Existence of solutions for a higher order non-local equation appearing in crack dynamics

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Abstract

In this paper, we prove the existence of non-negative solutions for a non-local higher order degenerate parabolic equation arising in the modeling of hydraulic fractures. The equation is similar to the well-known thin film equation, but the Laplace operator is replaced by a Dirichlet-to-Neumann operator, corresponding to the square root of the Laplace operator on a bounded domain with Neumann boundary conditions (which can also be defined using the periodic Hilbert transform). In our study, we have to deal with the usual difficulty associated to higher order equations (e.g. lack of maximum principle). However, there are important differences with, for instance, the thin film equation: First, our equation is nonlocal; Also the natural energy estimate is not as good as in the case of the thin film equation, and does not yields, for instance, boundedness and continuity of the solutions (our case is critical in dimension 1 in that respect).

Keywords: Hydraulic fractures, Higher order equation, Non-local equation, Thin film equation, Non-negative solutions, periodic Hilbert transform

MSC: 35G25, 35K25, 35A01, 35B09

1 Introduction

This paper is devoted to the following problem:

$$\begin{cases} u_t + (u^n I(u)_x)_x = 0 & \text{for } x \in \Omega, \quad t > 0 \\ u_x = 0, \ u^n I(u)_x = 0 & \text{for } x \in \partial\Omega, \quad t > 0 \\ u(x,0) = u_0(x) & \text{for } x \in \Omega \end{cases} \tag{1}$$

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where Ω is a bounded interval in $\mathbb{R}$, n is a positive real number and I is a non-local elliptic operator of order 1 satisfying $I \circ I = -\Delta$; the operator I will be defined precisely in Section 3 as the square root of the Laplace operator with Neumann boundary conditions. When $\Omega = \mathbb{R}$, it reduces to $I = -(-\Delta)^{1/2}$. In the sequel, we will always take $\Omega = (0, 1)$.

When $n = 3$, this equation arises in the modeling of hydraulic fractures. In that case, u represents the opening of a rock fracture which is propagated in an elastic material due to the pressure exerted by a viscous fluid which fills the fracture (see Section 2 for details). Such fractures occur naturally, for instance in volcanic dikes where magma causes fracture propagation below the surface of the earth, or can be deliberately propagated in oil or gas reservoirs to increase production. There is a significant amount of work involving the mathematical modeling of hydraulic fractures, which is beyond the scope of this article. The model that we consider in our paper, which corresponds to very simple fracture geometry, was developed independently by Geertsma and De Klerk [26] and Khristianovic and Zheltov [43]. Spence and Sharp [41] initiated the work on self-similar solutions and asymptotic analyses of the behavior of the solutions of (1) near the tip of the fracture (i.e. the boundary of the support of u). There is now an abundant literature that has extended this formal analysis to various regimes (see for instance [1], [2], [33] and reference therein). Several numerical methods have also been developed for this model (see in particular Peirce et al. [36], [37], [39] and [38]). However, to our knowledge, there are no rigorous existence results for general initial data. This paper is thus a first step toward a rigorous analysis of (1).

From a mathematical point of view, the equation under consideration:

$$u_t + (u^n I(u)_x)_x = 0 \tag{2}$$

is a non-local parabolic degenerate equation of order 3. It is closely related to the thin film equation which corresponds to the case $I = \partial_{xx}$:

$$u_t + (u^n u_{xxx})_x = 0 \tag{3}$$

(note that the porous media equation corresponds to the case $I(u) = u$). In particular, like the thin-film equation, Equation (2) lacks a comparison principle, and the existence of a non-negative solution (for non-negative initial data) is thus non-trivial (it is well known that non-negative initial data may generate changing sign solutions of the fourth order equation $\partial_t h + \partial_{xxxx} h = 0$).

However, compared with (3), the analysis of (2) presents some additional difficulties: First, the operator I is non-local and the algebra is not as simple as with the Laplace operator. Second, because of the lower order of the operator I, the natural regularity given by the energy inequality ($u \in H^{\frac{1}{2}}$ rather than $u \in H^1$) does not give the boundedness and continuity of weak solutions even in dimension 1.

A remarkable feature of (2) and (3) is that the degeneracy of the diffusion coefficient permits the existence of non-negative solutions. In the case of the thin film equation (3), the existence of non-negative weak solutions was first addressed by F. Bernis and A. Friedman [11] for $n > 1$. Further results were later obtained, by similar technics, in particular by E. Beretta, M. Bertsch and R. Dal Passo [5] and A. Bertozzi and M. Pugh [13, 14]. Results in higher dimension were obtained more recently (see [28, 27, 20]).

As in the case of the thin film equation, our approach to prove the existence of solutions for (1) relies on a regularization-stability argument, and the main tools are integral inequalities which we present now.

Integral inequalities. Besides the conservation of mass, the solutions of (2) satisfy two important inequalities (that have a counterpart for the thin film equation): Assuming that we have $\Omega = \mathbb{R}$ and $I = -(-\Delta)^{1/2}$ (for the time being), we can indeed easily show that the solutions of (2) satisfy the energy inequality

$$-\int_\Omega u(t)I(u(t))\,dx + \int_0^T \int_\Omega u^n (I(u)_x)^2\,dx\,dt \leq -\int_\Omega u_0 I(u_0)\,dx \tag{4}$$

(where $-\int uI(u)\,dx$ is the homogeneous $H^{1/2}$ norm) and an entropy like inequality

$$\int_\Omega G(u(t))\,dx - \int_0^T \int_\Omega u_x I(u)_x\,dx\,dt = \int_\Omega G(u_0)\,dx \tag{5}$$

where $G''(s) = \frac{1}{s^n}$.

The energy inequality (4) controls the $L^\infty(0,T;H^{1/2}(\Omega))$ norm of the solutions (instead of $L^\infty(0,T;H^1(\Omega))$ for the thin film equation). We see here that the order $1/2$ for the operator I is critical in dimension 1 in the sense that we are just short of a $L^\infty((0,T)\times\Omega)$ estimate and continuity of the solutions. Because of that fact, many of the arguments used in the analysis of the thin film equation do not apply directly to our case.

