{U}$ by

$$\begin{aligned} \mathscr{H}(U|\tilde{U}) &= \mathscr{H}(U) - \mathscr{H}(\tilde{U}) - D\mathscr{H}(\tilde{U})(U - \tilde{U}) \\ &= \rho(u - \tilde{u})^2 + p(\rho|\tilde{\rho}), \end{aligned}$$

where $p(\rho|\tilde{\rho})$ is the relative entropy associated to $\frac{1}{\gamma-1}\rho^\gamma$:

$$p(\rho|\tilde{\rho}) = \frac{1}{\gamma - 1}\rho^\gamma - \frac{1}{\gamma - 1}\tilde{\rho}^\gamma - \frac{\gamma}{\gamma - 1}\tilde{\rho}^{\gamma - 1}(\rho - \tilde{\rho}).$$

Note that, since p is strictly convex, $p(\rho|\tilde{\rho})$ is nonnegative for every ρ and $p(\rho|\tilde{\rho}) = 0$ if and only if $\rho = \tilde{\rho}$.

We recall that $\overline{\rho}(x)$ and $\overline{u}(x)$ are smooth functions satisfying (4) and (5), and we denote

$$\overline{U} = \begin{bmatrix} \overline{\rho} \\ \overline{\rho}\, \overline{u} \end{bmatrix}.$$

It is easy to check that there exists a positive constant C (depending on $\inf \overline{\rho}$) such that for every ρ and for every $x \in \mathbb{R}$, we have

$$\rho + \rho^\gamma \leq C[1 + p(\rho|\overline{\rho})]\,, \tag{12}$$

$$\liminf_{\rho \to 0} p(\rho|\overline{\rho}) \geq C^{-1}\,. \tag{13}$$

The first inequality we will use in the proof of Theorem 2.1 is the usual relative entropy inequality for compressible Navier-Stokes equations:

Lemma 3.1 *Let ρ, u be a solution of (1)-(2)-(3) satisfying the entropy inequality*

$$\partial_t \mathscr{H}(U) + \partial_x[F(U) - \mu(\rho)uu_x] + \mu(\rho)|u_x|^2 \leq 0, \tag{14}$$

Assume that the initial data (ρ_0, u_0) satisfies

$$\int_{\mathbb{R}} \mathscr{H}(U_0|\overline{U})\,dx = \int_{\mathbb{R}} \left[\rho_0 \frac{(u_0 - \overline{u})^2}{2} + p(\rho_0|\overline{\rho})\right] dx < +\infty. \tag{15}$$

Then, for every $T > 0$, there exists a positive constant $C(T)$ such that

$$\sup_{[0,T]} \int_{\mathbb{R}} \left[\rho \frac{(u - \overline{u})^2}{2} + p(\rho|\overline{\rho})\right] dx + \int_0^T \int_{\mathbb{R}} \mu(\rho)|u_x|^2\,dx\,dt \leq C(T). \tag{16}$$

The constant $C(T)$ depends only on $T > 0$, $\overline{U}$, the initial value U_0, γ, and on the constant C appearing in (7).

Note that when both $\overline{\rho}$ and ρ_0 are bounded above and below away from zero, it is easy to check that

$$p(\rho_0|\overline{\rho}) \leq C(\rho_0 - \overline{\rho})^2$$

and thus (15) holds under the assumptions of Theorem 2.1.

Proof of Lemma 3.1. First, we have (by a classical but tedious computation, see [Daf79]):

$$\begin{aligned}
\partial_t \mathscr{H}(U|\overline{U}) \quad = \quad & \big[\partial_t \mathscr{H}(U) + \partial_x(F(U) - \mu(\rho)u\partial_x u)\big] - \partial_t \mathscr{H}(\overline{U}) \\
& - \partial_x[F(U) - \mu(\rho)u\partial_x u] + \partial_x[DF(\overline{U})(U - \overline{U})] \\
& - D^2\mathscr{H}(\overline{U})[\partial_t \overline{U} + \partial_x A(\overline{U})](U - \overline{U}) \\
& - D\mathscr{H}(\overline{U})[\partial_t U + \partial_x A(U)] \\
& + D\mathscr{H}(\overline{U})[\partial_t \overline{U} + \partial_x A(\overline{U})] \\
& + D\mathscr{H}(\overline{U})\partial_x[A(U|\overline{U})],
\end{aligned}$$

where the relative flux is defined by

$$\begin{aligned}
A(U|\overline{U}) \quad &= \quad A(U) - A(\overline{U}) - DA(\overline{U}) \cdot (U - \overline{U}) \\
&= \quad \begin{bmatrix} 0 \\ \rho(u - \overline{u})^2 + (\gamma - 1)p(\rho|\overline{\rho}) \end{bmatrix}.
\end{aligned}$$

Since U is a solution of (1)-(2)-(3) and satisfies the entropy inequality, and using the fact that $\overline{U} = (\overline{\rho}, \overline{\rho u})$ satisfies (4) and (5) (and in particular $\partial_t \overline{U} = 0$), we deduce

$$\begin{aligned}
\partial_t \mathscr{H}(U|\overline{U}) \quad \leq \quad & -\mu(\rho)|\partial_x u|^2 \\
& -D^2\mathscr{H}(\overline{U})[\partial_x A(\overline{U})](U-\overline{U}) \\
& -D_2\mathscr{H}(\overline{U})[\partial_x(\mu(\rho)\partial_x u)] \\
& +D\mathscr{H}(\overline{U})\partial_x[A(U|\overline{U})] \\
& +D\mathscr{H}(\overline{U})[\partial_x A(\overline{U})] \\
& -\partial_x[F(U)-\mu(\rho)uu_x]+\partial_x[DF(\overline{U})(U-\overline{U})],
\end{aligned}$$

where $D_2\mathscr{H}(\overline{U})=\overline{u}$. We now integrate with respect to $x\in\mathbb{R}$, using the fact that supp $(\partial_x\overline{U})\in[-1,1]$, and we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\mathscr{H}(U|\overline{U})\,dx+\int_{\mathbb{R}}\mu(\rho)|\partial_x u|^2 \\
&\quad\leq -\int_{-1}^{1} D^2\mathscr{H}(\overline{U})[\partial_x A(\overline{U})](U-\overline{U})\,dx \\
&\qquad+\int_{-1}^{1}(\partial_x\overline{u})\mu(\rho)\partial_x u\,dx \\
&\qquad-\int_{-1}^{1}\partial_x[D\mathscr{H}(\overline{U})]A(U|\overline{U})\,dx \\
&\qquad-\int_{-1}^{1}\partial_x[D\mathscr{H}(\overline{U})]A(\overline{U})\,dx.
\end{aligned}$$