Next, we observe that as in the case of the thin film equation, the entropy inequality (5) provides some control on some negative power of u for $n > 2$. Indeed, we can take

$$G(s) = \int_1^s \int_1^r \frac{1}{t^n}\,dt\,dr$$

so that G is a nonnegative convex function satisfying $G'(1) = 0$ and $G(1) = 0$. This yields:

$$G(s) = \begin{cases} s\ln s - s + 1 & \text{when } n = 1 \\[2ex] -\dfrac{s^{2-n}}{(2-n)(n-1)} + \dfrac{s}{n-1} + \dfrac{1}{2-n} & \text{when } 1 < n < 2 \\[2ex] \ln\dfrac{1}{s} + s - 1 & \text{when } n = 2 \\[2ex] \dfrac{1}{(n-2)(n-1)}\dfrac{1}{s^{n-2}} + \dfrac{s}{n-1} - \dfrac{1}{n-2} & \text{when } n > 2. \end{cases} \tag{6}$$

Note that $G(0) = +\infty$ when $n \geq 2$, while G is bounded in a neighborhood of 0 for $n \in [1, 2)$. This will be key in proving the non-negativity of the solution. The entropy equality also gives some control on the $L^2(0, T; H^{3/2}(\Omega))$ norm of the solution which will be crucial in getting the necessary compactness in the construction of the solution. However, in order to make use of this inequality, we need $\int_\Omega G(u_0)\, dx$ to be finite, which, when $n \geq 2$ prohibits compactly supported initial data.

Besides those two inequalities, there are several other integral estimates that have proved extremely useful in the study of the thin film equation. The simplest one are local versions of (4) and (5). However, because of the nonlocal character of the operator I, it seems very delicate to establish similar inequalities for (2).

Another crucial estimate in the analysis of (3), established by Bernis [10], is the following:

$$\int (u^{\frac{n+2}{2}}_{xxx})^2\, dx \leq C \int u^n u_{xxx}^2\, dx$$

for $n \in (\frac{1}{2}, 3)$. Such an inequality yields important estimates from the dissipation in the energy inequality (despite the degeneracy of the diffusion coefficient). Again it is not clear what would play the role of this inequality in our situation. The same remark holds for the so called α-entropy [11, 5, 14, 15]: for $\alpha \in (\max(-1, \frac{1}{2} - n), 2 - n)$, $\alpha \neq 0$, it can be proved that the solutions of the thin film equation (3) satisfy:

$$\frac{1}{\alpha+1} \int u^{\alpha+1}(\cdot, T)\, dx + C \int_0^T \int \left(|\partial_x u^{\frac{\alpha+n+1}{4}}|^4 + |\partial_{xx} u^{\frac{\alpha+n+1}{2}}|^2 \right) dx\, dt \leq \frac{1}{\alpha+1} \int u_0^{\alpha+1}\, dx.$$

These last two inequalities are essential in establishing many qualitative properties of the solutions, such as finite speed expansion of the support and waiting time phenomenon. Though we expect such properties to hold for (2) as well, it is not clear at this point how to deal with the non local character of I.

Finally, let us comment on the power of the diffusion coefficient u^n. Interestingly, the power $n = 3$, which is the physically relevant power in our model, is critical in many results for the thin film equation. In particular many existence and regularity results (as well as waiting time results) are only valid for $n \in (0, 3)$. It is actually believed that for $n \geq 3$, (and it is proved for $n \geq 4$) the support of the solutions of (3) does not expand. It is not clear what would be the critical exponent for (2), though numerical results suggest that for $n = 3$, the support of the solutions of (2) does expand for all time.

Main results. A weak formulation of (2) is given by

$$\iint_Q u\, \partial_t \varphi\, dx\, dt + \iint_Q u^n \partial_x I(u)\, \partial_x \varphi\, dx\, dt = -\int_\Omega u_0 \varphi(0, \cdot)\, dx$$

for all $\varphi \in \mathcal{D}(\overline{Q})$ where Q denotes $\Omega\times(0,T)$. However, because of the degeneracy of the coefficient u^n, it is difficult to give a meaning to the term $u^n\partial_x I(u)$. We thus perform an additional integration by parts to get the following weak formulation of (2):

$$\iint_Q u\,\partial_t\varphi\,dx\,dt - \iint_Q nu^{n-1}\partial_x u\,I(u)\,\partial_x\varphi\,dx\,dt - \iint_Q u^n\,I(u)\,\partial_{xx}\varphi\,dx\,dt = -\int_\Omega u_0\varphi(0,\cdot)\,dx \quad (7)$$

for all $\varphi \in \mathcal{D}(\overline{Q})$ satisfying $\partial_x\varphi|_{\partial\Omega} = 0$.

We are going to prove the following existence theorem:

Theorem 1. *Assume $n \geq 1$. For any non-negative initial condition $u_0 \in H^{\frac{1}{2}}(\Omega)$ such that*

$$\int_\Omega G(u_0)\,dx < \infty, \quad (8)$$

there exists a non-negative function $u \in L^\infty(0,T,H^{\frac{1}{2}}(\Omega))$ such that

$$u \in L^2(0,T,H^{\frac{3}{2}}_N(\Omega)) \quad (9)$$

which satisfies (7) for all $\varphi \in \mathcal{D}(\overline{Q})$ satisfying $\partial_x\varphi|_{\partial\Omega} = 0$.
Furthermore u satisfies, for almost every $t \in (0,T)$

$$\int_\Omega u(t,\cdot)\,dx = \int_\Omega u_0\,dx, \quad (10)$$

$$\|u(t,\cdot)\|^2_{H^{\frac{1}{2}}(\Omega)} + 2\int_0^t\int_\Omega g^2\,dx\,ds \leq \|u_0\|^2_{H^{\frac{1}{2}}(\Omega)}, \quad (11)$$

where the function $g \in L^2(Q)$ satisfies $g = \partial_x(u^{\frac{n}{2}}I(u)) - \frac{n}{2}u^{\frac{n-2}{2}}\partial_x u\;I(u)$ in $\mathcal{D}'(\Omega)$, and

$$\int_\Omega G(u(x,t))\,dx + ||u||^2_{L^2(0,t;\dot{H}^{\frac{3}{2}}_N(\Omega))} \leq \int_\Omega G(u_0(x))\,dx. \quad (12)$$

We recall that the function $G : (0,\infty) \to \mathbb{R}_+$ is given by (6). The space $H^{\frac{3}{2}}_N(\Omega)$ appearing in (9) will be defined precisely in Section 3. In particular, the following characterization will be given:

$$H^{\frac{3}{2}}_N(\Omega) = \left\{u \in H^{\frac{3}{2}}(\Omega)\,;\;\int_\Omega \frac{u_x^2}{d(x)}\,dx < \infty\right\}$$

where $d(x)$ denotes the distance to $\partial\Omega$. Condition (9) thus implies that u satisfies $u_x = 0$ on $\partial\Omega$ in some weak sense.

Note that at least formally, we have $g = u^{\frac{n}{2}}\partial_x I(u)$ (though we do not have enough regularity on u to give a meaning to this product in general). Finally, we point out that we have $H_N^{\frac{3}{2}}(\Omega) \subset W^{1,p}(\Omega)$ for all $p < \infty$ and so every terms in (7) makes sense.