Writing

$$\left|\int_{-1}^{1}(\partial_x\overline{u})\mu(\rho)\partial_x u\,dx\right|\leq\|\partial_x\overline{u}\|_{L^\infty}^2\int_{-1}^{1}\mu(\rho)\,dx+\frac{1}{2}\int_{-1}^{1}\mu(\rho)|\partial_x u|^2\,dx\,,$$

it follows that there exists a constant C depending on $||\overline{U}||_{W^{1,\infty}}$ such that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\mathscr{H}(U|\overline{U})\,dx+\frac{1}{2}\int_{\mathbb{R}}\mu(\rho)\,|\partial_x u|^2 \\
&\quad\leq C+C\int_{-1}^{1}|U-\overline{U}|\,dx+C\int_{-1}^{1}|A(U|\overline{U})|\,dx \\
&\qquad+C\int_{-1}^{1}\mu(\rho)\,dx.
\end{aligned}\tag{17}$$

To conclude, we need to show that the right hand side can be controlled by $\mathscr{H}(U|\overline{U})$. First, we note that

$$|A(U|\overline{U})|\leq\max(1,(\gamma-1))\mathscr{H}(U|\overline{U}),$$

and that (12) and (7) yield

$$\int_{-1}^{1}\mu(\rho)\,dx\leq C+\int_{\mathbb{R}}p(\rho|\overline{\rho})\,dx.\tag{18}$$

Next, using (12) we get:

$$\begin{aligned}
\int_{-1}^{1} |U-\overline{U}|\,dx &\leq \int_{-1}^{1} |\rho-\overline{\rho}|\,dx + \int_{-1}^{1} \rho|u-\overline{u}|\,dx + \int_{-1}^{1} |\overline{u}(\rho-\overline{\rho})|\,dx\\
&\leq C\int_{-1}^{1}(1+p(\rho|\overline{\rho}))\,dx\\
&\quad + \left(\int_{-1}^{1}\rho\,dx\right)^{1/2}\left(\int_{-1}^{1}\rho(u-\overline{u})^2\,dx\right)^{1/2}\\
&\leq C\int_{-1}^{1}(1+p(\rho|\overline{\rho}))\,dx\\
&\quad + \left(\int_{-1}^{1}(1+p(\rho|\overline{\rho}))\,dx\right)^{1/2}\left(\int_{-1}^{1}\mathscr{H}(U|\overline{U})\,dx\right)^{1/2}\\
&\leq C\int_{-1}^{1}\mathscr{H}(U|\overline{U})\,dx + C.
\end{aligned}$$

So (17) becomes

$$\frac{d}{dt}\int_{\mathbb{R}}\mathscr{H}(U|\overline{U})\,dx + \frac{1}{2}\int_{\mathbb{R}}\mu(\rho)\,|\partial_x u|^2 \leq C + C\int_{-1}^{1}|\mathscr{H}(U|\overline{U})|\,dx,$$

and Gronwall's lemma gives Lemma 3.1.

For further reference, we note that this also implies

$$\frac{d}{dt}\int_{\mathbb{R}}\mathscr{H}(U|\overline{U})\,dx \leq C(T). \tag{19}$$

□

Unfortunately, it is a well-known fact that the estimates provided by Lemma 3.1 are not enough to prove the stability of the solutions of (1)-(2)-(3). The key tool of this paper is thus the following lemma:

Lemma 3.2 *Assume that $\mu(\rho)$ is a C^2 function, and let (ρ,u) be a solution of (1)-(2)-(3) such that*

$$u-\overline{u}\in L^2((0,T);H^2(\mathbb{R})),\quad \rho-\overline{\rho}\in L^\infty((0,T);H^1(\mathbb{R})),\quad 0<m\leq\rho\leq M. \tag{20}$$

Then there exists $C(T)$ such that the following inequality holds:

$$\sup_{[0,T]}\int\left[\frac{1}{2}\rho\big|(u-\overline{u})+\partial_x(\varphi(\rho))\big|^2 + p(\rho|\overline{\rho})\right]dx + \int_0^T\int_{\mathbb{R}}\partial_x(\varphi(\rho))\partial_x(\rho^\gamma)\,dx\,dt \leq C(T), \tag{21}$$

with φ such that

$$\varphi'(\rho) = \frac{\mu(\rho)}{\rho^2}. \tag{22}$$

The constant $C(T)$ depends only on $T > 0$, $(\overline{\rho}, \overline{u})$, the initial value U_0, γ, and on the constant C appearing in (7).

Since the viscosity coefficient $\mu(\rho)$ is non-negative, (22) implies that $\varphi(\rho)$ is increasing. The lemma thus implies that

$$\sup_{[0,T]} \int \left[\frac{1}{2}\rho\big|(u-\overline{u}) + \partial_x(\varphi(\rho))\big|^2 + p(\rho|\overline{\rho}) \right] dx \leq C(T),$$

which, together with Lemma 3.1, yields

$$||\sqrt{\rho}\partial_x(\varphi(\rho))||_{L^\infty(0,T;L^2(\Omega))} = 2||\mu(\rho)(\rho^{-1/2})_x||_{L^\infty(0,T;L^2(\Omega))} \leq C(T).$$

This inequality will be the corner stone of the proof of Theorem 2.1 which is detailed in the next section.

As mentioned in the introduction, Lemma 3.2 relies on a new mathematical entropy inequality that was first derived by Bresch and Desjardins in [BD02] and [BD04] in dimension 2 and higher. Of course, the computations are much simpler in dimension 1.

We stress out the fact that it is important to know exactly what regularity is needed on ρ and u to establish this inequality. Indeed, unlike inequality (16) which is quite classical, there is no obvious way to regularize the system of equations (1)-(2)-(3) while preserving the structure necessary to derive (21). Fortunately, it turns out that (20), which is the natural regularity for strong solutions, is enough to justify the computations, as we will see in the proof.

Proof. We have to show that

$$\frac{d}{dt}\int \left[\frac{1}{2}\rho|u-\overline{u}|^2 + \rho(u-\overline{u})(\varphi(\rho))_x + \frac{1}{2}\rho(\varphi(\rho))_x^2 \right] dx + \frac{d}{dt}\int p(\rho|\overline{\rho})\, dx$$

is bounded

Step 1. From the proof of the previous lemma (see (19)), we already know that:

$$\frac{d}{dt}\int \left[\frac{1}{2}\rho|u-\overline{u}|^2 \right] dx + \frac{d}{dt}\int p(\rho|\overline{\rho})\, dx \leq C(T). \tag{23}$$

Step 2. Next we show that:

$$\begin{aligned} \frac{d}{dt}\int \rho\frac{(\varphi(\rho))_x^2}{2}\, dx &= -\int \rho^2\varphi'(\rho)(\varphi(\rho))_x u_{xx}\, dx \\ &\quad - \int (2\rho\varphi'(\rho) + \rho^2\varphi''(\rho))\rho_x(\varphi(\rho))_x u_x\, dx. \end{aligned} \tag{24}$$

This follows straightforwardly from (1) when the second derivatives of the density are bounded in $L^2((0,T)\times\mathbb{R})$. In that case, it is worth mentionning that the right hand side can be rewritten as

$$\int \rho^2\varphi'(\rho)(\varphi(\rho))_{xx}u_x\,dx.$$

However, we do not have any bounds on ρ_{xx}. It is thus important to justify the derivation of (24):

First, we point out that (24) makes sense when (ρ,u) satisfies only (20) (we recall that since $\overline{\rho}$ and $\overline{u}$ are constant outide $(-1,1)$, (20) implies that $\rho_x \in L^\infty((0,T);L^2(\mathbb{R}))$ and $u_x \in L^2((0,T);H^1(\mathbb{R}))$). Moreover, we note that the computation only makes use of the continuity equation. The rigorous derivation of (24) (under assumption (20)) can thus be achieved by carefully regularizing the continuity equation. The details are presented in the appendix (see Lemma A.1).