For $n \geq 2$, condition (8) requires in particular that $\mathrm{Supp}(u_0) = \Omega$ and inequality (12) implies that this remains true for all positive time. This is a serious restriction since the case of compactly supported initial data is physically the most interesting (see Section 2). We hope to be able to get rid of this assumption in a further work.

For $n > 3$, we can actually show that condition (8) requires $u(\cdot, t)$ to be strictly positive for a.e. $t \in (0, T)$. In fact, we can prove:

Theorem 2. *When $n > 3$, there exists a set $P \subset (0, T)$ such that $|(0, T) \setminus P| = 0$ and the solution u given by Theorem 1 satisfies*

$$u(\cdot, t) \in \mathcal{C}^\alpha(\Omega) \text{ for all } t \in P \text{ and for all } \alpha < 1,$$

and $u(\cdot, t)$ is strictly positive in Ω. Finally, u solves

$$\partial_t u + \partial_x J = 0 \qquad \text{in } \mathcal{D}'(\Omega),$$

where

$$J(\cdot, t) = u^n \partial_x I(u) \in L^1(\Omega) \quad \text{for all } t \in P.$$

Organization of the article. The paper is organized as follows: In Section 2, we give a brief description of the mathematical modeling of hydraulic fracture which gives rise to equation (2) with $n = 3$. In Section 3, we introduce the functional analysis tools that will be needed to prove Theorem 1. In particular, the non-local operator $I(u)$ is defined, first using a spectral decomposition, then as a Dirichlet-to-Neuman map. An integral representation for I, using the periodic Hilbert transform is also given. Section 4 is devoted to the study of a regularized equation while the proof of Theorem 1 is given in Sections 5 (for the case $n \geq 2$) and 6 (for the case $n \in [1, 2)$). Theorem 2 is proved in Section 7.

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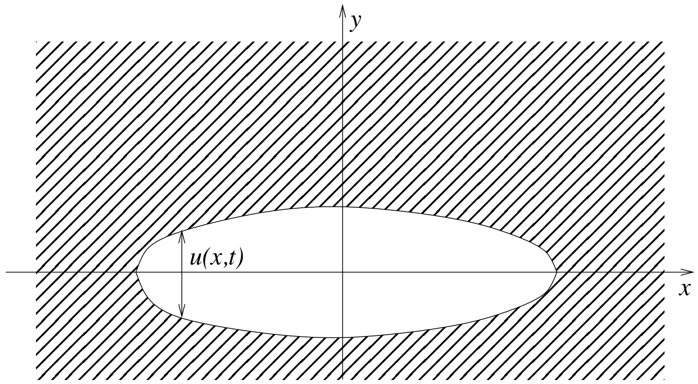
2 The physical model

When $n = 3$, Equation (2) can be used to model the propagation of an impermeable KGD fracture driven by a viscous fluid in a uniform elastic medium under condition of plane strain. More precisely, denoting by (x, y, z) the standard coordinates in $\mathbb{R}^3$, we consider a fracture which is invariant with respect



to one variable (say z) and symmetric with respect to another direction (say y). The fracture can then be entirely described by its opening $u(x,t)$ in the y direction (see Figure 2). Since it assumes that the fracture is an infinite strip whose cross-sections are in a state of plane strain, this model is only applicable to rectangular planar fracture with large aspect ratio.

We now briefly describe the main steps of the derivation of (1).

2.1 Conservation of mass and Poiseuille law

The conservation of mass for the fluid inside the fracture, averaged with respect to y yields:

$$\partial_t(\rho u) + \partial_x q = 0 \quad \text{in } \mathbb{R} \tag{13}$$

where ρ is the density of the fluid (which is assumed to be constant) and $q = q(x,t)$ denotes the fluid flux. This flux is given by

$$q = \rho u \overline{v} \tag{14}$$

where $\overline{v}$ is the y-averaged horizontal component of the velocity of the fluid

$$\overline{v} = \frac{1}{u} \int_{-u/2}^{u/2} v_H(t,x,y)\, dy.$$

Under the lubrification approximation, Navier-Stokes equations reduce to

$$\mu \frac{\partial^2 v_H}{\partial y^2}(t,x,y) = \partial_x p(x,t)$$

where p denotes the pressure of the fluid at a point x and μ denotes the viscosity coefficient. Assuming a no-slip boundary condition $v = 0$ at $y = \pm u/2$, we deduce

$$v_H(t,x,y) = \frac{1}{\mu} \partial_x p \left[\frac{1}{2} y^2 - \frac{1}{8} u^2 \right] \quad \text{for } -u \le y \le u$$

and so

$$\overline{v}(x,t) = -\frac{u^2}{12\mu}\partial_x p(x,t)$$

Using (14), we deduce *Poiseuille law*

$$q = -\frac{u^3}{12\mu}\partial_x p\,. \tag{15}$$

Together with (13), this implies

$$\partial_t u - \partial_x\left(\frac{u^3}{12\mu}\partial_x p\right) = 0.$$

In order to obtain (2), it only remains to express the pressure p as a function of the displacement u (i.e. $p = -I(u)$).

2.2 The pressure law

For a state of plane strain, the elasticity equation expresses the pressure as a function of the fracture opening. After a rather involved computation, one can derive the following nonlocal expression (see [19]):

$$p(x,t) = -\frac{E'}{4\pi}\int_{\mathbb{R}}\frac{\partial_x u(z,t)}{z-x}\,dz$$

where E' denotes Young's modulus. Denoting by $\mathcal{H}$ the Hilbert transform, we can rewrite this formula as

$$p(x,t) = \frac{E'}{4}\mathcal{H}(\partial_x u) = \frac{E'}{4}(-\Delta)^{1/2}u(x,t)$$

where $(-\Delta)^{1/2}$ is the half-Laplace operator, defined, for instance, using the Fourier transform by $\mathcal{F}((-\Delta)^{1/2}u) = |\xi|\mathcal{F}(u)$.

It is well known that the half Laplace operator can also be defined as a Dirichlet-to-Neumann map for the harmonic extension. More precisely, the pressure is given by

$$p(x) = \frac{E'}{4}\partial_y v(x,0) \tag{16}$$

where v solves

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}\times(0,+\infty), \\ v(x,0) = u(x,t), & \text{on } \mathbb{R}\,. \end{cases} \tag{17}$$

By taking advantage of the symmetry of the problem, the function $v(x,y)$ can be interpreted as the displacement of the rock. Denoting $I(u) = -(-\Delta)^{1/2}u$, we deduce $p = -\frac{E'}{4}I(u)$ and so

$$\partial_t u + \frac{E'}{48\mu}\partial_x\left(u^3\partial_x I(u)\right) = 0.$$

A technical assumption. In order to reduce the technicality of the analysis, we will assume that the crack is periodic with respect to x. Since we expect compactly supported initial data to give rise to compactly supported solutions whose supports expand with finite speed, this is a reasonnable physical assumption. We also assume that the initial crack is even with respect to $x = 0$ and we look for solutions that are also even.

By making such assumptions (periodicity and evenness), we can replace (17) with the following boundary value problem

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega \times (0, \infty) \\ v_\nu = 0 & \text{on } \partial\Omega \times (0, \infty) \\ v(x, 0) = u(x) & \text{on } \Omega \end{cases}$$

with $\Omega = (0, 1)$ if the period of the initial crack is 2. The cylinder $\Omega \times (0, +\infty)$ is denoted by C in the remaining of the paper.

Mathematical definition of the pressure. It turns out that it is easier to define first the operator I by using the spectral decomposition of the Laplace operator: We take $\{\lambda_k, \varphi_k\}$ the eigenvalues and corresponding eigenvectors of the Laplacian operator in Ω with Neumann boundary conditions on $\partial\Omega$:

$$\begin{cases} -\Delta \varphi_k = \lambda_k \varphi_k & \text{in } \Omega \\ \partial_\nu \varphi_k = 0 & \text{on } \partial\Omega. \end{cases}$$

We then define the operator I by

$$I(u) := \sum_{k=0}^{\infty} c_k \varphi_k(x) \mapsto -\sum_{k=0}^{\infty} c_k \lambda_k^{\frac{1}{2}} \varphi_k(x)$$

which maps $H^1(\Omega)$ onto $L^2(\Omega)$. We will prove that this operator can be characterized as a Dirichlet-to-Neumann map (see Proposition 2) and that it has an integral representation as well (see Proposition 3).

2.3 Boundary conditions

The opening u is solution of (2) in its support. The model must thus be supplemented with some *boundary conditions* at the tip of the fracture.

Assuming that $\mathrm{Supp}(u(t, \cdot)) = [-\ell(t), \ell(t)]$, it is usually assumed that

$$u = 0, \qquad u^3 \partial_x p = 0 \qquad \text{at } x = \pm\ell(t)$$

which ensures zero width and zero fluid loss at the tip.

We point out that the model is a free boundary problem, since the support $[-\ell(t), \ell(t)]$ of the fracture is not known a priori. Since the equation is of order 3, those two conditions are not enough to fully determine a solution. In fact, there should be an additional condition which takes into account the energy required to break the rock at the tip of the crack. Consistent with linear elastic fracture propagation, we can assume that the rock toughness K_{Ic} equals the stress intensity factor K_I. When the crack propagation is determined by the toughness of the rock, a formal asymptotic analysis of fracture profile at the tip (see [1, 23]) then shows that

$$u(x,t) \sim \frac{K'}{E'}\sqrt{\ell(t)-x} \qquad \text{as } x \to \ell(t) \tag{18}$$

with $K' = 4\sqrt{\frac{2}{\pi}}K_{Ic}$ (and a similar condition for $x \to -\ell$). One can now take this condition on the profile of u at the tip as the missing free boundary condition. The resulting free boundary problem is clearly very delicate to study (remember that (1) is a third order degenerate non-local parabolic equation).

A particular case which is simpler and still interesting is the case of zero toughness ($K_{Ic} = 0$). This is relevant mainly if there is a pre-fracture (i.e. the rock is already cracked, even though $u = 0$ outside the initial fracture). Mathematically speaking, this means that Equation (1) is now satisfied everywhere in $\mathbb{R}$ even though u is expected to have compact support. No free boundary conditions are necessary. This is the problem that we are considering in this paper.

Note that one should then have $\lim_{x\to\ell}(\ell(t)-x)^{-1/2}u(x,t) = 0$ at the tip of the crack. In fact, formal arguments show that the asymptotic behavior of the fracture opening near the fracture tip should be proportional to $(\ell(t)-x)^{2/3}$ (see [1, 23]).

This approach is very similar to what is usually done with the porous media equation, and it has been used successfully in the case of the thin film equation to prove the existence of solutions with zero contact angle (in that case, we speak of precursor film, or pre-wetting). The study of the free boundary problem with free boundary condition (18) would be considerably more difficult (one would expect the gradient flow approach developed by F. Otto [35] for the thin film equation with non zero contact angle to yield some result when $n = 1$).

3 Preliminaries

In this section, we define the operator I and give the functional analysis results that will play an important role in the proof of the main theorem. A very similar operator, with Dirichlet boundary conditions rather than Neumann boundary conditions, was studied by Cabré and Tan [17]. This section follows their analysis very closely.

3.1 Functional spaces

The space $H^s_N(\Omega)$. We denote by $\{\lambda_k, \varphi_k\}_{k=0,1,2...}$ the eigenvalues and corresponding eigenfunctions of the Laplace operator in Ω with Neumann boundary conditions on $\partial\Omega$:

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k & \text{in } \Omega \\ \partial_\nu\varphi_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

normalized so that $\int_\Omega \varphi_k^2\, dx = 1$. When $\Omega = (0, 1)$, we have

$$\lambda_0 = 0\,, \qquad \varphi_0(x) = 1$$

and

$$\lambda_k = (k\pi)^2\,, \qquad \varphi_k(x) = \sqrt{2}\cos(k\pi x) \qquad k = 1,\, 2,\, 3,\, \ldots$$

The φ_k's clearly form an orthonormal basis of $L^2(\Omega)$. Furthermore, the φ_k's also form an orthogonal basis of the space $H^s_N(\Omega)$ defined by

$$H^s_N(\Omega) = \left\{ u = \sum_{k=0}^{\infty} c_k \varphi_k\,;\, \sum_{k=0}^{\infty} c_k^2(1 + \lambda_k^s) < +\infty \right\}$$

equipped with the norm

$$||u||^2_{H^s_N(\Omega)} = \sum_{k=0}^{\infty} c_k^2(1 + \lambda_k^s)$$

or equivalently (noting that $c_0 = \|u\|_{L^1(\Omega)}$ and $\lambda_k \geq 1$ for $k \geq 1$):

$$||u||^2_{H^s_N(\Omega)} = \|u\|^2_{L^1(\Omega)} + \|u\|^2_{\dot{H}^s_N(\Omega)}$$

where the homogeneous norm is given by:

$$\|u\|^2_{\dot{H}^s_N(\Omega)} = \sum_{k=1}^{\infty} c_k^2 \lambda_k^s.$$

A characterisation of $H^s_N(\Omega)$. The precise description of the space $H^s_N(\Omega)$ is a classical problem.