Step 3. Next, we evaluate the derivative of the cross-product:

$$\begin{aligned}
&\frac{d}{dt}\int \rho(u-\overline{u})\partial_x(\varphi(\rho))\,dx\\
&\qquad=\int \partial_x(\varphi(\rho))\partial_t(\rho(u-\overline{u}))\,dx+\int \rho(u-\overline{u})\partial_t\partial_x(\varphi(\rho))\,dx\\
&\qquad=\int \partial_x(\varphi(\rho))\partial_t(\rho(u-\overline{u}))\,dx-\int(\rho(u-\overline{u}))_x\varphi'(\rho)\partial_t\rho\,dx\,. \qquad (25)
\end{aligned}$$

Multiplying (2) by $\partial_x\varphi(\rho)$, we get:

$$\begin{aligned}
\int \partial_x(\varphi(\rho))\partial_t(\rho(u-\overline{u}))\,dx &= \int \partial_x(\varphi(\rho))\partial_t(\rho u)\,dx-\int \partial_x(\varphi(\rho))(\partial_t\rho)\overline{u}\,dx\\
&= \int(\varphi(\rho))_x(\mu(\rho)u_x)_x\,dx\\
&\quad -\int \partial_x(\varphi(\rho))\partial_x(\rho^\gamma)\,dx\\
&\quad -\int \partial_x(\varphi(\rho))\partial_x(\rho u^2)\,dx\\
&\quad +\int \varphi'(\rho)(\rho u)_x\rho_x\overline{u}\,dx.
\end{aligned}$$

The continuity equation easily yields:

$$\int(\rho(u-\overline{u}))_x\varphi'(\rho)\partial_t\rho\,dx=-\int((\rho u)_x)^2\varphi'(\rho)\,dx+\int(\rho\overline{u})_x(\rho u)_x\varphi'(\rho)\,dx.$$

Note that those equalities hold as soon as ρ and u satisfy (20).

Step 4. If φ and μ satisfy (22) then we have

$$\begin{aligned}
\int(\varphi(\rho))_x(\mu(\rho)u_x)_x\,dx &= \int \rho^2\varphi'(\rho)(\varphi(\rho))_x u_{xx}\,dx\\
&\quad+\int(2\rho\varphi'(\rho)+\rho^2\varphi''(\rho))\rho_x(\varphi(\rho))_x u_x\,dx,
\end{aligned}$$

so (24) and (25) yields

$$\begin{aligned}
&\frac{d}{dt}\left\{\int \rho(u-\overline{u})\partial_x\varphi(\rho)+\rho\frac{|\partial_x\varphi(\rho)|^2}{2}\,dx\right\}+\int \partial_x\varphi(\rho)\partial_x(\rho^\gamma)\,dx\\
&\qquad = -\int \partial_x(\varphi(\rho))\partial_x(\rho u^2)\,dx+\int ((\rho u)_x)^2\varphi'(\rho)\,dx\\
&\qquad\quad +\int (\rho u)_x\varphi'(\rho)[\rho_x\overline{u}-(\rho\overline{u})_x]\,dx\\
&\qquad = \int \varphi'(\rho)[-\rho_x(\rho u^2)_x+((\rho u)_x)^2]\,dx-\int (\rho u)_x\varphi'(\rho)\rho\overline{u}_x\,dx\\
&\qquad = \int \rho^2\varphi'(\rho)u_x^2\,dx-\int (\rho u)_x\varphi'(\rho)\rho\overline{u}_x\,dx,
\end{aligned}$$

and using (22), we deduce

$$\begin{aligned}
&\frac{d}{dt}\left\{\int \rho(u-\overline{u})\partial_x\varphi(\rho)+\rho\frac{|\partial_x\varphi(\rho)|^2}{2}\,dx\right\}+\int \partial_x\varphi(\rho)\partial_x(\rho^\gamma)\,dx\\
&\qquad = \int \mu(\rho)(u_x)^2\,dx-\int \mu(\rho)u_x\overline{u}_x\,dx-\int \rho\,\partial_x(\varphi(\rho))u\,\overline{u}_x\,dx. \quad (26)
\end{aligned}$$

Moreover, since $\overline{u}_x$ has support in $(-1,1)$ and using the bounds given by Lemma 3.1 and inequality (18), it is readily seen that the right hand side in this equality is bounded by:

$$\begin{aligned}
&C\int_{\mathbb{R}}\mu(\rho)|u_x|^2\,dx+C\int_{-1}^1\mu(\rho)\,dx+C\int \rho|\partial_x\varphi(\rho)|^2\,dx+C\int_{-1}^1\rho u^2\,dx\\
&\quad \le C\int \mu(\rho)|u_x|^2\,dx+C\int_{\mathbb{R}}p(\rho,\overline{\rho})\,dx+C\int_{\mathbb{R}}\rho|(u-\overline{u})+\partial_x\varphi(\rho)|^2\,dx+C(T).
\end{aligned}$$

Putting (26) and (23) together, we deduce

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left[\frac{1}{2}\rho\big|(u-\overline{u})+\partial_x(\varphi(\rho))\big|^2+p(\rho|\overline{\rho})\right]\,dx+\int_{\mathbb{R}}\partial_x(\varphi(\rho))\partial_x(\rho^\gamma)\,dx\\
&\quad \le C\int_{\mathbb{R}}\mu(\rho)|u_x|^2\,dx+C\int_{\mathbb{R}}\left[\frac{1}{2}\rho|(u-\overline{u})+\partial_x\varphi(\rho)|^2+p(\rho,\overline{\rho})\right]\,dx+C(T).
\end{aligned}$$

Finally, using the bounds on the viscosity from Lemma 3.1 and Gronwall's inequality we easily deduce (21). □

4 Proof of Theorem 2.1

In this section, we prove the existence part of Theorem 2.1.