Intuitively, for $s < 3/2$, the boundary condition $u_\nu = 0$ does not make sense, and one can show that (see Agranovich and Amosov [4] and references therein):

$$H^s_N(\Omega) = H^s(\Omega) \quad \text{ for all } 0 \leq s < \frac{3}{2}.$$

In particular, we have $H^{\frac{1}{2}}_N(\Omega) = H^{\frac{1}{2}}(\Omega)$ and we will see later that

$$\|u\|^2_{\dot{H}^{\frac{1}{2}}(\Omega)} = \int_\Omega \int_\Omega (u(y) - u(x))^2 \nu(x, y) dx dy$$

where $\nu(x, y)$ is a given positive function; see (23) below.

For $s > 3/2$, the Neumann condition has to be taken into account, and we have in particular

$$H^2_N(\Omega) = \{u \in H^2(\Omega)\,;\, u_\nu = 0 \text{ on } \partial\Omega\}$$

which will play a particular role in the sequel. More generally, a similar characterization holds for $3/2 < s < 7/2$. For $s > 7/2$, additional boundary conditions have to be taken into account.

The case $s = 3/2$ is critical (note that $u_\nu|_{\partial\Omega}$ is not well defined in that space) and one can show that

$$H_N^{\frac{3}{2}}(\Omega) \quad = \quad \left\{ u \in H^{\frac{3}{2}}(\Omega)\,;\, \int_\Omega \frac{u_x^2}{d(x)}\,dx < \infty \right\}$$

where $d(x)$ denotes the distance to $\partial\Omega$. A similar result appears in [17]; more precisely, such a characterization of $H_N^{\frac{3}{2}}(\Omega)$ can be obtained by considering functions u such that $u_x \in \mathcal{V}_0(\Omega)$ where $\mathcal{V}_0(\Omega)$ is defined in [17] as the equivalent of our space $H_N^{1/2}(\Omega)$ with Dirichlet rather than Neumann boundary conditions. We do not dwell on this issue since we will not need this result in our proof.

3.2 The operator I

As it is explained in the Introduction, the operator I is related to the computation of the pressure as a function of the displacement. From this point of view, the operator I is a Dirichlet-to-Neumann operator associated with the Laplacian. Since we study the problem in a periodic setting we explained that this yields to consider Neumann boundary conditions on a cylinder $C = \Omega \times (0, +\infty)$.

Spectral definition. It is convenient to begin with the spectral definition of the operator I: With λ_k and φ_k defined by (19), we define the operator

$$I : \sum_{k=0}^{\infty} c_k \varphi_k \longmapsto -\sum_{k=0}^{\infty} c_k \lambda_k^{\frac{1}{2}} \varphi_k \tag{20}$$

which clearly maps $H^1(\Omega)$ onto $L^2(\Omega)$ and $H_N^2(\Omega)$ onto $H^1(\Omega)$.

Dirichlet-to-Neuman map. With the spectral definition in hand, we are now going to show that I can also be defined as the Dirichlet-to-Neumann operator associated with the Laplace operator supplemented with Neumann boundary conditions.

To be more precise, we consider the following boundary problem in the cylinder $C = \Omega \times (0, +\infty)$:

$$\begin{cases} -\Delta v = 0 & \text{in } C, \\ v(x,0) = u(x) & \text{on } \Omega, \\ v_\nu = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{21}$$

We will show that we have

$$I(u) = \partial_y v(\cdot, 0).$$

We start with the following result which show the existence of a unique harmonic extension v:

Proposition 1 (Existence and uniqueness for (21))**.** *For all $u \in H_N^{\frac{1}{2}}(\Omega)$, there exists a unique extension $v \in H^1(C)$ solution of (21).*

Furthermore, if $u(x) = \sum_{k=1}^{\infty} c_k \varphi_k(x)$, then

$$v(x,y) = \sum_{k=1}^{\infty} c_k \varphi_k(x) \exp(-\lambda_k^{\frac{1}{2}} y). \tag{22}$$

Proof. We recall that $H_N^{\frac{1}{2}}(\Omega) = H^{\frac{1}{2}}(\Omega)$, and for a given $u \in H^{\frac{1}{2}}(\Omega)$ we consider the following minimization problem:

$$\inf \left\{ \int_C |\nabla w|^2 \, dx \, dy \, ; \; w \in H^1(C) \, , \; w(\cdot, 0) = u \text{ on } \Omega \right\} .$$

Using classical arguments, one can show that this problem has a unique minimizer v (note that the set of functions on which we minimize the functional is not empty). This minimizer is a weak solution of (21). In particular, it satisfies

$$\int_C \nabla v \cdot \nabla w \, dx \, dy = 0$$

for all $w \in H^1(\Omega)$ such that $w(\cdot, 0) = 0$ on Ω, which includes a weak formulation of the Neumann condition.

Finally, the representation formula (22) follows from a straightforward computation. Indeed, we have

$$\begin{aligned}
\int_0^{\infty} \int_{\Omega} |\nabla v|^2 \, dx \, dy &= \int_0^{\infty} \int_{\Omega} |\partial_x v|^2 + |\partial_y v|^2 \, dx \, dy \\
&= 2 \sum_{k=1}^{\infty} c_k^2 \lambda_k \int_0^{\infty} \exp(-2\lambda_k^{1/2} y) \, dy \\
&= 2 \sum_{k=1}^{\infty} c_k^2 \lambda_k \frac{1}{2\lambda_k^{1/2}} \\
&= \sum_{k=1}^{\infty} c_k^2 \lambda_k^{1/2} = ||u||^2_{\dot{H}^{\frac{1}{2}}(\Omega)}
\end{aligned}$$

which shows that v belongs to $H^1(C)$. The fact that v satisfies (21) is easy to check. $\square$

We can now show:

Proposition 2 (The operator I is of Dirichlet-to-Neumann type)**.** *For all $u \in H_N^2(\Omega)$, we have*

$$I(u)(x) = -\frac{\partial v}{\partial \nu}(x, 0) = \partial_y v(x, 0) \quad \text{for all } x \in \Omega,$$

where v is the unique harmonic extension solution of (21).

Furthermore $I \circ I(u) = -\Delta u$.