The proof relies on the following proposition:

Proposition 4.1 *Assume that the viscosity coefficient μ satisfies (6)-(7) and consider initial data (ρ_0, u_0) satisfying (8). Then for all $T > 0$, there exist some constants $C(T)$, $\underline{\kappa}(T)$ and $\overline{\kappa}(T)$ such that for any strong solution (ρ, u) of (1)-(2)-(3) with initial data (ρ_0, u_0), defined on $(0,T)$ and satisfying*

$$\begin{aligned}
&\rho - \overline{\rho} \in L^\infty(0,T,H^1(\mathbb{R}))\,, \qquad && \partial_t \rho \in L^2((0,T)\times\mathbb{R})\,,\\
&u - \overline{u} \in L^2(0,T;H^2(\mathbb{R}))\,, \qquad && \partial_t u \in L^2((0,T)\times\mathbb{R}),
\end{aligned}$$

with ρ and ρ^{-1} bounded, the following bounds hold

$$\begin{aligned}
&0 < \underline{\kappa}(T) \leq \rho(t) \leq \overline{\kappa}(T) \quad \forall t \in [0,T],\\
&\|\rho - \overline{\rho}\|_{L^\infty(0,T;H^1(\mathbb{R}))} \leq C(T),\\
&\|u - \overline{u}\|_{L^\infty(0,T;H^1(\mathbb{R}))} \leq C(T).
\end{aligned}$$

Moreover the constants $C(T)$, $\underline{\kappa}(T)$ and $\overline{\kappa}(T)$ depend on μ only through the constant C arising in (6) and (7).

Proof of Theorem 2.1. We define $\mu_n(\rho)$ to be the following positive approximation of the viscosity coefficient:

$$\mu_n(s) = \max(\mu(s), 1/n).$$

Notice that μ_n verifies

$$\mu \leq \mu_n \leq \mu + 1.$$

In particular μ_n satisfies (6) and (7) with some constants that are independent on n.

Next, for all $n > 0$, we let (ρ_n, u_n) be the strong solution of (1)-(2)-(3) with $\mu = \mu_n$:

$$\begin{aligned}
&\rho_t + (\rho u)_x = 0\\
&(\rho u)_t + (\rho u^2)_x + p(\rho)_x = (\mu_n(\rho)\, u_x)_x.
\end{aligned}$$

This solution exists at least for small time $(0, T_0)$ thanks to Proposition 2.1 (note that T_0 may depend on n). Proposition 4.1 then implies that for all $T > 0$ there exists $C(T)$, $\overline{\kappa}(T)$, and $\underline{\kappa}(T) > 0$, independent on n, such that

$$\begin{aligned}
&\underline{\kappa}(T) \leq \rho_n(t) \leq \overline{\kappa}(T) \quad \forall t \in [0,T],\\
&\|\rho_n - \overline{\rho}\|_{L^\infty(0,T;H^1(\mathbb{R}))} \leq C(T),\\
&\|u_n - \overline{u}\|_{L^\infty(0,T;H^1(\mathbb{R}))} \leq C(T).
\end{aligned}$$

In particular we can take $T_0 = \infty$ in Proposition 2.1 (for all n). Moreover, since the bound from below for the density is uniform in n for any $T > 0$, by taking n large enough (namely $n \geq 1/\underline{\kappa}(T)$), it is readily seen that (ρ_n, u_n) is a solution of (1)-(2)-(3) on $[0,T]$ with the non-truncated viscosity coefficient $\mu(\rho)$. From the uniqueness of the solution of Proposition 2.1, we see that, passing to the limit in n, we get the desired global solution of (1)-(2)-(3). □

The rest of this section is thus devoted to the proof of Proposition 4.1. First, we will show that ρ is bounded from above and from below uniformly by some positive constants. Then we will investigate the regularity of the velocity by some standard arguments for parabolic equations.

4.1 A priori estimates

Since the initial datum (ρ_0, u_0) satisfies (8), we have

$$\int \rho_0(u_0-\overline{u})^2\,dx < \infty \quad \text{ and } \quad \int_\Omega \rho_0|\partial_x(\varphi(\rho_0))|^2\,dx < +\infty.$$

Moreover, $(\rho,\, u)$ satisfies (20), so we can use the inequalities stated in Lemmas 3.1 and 3.2. We deduce the following estimates, which we shall use throughout the proof of Proposition (4.1):

$$\begin{array}{l} ||\sqrt{\rho}(u-\overline{u})||_{L^\infty(0,T);L^2(\Omega))} \leq C(T), \\ ||\rho||_{L^\infty(0,T;L^1_{\rm loc}\cap L^\gamma_{\rm loc}(\Omega))} \leq C(T), \\ ||\rho-\overline{\rho}||_{L^\infty(0,T;L^1(\Omega))} \leq C(T), \\ ||\sqrt{\mu(\rho)}(u)_x||_{L^2(0,T;L^2(\Omega))} \leq C(T). \end{array} \tag{27}$$

and

$$\begin{array}{l} ||\mu(\rho)\partial_x(\rho^{-1/2})||_{L^\infty(0,T;L^2(\Omega))} \leq C(T), \\ ||\sqrt{\mu(\rho)}\partial_x(\rho^{\gamma/2-1/2})||_{L^2(0,T;L^2(\Omega))} \leq C(T). \end{array} \tag{28}$$

4.2 Uniform bounds for the density.

The first proposition shows that no vacuum states can arise:

Proposition 4.2 *For every $T>0$, there exists a constant $\underline{\kappa}(T)>0$ such that*

$$\rho(x,t) \geq \underline{\kappa}(T) \qquad \forall (x,t)\in\mathbb{R}\times[0,T].$$

The proof of this proposition will follow from two lemmas. First we have:

Lemma 4.1 *For every $T>0$, There exists $\delta>0$ and $R(T)$ such that for every $x_0\in\mathbb{R}$ and $t_0>0$, there exists $x_1\in[x_0-R(T), x_0+R(T)]$ with*

$$\rho(x_1,t_0) > \delta.$$

This nice result can be found in [Hof98]. We give a proof of it for the sake of completeness.
Proof. Let $\delta>0$ be such that

$$p(\rho|\overline{\rho}) \geq \frac{C^{-1}}{2} \qquad \forall \rho<\delta$$

(such a δ exists thanks to (13)). Then, if

$$\sup_{x\in[x_0-R,x_0+R]} \rho(x,t_0) < \delta$$

we have

$$\int p(\rho|\overline{\rho})\, dx \geq C^{-1}R$$

and since the integral in the left hand side is bounded by a constant (see Lemma 3.1), a suitable choice of R leads to a contradiction. $\square$

Lemma 4.2 *Let*

$$w(x,t) = \inf(\rho(x,t),1) = 1-(1-\rho(x,t))_+.$$

Then there exists $\varepsilon > 0$ and a constant $C(T)$ such that

$$||\partial_x w^{-\varepsilon}||_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C(T).$$

Proof. We have

$$\partial_x w = \partial_x \rho 1_{\{\rho\leq 1\}}.$$

In particular (28) gives

$$||\frac{\mu(w)}{w^{3/2}} w_x||_{L^\infty(0,T;L^2(\Omega))} \leq C$$

so using (6) we deduce:

$$||w^{\alpha-3/2}\partial_x w||_{L^\infty(0,T;L^2(\Omega))} = ||\partial_x w^{\alpha-1/2}||_{L^\infty(0,T;L^2(\Omega))} \leq C,$$

and the result follows with $\varepsilon = 1/2-\alpha > 0$. $\square$

Proof of Proposition 4.2. Together with Sobolev-Poincaré inequality, Lemma 4.1 and 4.2 yield that $w^{-\varepsilon}$ is bounded in $L^\infty((0,T)\times\mathbb{R})$:

$$w^{-\varepsilon}(x,t) \leq C(T) \qquad \forall (x,t)\in\mathbb{R}\times(0,T).$$

This yields Proposition 4.2 with $\underline{\kappa}(T) = C(T)^{-1/\varepsilon}$. $\square$

Next, we find a bound for the density in L^∞:

Proposition 4.3 *For every $T>0$, there exist a constant $\overline{\kappa}(T)$ such that*

$$\rho(x,t) \leq \overline{\kappa}(T) \qquad \forall (x,t)\in\mathbb{R}\times(0,T).$$

Let $s = (\gamma-1)/2$, then (21) with (6) and (22) yields $\partial_x(\rho^s)$ bounded in $L^2((0,T)\times\mathbb{R})$. Moreover, for every compact subset K of $\mathbb{R}$, we have

$$\begin{aligned}
\int_K |\partial_x \rho^s|\, dx &= \int_K |\rho^{s-1}\partial_x\rho|\, dx \\
&\leq \left(\int_K \rho^{1+2s}\, dx\right)^{1/2}\left(\int_K \frac{1}{\rho^3}(\partial_x\rho)^2\, dx\right)^{1/2} \\
&\leq \left(\int_K \rho^{\gamma}\, dx\right)^{1/2}\left(\int_K \rho\varphi'(\rho)^2(\partial_x\rho)^2\, dx\right)^{1/2}
\end{aligned}$$

and so using (12) we get

$$\int_K |\partial_x \rho^s| \, dx \le C \left(|K| + \int_K p(\rho|\overline{\rho}) \, dx \right)^{1/2} \left(\int_K \rho \varphi'(\rho)^2 (\partial_x \rho)^2 \, dx \right)^{1/2}.$$

Since

$$\rho^s \le 1 + \rho^\gamma$$

we deduce that

$$\rho^s \text{ is bounded in } L^\infty(0,T; W^{1,1}_{\text{loc}}(\mathbb{R})),$$

and the $W^{1,1}(K)$ norm of $\rho^s(t,\cdot)$ only depends on $|K|$. Sobolev imbedding thus yields Proposition 4.3. □

Proposition 4.4 *There exists a constant $C(T)$ such that*

$$\|\rho(x,t) - \overline{\rho}(x)\|_{L^\infty(0,T;H^1(\mathbb{R}))} \le C(T).$$

Proof. Proposition 4.3 yields

$$\begin{aligned}
\int (\partial_x \rho)^2 \, dx &\le \overline{\kappa}^3 \int \frac{1}{\rho^3} (\partial_x \rho)^2 \, dx \\
&\le \overline{\kappa}^3 \int \frac{\rho}{(\mu(\rho))^2} (\phi'(\rho))^2 (\partial_x \rho)^2 \, dx \\
&\le \frac{\nu \overline{\kappa}^3}{\inf(1, \underline{\kappa}^{2\alpha})} \int \rho (\partial_x \phi(\rho))^2 \, dx \\
&\le C(T).
\end{aligned}$$

And the result follows. □

4.3 Uniform bounds for the velocity

Proposition 4.5 *There exists a constant $C(T)$ such that*

$$\|u - \overline{u}\|_{L^2(0,T;H^2(\mathbb{R}))} \le C(T)$$

and

$$\|\partial_t u\|_{L^2(0,T;L^2(\mathbb{R}))} \le C(T).$$

In particular, $u - \overline{u} \in C^0(0,T;H^1(\mathbb{R}))$.

Proof. First, we show that $u - \overline{u}$ is bounded in $L^2(0,T;H^1(\mathbb{R}))$. Since $\rho \ge \kappa > 0$, and using (6), it is readily seen that there exists a constant $\nu' > 0$ such that

$$\mu(\rho(x,t)) \ge \nu' \qquad \forall (x,t) \in \mathbb{R} \times [0,T],$$

and so (16) gives

$$\partial_x u \text{ is bounded in } L^2((0,T) \times \mathbb{R})$$

and

$$u - \overline{u} \text{ is bounded in } L^\infty(0,T;L^2(\mathbb{R})).$$

Therefore $u - \overline{u}$ is bounded in $L^2(0,T;H^1(\mathbb{R}))$.

Note that this implies that $\partial_t \rho$ is bounded in $L^2((0,T)\times\mathbb{R})$. Since $\rho - \overline{\rho}$ is bounded in $L^\infty(0,T;H^1)$, it follows (see [Ama00] for example) that

$$\rho \in \mathcal{C}^{s_0}((0,T)\times\mathbb{R})$$

for some $s_0 \in (0,1)$.

Next, we rewrite (2) as follows:

$$\partial_t u - \left(\frac{\mu(\rho)}{\rho}u_x\right)_x = -\gamma\rho^{\gamma-2}\rho_x + \overline{u}u_x + (\partial_x(\varphi(\rho)) - (u-\overline{u}))u_x \tag{29}$$

where we recall that φ, which is defined by $\varphi'(\rho) = \mu(\rho)/\rho^2$, is the function arising in the new entropy inequality (see Lemma 3.2).

In order to deduce some bounds on u, we need to control the right hand side of (29). The first term, $\rho^{\gamma-2}\rho_x$, is bounded in $L^\infty(0,T;L^2(\mathbb{R}))$ (thanks to Proposition 4.4). The second term is bounded in $L^2((0,T)\times\mathbb{R})$ since $\overline{u}$ is in L^∞. For the last part, we write (using Hölder inequality and interpolation inequality):

$$\begin{aligned}
&||(\partial_x(\varphi(\rho)) - (u-\overline{u})u_x||_{L^2(0,T;L^{4/3}(\mathbb{R}))} \\
&\quad \le ||\partial_x(\varphi(\rho)) - (u-\overline{u})||_{L^\infty(0,T;L^2(\mathbb{R}))}||u_x||_{L^2(0,T;L^4(\mathbb{R}))} \\
&\quad \le ||\partial_x(\varphi(\rho)) - (u-\overline{u})||_{L^\infty(L^2)}||u_x||^{2/3}_{L^2(L^2)}||u_x||^{1/3}_{L^2(0,T;W^{1,4/3}(\mathbb{R}))} \\
&\quad \le C||u_x||^{1/3}_{L^2(0,T;W^{1,4/3}(\mathbb{R}))}
\end{aligned}$$

(here we make use of (21) and Proposition 4.2). So regularity results for parabolic equation of the form (29) (note that the diffusion coefficient is in $\mathcal{C}^{s_0}((0,T)\times\mathbb{R})$) yield

$$||u_x||_{L^2(0,T;W^{1,4/3}(\mathbb{R}))} \le C||u_x||^{1/3}_{L^2(0,T;W^{1,4/3}(\mathbb{R}))} + C,$$

and so

$$||u_x||_{L^2(0,T;W^{1,4/3}(\mathbb{R}))} \le C.$$

Using Sobolev inequalities, it follows that u_x is bounded in $L^2(0,T;L^\infty(\mathbb{R}))$.