Proof. This follows from a direct computation using (22). Furthermore, if u is in $H^2_N(\Omega)$, we know that $\sum_{k=0}^{\infty} c_k^2 \lambda_k^2 < \infty$. It is now easy to derive the following equality

$$I(I(u)) = \sum_{k=0}^{\infty} c_k \lambda_k \varphi_k(x) = -\Delta u.$$

□

Integral representation. The operator I can also be represented as a singular integral operator. Indeed, we will prove the following

Proposition 3. *Consider a smooth function $u : \Omega \to \mathbb{R}$. Then for all $x \in \Omega$,*

$$I(u)(x) = \int_\Omega (u(y) - u(x))\nu(x, y)dy$$

where $\nu(x, y)$ is defined as follows: for all $x, y \in \Omega$,

$$\nu(x, y) = \frac{\pi}{2}\left(\frac{1}{1 - \cos(\pi(x - y))} + \frac{1}{1 - \cos(\pi(x + y))}\right). \tag{23}$$

Proof. We use the Dirichlet-to-Neumann definition of I. Let v denote the solution of (21). Then v is the restriction to $(0, 1)$ of the unique solution w of (21) where Ω is replaced with $(-1, 1)$ and u is replaced by its even extension to $(-1, 1)$. In particular, w is even with respect to x. Then there exists a holomorphic function W defined in the cylinder $(-1, 1)\times(0, +\infty)$ such that $w = \mathrm{Re}(W)$. Next, we consider the holomorphic function $z \mapsto e^{i\pi z} = e^{-\pi y}e^{i\pi x}$ defined on the cylinder $(-1, 1)\times(0, +\infty)$ with values into the unit disk $D_1 = \{(x, y) : x^2+y^2 < 1\}$. If z denotes the complex variable $x + iy$, then a new holomorphic function W_0 is obtained by the following formula

$$W(z) = W_0(e^{i\pi z}).$$

In particular, W_0 is defined and harmonic in D_1. This implies that the function W_0 can be represented by the Poisson integral. Precisely,

$$W_0(Z) = \frac{1 - |Z|^2}{2\pi}\int_{\partial D_1} \frac{W_0(Y)}{|Y - Z|^2} d\sigma(Y).$$

This implies that for all $z \in C$,

$$W(z) = \frac{1 - e^{-2\pi y}}{2\pi}\int_{-1}^{1} \frac{W(\theta)}{|e^{i\pi\theta} - e^{-\pi y}e^{i\pi x}|^2}\pi d\theta$$

and we finally obtain

$$w(x, y) = \frac{1 - e^{-2\pi y}}{2}\int_{-1}^{1} \frac{w(\theta, 0)}{|e^{i\pi\theta} - e^{-\pi y}e^{i\pi x}|^2} d\theta.$$

Taking $w = 1$, we get in particular the following equality:

$$1 = \frac{1 - e^{-2\pi y}}{2} \int_{-1}^{1} \frac{1}{|e^{i\pi\theta} - e^{-\pi y} e^{i\pi x}|^2} d\theta.$$

We deduce:

$$\frac{w(x,y) - w(x,0)}{y} = \frac{1 - e^{-2\pi y}}{2y} \int_{-1}^{1} \frac{w(\theta,0) - w(x,0)}{|e^{i\pi\theta} - e^{-\pi y} e^{i\pi x}|^2} d\theta$$

which implies (letting y go to zero):

$$\partial_y w(x,0) = \pi \int_{-1}^{1} \frac{w(\theta,0) - w(x,0)}{|e^{i\pi\theta} - e^{i\pi x}|^2} d\theta.$$

The integral on the right hand side of the previous equality is understood in the sense of the principal value of the associated distribution. We finally use the fact that w is even in x and is equal to u on Ω to obtain the following singular integral representation of $I(u)$:

$$I(u)(x) = \pi \int_0^1 (u(\theta,0) - u(x,0)) \left(\frac{1}{|1 - e^{i\pi(x-\theta)}|^2} + \frac{1}{|1 - e^{i\pi(x+\theta)}|^2} \right) d\theta.$$

$\square$

The space $H^{-\frac{1}{2}}(\Omega)$. The space $H^{-\frac{1}{2}}(\Omega)$ is defined as the topological dual space of $H^{\frac{1}{2}}(\Omega)$. It is classical that for any $u \in H^{-\frac{1}{2}}(\Omega)$, there exists $u_1 \in L^2(\Omega)$ and $u_2 \in H^{\frac{1}{2}}(\Omega)$ such that $u = u_1 + \partial_x u_2$ (in the sense of distributions). We will also use repeatedly the following elementary lemma, whose proof is left to the reader:

Lemma 1. *If $u \in H^{\frac{1}{2}}(\Omega)$, then the distribution $I(u)$ is in $H^{-\frac{1}{2}}(\Omega)$ and for all $v \in H^{\frac{1}{2}}(\Omega)$,*

$$\langle I(u), v \rangle_{H^{-\frac{1}{2}}(\Omega), H^{\frac{1}{2}}(\Omega)} = -\sum_{k=0}^{+\infty} \lambda_k^{\frac{1}{2}} c_k d_k$$

where $u = \sum_{k=0}^{+\infty} c_k \varphi_k$ and $v = \sum_{k=0}^{+\infty} d_k \varphi_k$. In particular,

$$-\langle I(u), u \rangle_{H^{-\frac{1}{2}}(\Omega), H^{\frac{1}{2}}(\Omega)} = ||u||^2_{\dot{H}^{\frac{1}{2}}(\Omega)}.$$

Important equalities. The semi-norms $|| \cdot ||_{\dot{H}^{\frac{1}{2}}(\Omega)}$, $|| \cdot ||_{\dot{H}^1(\Omega)}$, $|| \cdot ||_{\dot{H}^{\frac{3}{2}}(\Omega)}$ and $|| \cdot ||_{\dot{H}^2_N(\Omega)}$ are related to the operator I by equalities which will be used repeatedly.

Proposition 4 (The operator I and several semi-norms)**.**

For all $u \in H^{\frac{1}{2}}(\Omega)$, we have

$$\frac{1}{2}\int_\Omega\int_\Omega (u(x)-u(y))^2\nu(x,y)dxdy = ||u||^2_{\dot{H}^{\frac{1}{2}}(\Omega)}.$$

For all $u \in H^1(\Omega)$, we have

$$\int_\Omega (I(u))^2 dx = \|u\|^2_{\dot{H}^1(\Omega)}.$$

For all $u \in H^2_N(\Omega)$, we have

$$-\int_\Omega I(u)_x u_x \, dx = ||u||^2_{\dot{H}^{\frac{3}{2}}_N(\Omega)}.$$

For all $u \in H^2_N(\Omega)$, we have

$$\int_\Omega (\partial_x I(u))^2 \, dx = \|u\|^2_{\dot{H}^2_N(\Omega)}.$$

Remark 1. Note that $I(u)_x \neq I(u_x)$.

Proof. The two first equalities are easily derived form the definition of I, definitions of the semi-norms, the integral representation of I and the fact that $\nu(x,y) = \nu(y,x)$.

In order to prove the third and fourth equalities, we first remark that $\partial_x\varphi_k = -\lambda_k^{\frac{1}{2}}\sin(k\pi x)$ form an orthogonal basis of $L^2(\Omega)$.