Finally, we can now see that the right hand side in (29) is bounded in $L^2(0,T;L^2(\mathbb{R}))$, and classical regularity results for parabolic equations give

$$u - \overline{u} \text{ is bounded in } L^2(0,T;H^2(\mathbb{R}))$$

and

$$\partial_t u \text{ is bounded in } L^2(0,T;L^2(\mathbb{R})),$$

which concludes the proof. $\square$

It is now readily seen that Proposition 4.1 follows from Propositions 4.2, 4.3, 4.4 and 4.5.

5 Uniqueness

In this last section, we establish the uniqueness of the global strong solution in a large class of weak solutions satisfying the usual entropy inequality. This result can be rewritten as follows:

Proposition 5.1 *Assume that*

$$\mu(\rho) \geq \nu > 0 \quad \textit{for all } \rho \geq 0,$$

and that there exists a constant C such that

$$|\mu(\rho) - \mu(\tilde{\rho})| \leq C|\rho - \tilde{\rho}| \quad \textit{for all } \rho,\ \tilde{\rho} \geq 0.$$

Assume moreover that $\gamma \geq 2$, and let (ρ, u) be the solution of (1)-(2)-(3) given by Theorem 2.1.

If $(\tilde{\rho}, \tilde{u})$ is a weak solution of (1)-(2)-(3) with initial data (ρ_0, u_0) and satisfying the entropy inequality (14) and relative entropy bound (16), and if

$$\lim_{x\to\pm\infty} (\tilde{\rho} - \rho_{\pm}) = 0\,, \qquad \lim_{x\to\pm\infty} (\tilde{u} - u_{\pm}) = 0,$$

then

$$(\tilde{\rho}, \tilde{u}) = (\rho, u).$$

Notice that we do not need to assume that $\tilde{\rho}$ does not vanish. This Proposition will be a consequence of the following Lemma:

Lemma 5.1 *Let $\tilde{U} = (\tilde{\rho}, \tilde{\rho}\tilde{u})$ be a weak solution of (1)-(2)-(3) satisfying the inequality (14) and let $U = (\rho, \rho u)$ be a strong solution of (1)-(2)-(3) satisfying the equality (10). Assume moreover that $\tilde{U}$ and U are such that*

$$\lim_{x\to\pm\infty} (\tilde{\rho} - \rho) = 0\,, \qquad \lim_{x\to\pm\infty} (\tilde{u} - u) = 0. \tag{30}$$

Then we have:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} \mathscr{H}(\tilde{U}|U)\,dx + \int_{\mathbb{R}} \mu(\tilde{\rho})\,[\partial_x(\tilde{u} - u)]^2 \\
&\leq C\int |\partial_x u| \mathscr{H}(\tilde{U}|U)\,dx \\
&\quad - \int_{\mathbb{R}} \partial_x u[\mu(\tilde{\rho}) - \mu(\rho)]\,[\partial_x(\tilde{u} - u)]\,dx \\
&\quad + \int_{\mathbb{R}} \frac{\partial_x(\mu(\rho)\partial_x u)}{\rho}\,(\tilde{\rho} - \rho)\,(u - \tilde{u})\,dx.
\end{aligned} \tag{31}$$

The proof of this lemma relies only on the structure of the equation and not on the properties of the solutions. We postpone it to the end of this section.

Proof of Proposition 5.1. In order to prove Proposition 5.1, we have to show that the last two terms in (31) can be controled by the relative entropy $\mathscr{H}(\tilde{U}|U)$ and the viscosity. Since $\gamma \geq 2$ and $\rho \geq \overline{\kappa} > 0$, we note that there exists C such that

$$p(\tilde{\rho}|\rho) \geq C|\tilde{\rho}-\rho|^2 \quad \text{for all } \tilde{\rho} \geq 0.$$

Then, we can write

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \partial_x u[\mu(\tilde{\rho})-\mu(\rho)]\,[\partial_x(\tilde{u}-u)]\,dx\right| \\
&\quad \leq C\|\partial_x u\|_{L^\infty(\mathbb{R})}\left(\int_{\mathbb{R}}|\tilde{\rho}-\rho|^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}}|\partial_x(\tilde{u}-u)|^2\,dx\right)^{1/2} \\
&\quad \leq C\|\partial_x u\|^2_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|\tilde{\rho}-\rho|^2\,dx + \frac{1}{4}\int_{\mathbb{R}}\mu(\tilde{\rho})|\partial_x(\tilde{u}-u)|^2\,dx \\
&\quad \leq C\|\partial_x u\|^2_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}\mathscr{H}(\tilde{U}|U)\,dx + \frac{1}{4}\int_{\mathbb{R}}\mu(\tilde{\rho})|\partial_x(\tilde{u}-u)|^2\,dx
\end{aligned}$$

which does the trick for the first of the last two terms in (31). For the last term, we see that if we had $\partial_x(\mu(\rho)\partial_x u)$ bounded in $L^\infty((0,T)\times\mathbb{R})$, a similar computation would apply. However, writing

$$\partial_x(\mu(\rho)\partial_x u) = \mu'(\rho)(\partial_x\rho)\,(\partial_x u) + \mu(\rho)\partial_{xx}u$$

it is readily seen that $\partial_x(\mu(\rho)\partial_x u)$ is only bounded in $L^2((0,T)\times\mathbb{R})$. For that reason, we need to control $|\tilde{u}-u|$ in L^∞, which is made possible by the following Lemma:

Lemma 5.2 *Let $\tilde{\rho} \geq 0$ be such that $\int p(\tilde{\rho}|\overline{\rho})\,dx < +\infty$. Then there exists a constant C (depending on $\int p(\tilde{\rho}|\overline{\rho})\,dx$) such that for any regular function h:*

$$\|h\|_{L^\infty(\mathbb{R})} \leq C\left(\int_{\mathbb{R}}\tilde{\rho}|h|^2\,dx\right)^{1/2} + C\left(\int_{\mathbb{R}}|h_x|^2\,dx\right)^{1/2}.$$