In order to prove the fourth equality, we first write

$$\partial_x(I(u)) = -\sum_{k=1}^{\infty} c_k\lambda_k^{\frac{1}{2}}\partial_x\varphi_k \quad \text{in } L^2(\Omega)$$

from which we deduce

$$\begin{aligned}
\int_\Omega (I(u)_x)^2 \, dx &= \sum_{k=1}^{\infty} c_k^2\lambda_k \int_\Omega (\partial_x\varphi_k)^2 \, dx \\
&= \sum_{k=1}^{\infty} c_k^2\lambda_k \int_\Omega \varphi_k(-\partial_{xx}\varphi_k) \, dx \\
&= \sum_{k=0}^{\infty} c_k^2\lambda_k^2 = ||u||^2_{\dot{H}^2_N}.
\end{aligned}$$

As far as the third equality is concerned, we note that

$$u_x = \sum_{k=0}^{\infty} c_k\partial_x\varphi_k \quad \text{in } L^2(\Omega).$$

We then have

$$\begin{aligned}
-\int_\Omega I(u)_x u_x\,dx &= \sum_{k=0}^{\infty} c_k^2 \lambda_k^{\frac{1}{2}} \int_\Omega (\partial_x \varphi_k)^2\,dx \\
&= -\sum_{k=0}^{\infty} c_k^2 \lambda_k^{\frac{1}{2}} \int_\Omega \varphi_k \partial_{xx}\varphi_k\,dx \\
&= \sum_{k=0}^{\infty} c_k^2 \lambda_k^{\frac{1}{2}} \int_\Omega \lambda_k \varphi_k^2\,dx \\
&= \sum_{k=0}^{\infty} c_k^2 \lambda_k^{\frac{3}{2}} = ||u||^2_{\dot{H}^{\frac{3}{2}}(\Omega)}.
\end{aligned}$$

□

3.3 The problem $-I(u) = g$

We conclude this preliminary section by giving a few results about the following problem:

$$\begin{gathered}
\text{For a given } g \in L^2(\Omega), \text{ find } u \in H^1(\Omega) \text{ such that} \\
-I(u) = g.
\end{gathered} \tag{24}$$

Note that $\int_\Omega I(u)\,dx = 0$ for all $u \in H^1(\Omega)$ (since $\int_\Omega \varphi_k\,dx = 0$ for all $k \geq 1$) and so a necessary condition for the existence of a solution to (24) is

$$\int_\Omega g(x)\,dx = 0.$$

Note also that there is no uniqueness since if u is a solution then $u + C$ is also a solution for any constant C. We may however expect a unique solution if we add the further constraint $\int u\,dx = 0$. Indeed, a weak solution $u \in H^{\frac{1}{2}}(\Omega)$ for $g \in H^{-\frac{1}{2}}(\Omega)$ can be found using Lax-Milgram theorem in $\{u \in H^{\frac{1}{2}}(\Omega)\,;\, \int_\Omega u\,dx = 0\}$ equipped with the norm $||u||_{\dot{H}^{\frac{1}{2}}(\Omega)}$. Alternatively, we can use the spectral framework: For $g \in L^2(\Omega)$ such that $\int_\Omega g(x)\,dx = 0$, we have

$$g = \sum_{k=1}^{\infty} d_k \varphi_k \quad \text{with} \quad \sum_{k=1}^{\infty} d_k^2 < \infty.$$

We can then write:

$$u = I^{-1}(g) := \sum_{k=1}^{\infty} \frac{d_k}{\lambda_k^{\frac{1}{2}}} \varphi_k \tag{25}$$

which clearly lies in $H^1(\Omega)$ and satisfies $\int_\Omega u\,dx = 0$. The fact that the φ_k's form an orthogonal basis of $L^2(\Omega)$ implies that there is only one solution of (24) such that $\int_\Omega u\,dx = 0$. Finally it is clear from (25) that further regularity on g will imply further regularity on u. We sum up this discussion in the following statement

Theorem 3. *For all $g \in L^2(\Omega)$ such that $\int_\Omega g\,dx = 0$, there exists a unique function $u \in H^1(\Omega)$ such that $-I(u) = g$ in $L^2(\Omega)$ and $\int_\Omega u\,dx = 0$. Furthermore, if g is in $H^1(\Omega)$, then $u \in H^2_N(\Omega)$.*

We will also use the following corollary of the previous theorem

Corollary 1. *For all $g \in L^2(\Omega)$, there exists a unique solution $u \in H^1(\Omega)$ of*

$$-I(v) + \int_\Omega v\,dx = g.$$

Proof. Set $m = \int_\Omega g(x)\,dx$ and consider $\tilde{g} = g - m$. Then $\tilde{g} \in L^2(\Omega)$ and $\int_\Omega \tilde{g}dx = 0$. There is a (unique) $u \in H^1(\Omega)$ such that

$$-I(u) = g - m, \quad \int_\Omega u(x)\,dx = 0.$$

We then set $v = u + m$. Then $\int_\Omega v\,dx = m$ and

$$-I(v) = -I(u) = g - m = g - \int_\Omega v\,dx.$$

As far as uniqueness is concerned, if we consider two solutions v_1 and v_2 then we have

$$\int_\Omega v_1 dx = \int_\Omega v_2 dx = \int_\Omega g$$

and this implies that $w = v_1 - v_2$ satisfies $-I(w) = 0$. The uniqueness of the solution given by Theorem 3 implies that $w = 0$ and the proof is complete. □

4 A regularized problem

We now turn to the proof of Theorem 1. The degeneracy of the diffusion coefficient is a major obstacle to the development of a variational argument. As in [11], the existence of solution for (2) is thus obtained via a regularization approach: Given $\varepsilon > 0$, we consider

$$\partial_t u + \partial_x(f_\varepsilon(u)\partial_x I(u)) = 0, \quad t \in (0,T), x \in \Omega \tag{26}$$

where

$$f_\varepsilon(s) = {s_+}^n + \varepsilon$$

(with $s_+ = \max(s, 0)$), with the initial condition

$$u(0,x) = u_0(x) \tag{27}$$

and boundary conditions

$$u_x = 0\,, \quad f_\varepsilon(u)\partial_x(I(u)) = 0 \quad \text{ on } \partial\Omega.$$

The first step in the proof of Theorem 1, is to prove the following proposition:

Proposition 5 (Existence of solution for the regularized problem)**.** *For all $u_0 \in H^{\frac{1}{2}}(\Omega)$ and for all $T > 0$, there exists a unique function u^ε such that*

$$u^\varepsilon \in L^\infty(0,T;H^{\frac{1}{2}}(\Omega)) \cap L^2(0,T;H^2_N(\Omega))$$

solution of

$$\iint_Q u^\varepsilon \partial_t \varphi \, dx\, dt + \iint_Q f_\varepsilon(u^\varepsilon)\partial_x I(u^\varepsilon)\partial_x \varphi \, dx\, dt = -\int_\Omega u_0 \varphi(0,\cdot)\, dx \tag{28}$$

for all $\varphi \in \mathcal{C}^1_c([0,T), H^1(\Omega))$ with $Q = \Omega \times (0,T)$.