Using Lemma 5.2 with $h=\tilde{u}-u$, we deduce:

$$\begin{aligned}
&\int_{\mathbb{R}}\frac{\partial_x(\mu(\rho)\partial_x u)}{\rho}\,(\tilde{\rho}-\rho)\,(u-\tilde{u})\,dx \\
&\leq \left\|\frac{\partial_x(\mu(\rho)\partial_x u)}{\rho}\right\|_{L^2(\mathbb{R})}\|\tilde{\rho}-\rho\|_{L^2(\mathbb{R})}\|u-\tilde{u}\|_{L^\infty(\mathbb{R})} \\
&\leq C\left\|\frac{\partial_x(\mu(\rho)\partial_x u)}{\rho}\right\|_{L^2(\mathbb{R})}\mathscr{H}(\tilde{U}|U)^{\frac{1}{2}}\left(\mathscr{H}(\tilde{U}|U)^{\frac{1}{2}} + \left(\int_R|\partial_x(u-\tilde{u})|^2\,dx\right)^{\frac{1}{2}}\right) \\
&\leq C\left\|\frac{\partial_x(\mu(\rho)\partial_x u)}{\rho}\right\|^2_{L^2(\mathbb{R})}\mathscr{H}(\tilde{U}|U) + \frac{1}{4}\int_R\mu(\tilde{\rho})|\partial_x(u-\tilde{u})|^2\,dx.
\end{aligned}$$

So (31) becomes

$$\frac{d}{dt}\int_{\mathbb{R}} \mathscr{H}(\tilde{U}|U)\,dx + \frac{1}{2}\int_{\mathbb{R}} \mu(\tilde{\rho})\,[\partial_x(\tilde{u}-u)]^2 \le C(t)\int \mathscr{H}(\tilde{U}|U)\,dx$$

where $C(t) \in L^1(0,T)$. Gronwall Lemma, together with the fact that

$$\mathscr{H}(\tilde{U}|U)(t=0) = 0$$

yields Proposition 5.1. □

Proof of Lemma 5.2. Using (13), we see that there exists some $\delta > 0$ and C such that

$$|\{x \in \mathbb{R}\,;\ \tilde{\rho} \le \delta\}| \le C\int_{\mathbb{R}} p(\tilde{\rho}|\overline{\rho})\,dx.$$

We take $R = C\int p(\tilde{\rho}|\overline{\rho})\,dx + 1$. Then, for every x_0 in $\mathbb{R}$, we know that in the interval $(x_0 - R/2, x_0 + R/2)$, $\tilde{\rho}$ is larger than δ is a set of measure at least 1 we denote by ω this set:

$$\omega = (x_0 - R/2, x_0 + R/2) \cap \{\tilde{\rho} \ge \delta\}.$$

Then, for all $x \in \omega$, we have

$$|h(x_0)| \le |h(x)| + \int_{x_0}^{x} |h_x(y)|\,dy \le |h(x)| + R^{1/2}\left(\int_{\mathbb{R}} |h_x(y)|^2\,dy\right)^{1/2}.$$

Integrating with respect to x in ω, we deduce:

$$\begin{aligned} |h(x_0)| &\le \frac{1}{|\omega|}\int_{\omega} |h|\,dx + R^{1/2}\left(\int_{\mathbb{R}} |h_x|^2\,dx\right)^{1/2} \\ &\le \frac{1}{|\omega|^{1/2}}\left(\int_{\omega} |h|^2\,dx\right)^{1/2} + R^{1/2}\left(\int_{\mathbb{R}} |h_x|^2\,dx\right)^{1/2}. \end{aligned}$$

Finally, since $\tilde{\rho} \ge \delta$ in ω, we have

$$|h(x_0)| \le \frac{1}{\delta^{1/2}|\omega|^{1/2}}\left(\int_{\omega} \tilde{\rho}|h|^2\,dx\right)^{1/2} + R^{1/2}\left(\int_{\mathbb{R}} |h_x|^2\,dx\right)^{1/2}$$

and since $|\omega| \ge 1$, the result follows. □

Proof of Lemma 5.1. To prove the lemma, it is convenient to note that the system (1-2)-(3) can be rewritten in the form

$$\partial_t U_i + \partial_x A_i(U) = \partial_x\big[B_{ij}(U)\partial_x\left(D_j\mathscr{H}(U)\right)\big],$$

where $B(U)$ is a positive symmetric matrix and $D\mathscr{H}$ denotes the derivative (with respect to U) of the entropy $\mathscr{H}(U)$ associated with the flux $A(U)$. The existence of such an entropy is equivalent to the existence of an entropy flux function F such that

$$D_j F(U) = \sum_i D_i \mathscr{H}(U) D_j A_i(U) \tag{32}$$

for all U. Then strong solutions of (1)-(2)-(3) satisfy

$$\partial t \mathscr{H}(U) + \partial_x F(U) - \partial_x (B(U)\partial_x D\mathscr{H}(U)) D\mathscr{H}(U) = 0.$$

In our case, we have

$$A(U) = \begin{bmatrix} m \\ \dfrac{m^2}{\rho} + \rho^\gamma \end{bmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 + \rho^\gamma \end{bmatrix}$$

and

$$B(U) = \mu(\rho) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Then, a carefull computation (using (32)) yields

$$\begin{aligned}
&\partial_t \mathscr{H}(\tilde{U}|U) = \\
&\quad = \big[\partial_t \mathscr{H}(\tilde{U}) + \partial_x (F(\tilde{U})) - \partial_x (B(\tilde{U})\partial_x D\mathscr{H}(\tilde{U})) D\mathscr{H}(\tilde{U})\big] \\
&\qquad - \big[\partial_t \mathscr{H}(U) + \partial_x F(U) - \partial_x (B(U)\partial_x D\mathscr{H}(U)) D\mathscr{H}(U)\big] \\
&\qquad - \partial_x [F(\tilde{U}) - F(U)] \\
&\qquad - D^2\mathscr{H}(U)[\partial_t U + \partial_x A(U) - \partial_x \left(B(U)\partial_x \left(D\mathscr{H}(U)\right)\right)](\tilde{U} - U) \\
&\qquad - D\mathscr{H}(U)[\partial_t \tilde{U} + \partial_x A(\tilde{U}) - \partial_x \left(B(\tilde{U})\partial_x \left(D\mathscr{H}(\tilde{U})\right)\right)] \\
&\qquad + D\mathscr{H}(U)[\partial_t U + \partial_x A(U) - \partial_x \left(B(U)\partial_x \left(D\mathscr{H}(U)\right)\right)] \\
&\qquad + \partial_x [DF(U)(\tilde{U} - U)] \\
&\qquad + D\mathscr{H}(U)\partial_x [A(\tilde{U}|U)] \\
&\qquad + \partial_x \left[B(\tilde{U})\partial_x D\mathscr{H}(\tilde{U}) - B(U)\partial_x D\mathscr{H}(U)\right]\left[D\mathscr{H}(\tilde{U}) - D\mathscr{H}(U)\right] \\
&\qquad + \partial_x (B(U)\partial_x D\mathscr{H}(U)) D\mathscr{H}(\tilde{U}|U),
\end{aligned}$$

where the relative flux is defined by

$$A(\tilde{U}|U) \quad = \quad A(\tilde{U}) - A(U) - DA(U)\cdot(\tilde{U} - U)$$

Using the fact that $\tilde{U}$ and U are solutions satisfying the natural entropy inequality and equality, we deduce

$$\begin{aligned}
&\partial_t \mathscr{H}(\tilde{U}|U) \\
&\quad \le -\partial_x [F(\tilde{U}) - F(U)] + \partial_x [DF(U)(\tilde{U} - U)] \\
&\qquad + D\mathscr{H}(U)\partial_x [A(\tilde{U}|U)] \\
&\qquad + \partial_x \left[B(\tilde{U})\partial_x D\mathscr{H}(\tilde{U}) - B(U)\partial_x D\mathscr{H}(U)\right]\left[D\mathscr{H}(\tilde{U}) - D\mathscr{H}(U)\right] \\
&\qquad + \partial_x (B(U)\partial_x D\mathscr{H}(U)) D\mathscr{H}(\tilde{U}|U),
\end{aligned}$$

Integrating with respect to x and using (30), we deduce

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} \mathscr{H}(\tilde{U}|U)\,dx \\
&\leq -\int \partial_x[D\mathscr{H}(U)]A(\tilde{U}|U)\,dx \\
&\quad -\int_{\mathbb{R}} \left[B(\tilde{U})\partial_x D\mathscr{H}(\tilde{U}) - B(U)\partial_x D\mathscr{H}(U)\right]\partial_x\left[D\mathscr{H}(\tilde{U}) - D\mathscr{H}(U)\right]dx \\
&\quad +\int_{\mathbb{R}} \partial_x[B(U)\partial_x D\mathscr{H}(U)]D\mathscr{H}(\tilde{U}|U)\,dx.
\end{aligned}$$

Finally, we check that

$$\partial_x[D\mathscr{H}(U)]A(\tilde{U}|U) = (\partial_x u)[\rho(u-\tilde{u})^2 + (\gamma-1)p(\rho|\tilde{\rho})]\,,$$

$$\partial_x[B(U)\partial_x D\mathscr{H}(U)]D\mathscr{H}(\tilde{U}|U) = \frac{\partial_x(\mu(\rho)\partial_x u)}{\rho}\,(\tilde{\rho}-\rho)\,(u-\tilde{u}),$$

and

$$\begin{aligned}
&\left[B(\tilde{U})\partial_x D\mathscr{H}(\tilde{U}) - B(U)\partial_x D\mathscr{H}(U)\right]\partial_x\left[D\mathscr{H}(\tilde{U}) - D\mathscr{H}(U)\right] \\
&\qquad = [\mu(\tilde{\rho})\partial_x\tilde{u} - \mu(\rho)\partial_x u]\,\partial_x\,[\tilde{u}-u] \\
&\qquad = \mu(\tilde{\rho})\,[\partial_x\tilde{u} - \partial_x u]^2 + \partial_x u[\mu(\tilde{\rho}) - \mu(\rho)]\partial_x\,[\tilde{u}-u]\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} \mathscr{H}(\tilde{U}|U)\,dx + \int_{\mathbb{R}} \mu(\tilde{\rho})\,[\partial_x(\tilde{u}-u)]^2 \\
&\leq C\int |\partial_x u|\mathscr{H}(\tilde{U}|U)\,dx \\
&\quad -\int_{\mathbb{R}} \partial_x u[\mu(\tilde{\rho}) - \mu(\rho)]\,[\partial_x(\tilde{u}-u)]\,dx \\
&\quad +\int_{\mathbb{R}} \frac{\partial_x(\mu(\rho)\partial_x u)}{\rho}\,(\tilde{\rho}-\rho)\,(u-\tilde{u})\,dx.
\end{aligned}$$

which gives the Lemma. □

A Proof of equality 24

Lemma A.1 *Let (ρ,u) satisfy (20) and*

$$\begin{cases} \partial_t\rho + \partial_x(\rho\,u) = 0 \\ \rho(x,0) = \rho_0(x). \end{cases} \tag{33}$$

Then (ρ,u) satisfies (24).

Proof. We denote by h_ε the convolution of any function h by a mollifier. Convoluting (33) by the mollifier, we get

$$\partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u) = r_\varepsilon$$

where

$$r_\varepsilon = \partial_x(\rho_\varepsilon u) - \partial_x(\rho u)_\varepsilon.$$

Since ρ_ε is now a smooth function, a straightforward computation yields:

$$\begin{aligned} &\frac{d}{dt}\int \rho_\varepsilon \frac{(\varphi(\rho_\varepsilon)_x)^2}{2}\,dx \\ &\quad - \int (\rho_\varepsilon)^2 \varphi'(\rho_\varepsilon)\varphi(\rho_\varepsilon)_x u_{xx}\,dx \\ &\quad - \int (2\rho_\varepsilon \varphi'(\rho_\varepsilon) + (\rho_\varepsilon)^2\varphi''(\rho_\varepsilon))(\rho_\varepsilon)_x \varphi(\rho_\varepsilon)_x u_x\,dx \\ &\qquad = \int \rho_\varepsilon \varphi(\rho_\varepsilon)_x (\varphi'(\rho_\varepsilon) r_\varepsilon)_x\,dx. \end{aligned} \tag{34}$$

In order to pass to the limit $\varepsilon \to 0$, we note that

$$\rho_\varepsilon - \overline{\rho} \longrightarrow \rho - \overline{\rho} \quad \text{in } L^\infty(0,T;H^1(\mathbb{R})) \text{ strong},$$

which is enough to take the limit in the left hand side of (34) (note that it implies the strong convergence in $L^\infty(0,T;L^\infty(\mathbb{R}))$). To show that the right hand side goes to zero, we only need to show that r_ε goes to zero in $L^2(0,T;H^1(\mathbb{R}))$ strong (and thus in $L^2(0,T;L^\infty(\mathbb{R}))$). We write

$$\begin{aligned} \partial_x r_\varepsilon &= 2[\partial_x\rho_\varepsilon \partial_x u - (\partial_x \rho\, \partial_x u)_\varepsilon] \\ &\quad + \rho_\varepsilon \partial_{xx} u - (\rho\, \partial_{xx} u)_\varepsilon \\ &\quad + \partial_{xx}\rho_\varepsilon u - (\partial_{xx}\rho\, u)_\varepsilon. \end{aligned}$$

The first two terms converge to zero thanks to the strong convergence of ρ_ε in $L^\infty(0,T;H^1(\mathbb{R}))$. For the last term, we note that

$$\partial_x \rho \in L^\infty(0,T;L^2(\mathbb{R})) \quad \text{and} \quad u \in L^2(0,T;W^{1,\infty}(\mathbb{R}))$$

so the strong convergence to zero in $L^2((0,T)\times\mathbb{R})$ follows from Lemma II.1 in DiPerna-Lions [DL89]. $\square$

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