Moreover, the function u^ε satisfies

$$\int_\Omega u^\varepsilon(x,t)\, dx = \int_\Omega u_0(x)\, dx \qquad a.e.\ t \in (0,T) \tag{29}$$

and

$$\|u^\varepsilon(t,\cdot)\|^2_{\dot H^{\frac{1}{2}}(\Omega)} + 2\int_0^t \int_\Omega f_\varepsilon(u^\varepsilon)(\partial_x I(u^\varepsilon))^2 \, dx\, ds \le \|u_0\|^2_{\dot H^{\frac{1}{2}}(\Omega)} \quad a.e.\ t \in (0,T). \tag{30}$$

Finally, if G_ε is a non-negative function such that $G''_\varepsilon(s) = \frac{1}{f_\varepsilon(s)}$, then u^ε satisfies for almost every $t \in (0,T)$

$$\int_\Omega G_\varepsilon(u^\varepsilon)(x,t)\, dx + \int_0^t \| \ u^\varepsilon(s)\|^2_{\dot H^{\frac{3}{2}}_N(\Omega)}\, ds \le \int_\Omega G_\varepsilon(u_0)\, dx. \tag{31}$$

Remark 2. Note that this result does not require condition (8) to be satisfied and is thus valid with compactly supported initial data. However, we will need condition (8) to get enough compactness on u^ε to pass to the limit $\varepsilon \to 0$ and complete the proof of Theorem 1.

There are several possible approaches to prove Proposition 5. In the next sections, we present a proof based on a time discretization of (28) and fairly classical monotonicity method (though the operator here is not monotone, but only pseudo-monotone).

4.1 Stationary problem

In order to prove Proposition 5, we first consider the following stationary problem (for $\tau > 0$):

For a given $g \in H^{\frac{1}{2}}(\Omega)$, find $u \in H^2_N(\Omega)$ such that

$$\begin{cases} u + \tau \partial_x(f_\varepsilon(u)\partial_x I(u)) &= g \quad \text{in } \Omega \\ \partial_x u = 0 \text{ and } \partial_x I(u) &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{32}$$

Once we prove the existence of a solution for (32), a simple time discretization method will provide the existence of a solution to (28). We are going to prove:

Proposition 6 (The stationary problem)**.** *For all $g \in H^{\frac{1}{2}}(\Omega)$, there exists $u \in H^2_N(\Omega)$ such that for all $\varphi \in H^1(\Omega)$,*

$$\frac{1}{\tau}\int_\Omega (u-g)\varphi\,dx - \int_\Omega f_\varepsilon(u)\,\partial_x I(u)\,\partial_x\varphi\,dx = 0\,. \tag{33}$$

Furthermore,

$$\int_\Omega u(x)\,dx = \int_\Omega g(x)\,dx\,, \tag{34}$$

$$\|u\|^2_{\dot H^{\frac{1}{2}}(\Omega)} + 2\tau\int_\Omega f_\varepsilon(u)(\partial_x I u)^2\,dx \leq \|g\|^2_{\dot H^{\frac{1}{2}}(\Omega)}\,, \tag{35}$$

and if $\int_\Omega G_\varepsilon(g)\,dx < \infty$ then

$$\int_\Omega G_\varepsilon(u)\,dx + \tau\|u\|^2_{\dot H^{\frac{3}{2}}_N(\Omega)} \leq \int_\Omega G_\varepsilon(g)\,dx. \tag{36}$$

In order to prove such a result, we have to reformulate (33):

New formulation of (33)**.** We are going to use classical variational methods to show the existence of a solution to (33). In order to work with a coercive non-linear operator, we need to take $\varphi = -I(v)$ as a test function. We note, however, that by doing that we would restrict ourself to test functions with zero mean value. In order to recover all test functions from $H^1(\Omega)$, we use Corollary 1 and consider

$$\varphi = -I(v) + \int_\Omega v\,dx \tag{37}$$

for some function $v \in H^2_N(\Omega)$. Let us emphasize the fact that Corollary 1 implies in particular that there is a one-to-one correspondence between $\varphi \in H^1(\Omega)$ and $v \in H^2_N(\Omega)$ satisfying (37).

Using (37), Equation (33) becomes:

$$\begin{aligned} -\int_\Omega u\,I(v)\,dx + \left(\int_\Omega u\,dx\right)\left(\int_\Omega v\,dx\right) + \tau\int_\Omega f_\varepsilon(u)\partial_x I(u)\partial_x I(v)\,dx \\ = -\int_\Omega g\,I(v)\,dx + \left(\int_\Omega g\,dx\right)\left(\int_\Omega v\,dx\right). \end{aligned} \tag{38}$$

We can now introduce the non-linear operator we are going to work with.

A non-linear operator. We define for all u and $v \in H^2_N(\Omega)$

$$A(u)(v) = -\int_\Omega u\,I(v)\,dx + \left(\int_\Omega u\,dx\right)\left(\int_\Omega v\,dx\right) + \tau\int_\Omega f_\varepsilon(u)\partial_x I(u)\partial_x I(v)\,dx.$$

One can now show that this non-linear operator is continuous, coercive and pseudo-monotone. Classical theorems imply the existence of a solution to the equation $A(u) = g$ for proper g's. More precisely, we have the following proposition:

Proposition 7 (Existence for the new problem)**.** *For all $g \in H^{\frac{1}{2}}(\Omega)$ there exists $u \in H^2_N(\Omega)$ such that*

$$A(u)(v) = -\int_\Omega g\, I(v)\, dx + \left(\int_\Omega g\, dx\right)\left(\int_\Omega v\, dx\right) \qquad \text{for all } v \in H^2_N(\Omega). \quad (39)$$

For the sake of readability, we postpone the proof of this rather technical proposition to Appendix A, and we turn to the proof of Proposition 6.

Proof of Proposition 6. For a given $g \in H^{\frac{1}{2}}(\Omega)$, Proposition 7 gives the existence of a solution $u \in H^2_N(\Omega)$ of (38). We recall that for any $\varphi \in H^1(\Omega)$, there exists $v \in H^2_N(\Omega)$ such that

$$\varphi = -I(v) + \int_\Omega v\, dx$$

and so equivalently, we have that u satisfies (33) for all $\varphi \in H^1(\Omega)$.

Next, we note that the mass conservation equality (34) is readily obtained by taking $v = 1$ as a test function in (38), while (35) follows by taking $v = u - \int_\Omega u\, dx$:

$$\begin{aligned}\|u\|^2_{\dot H^{\frac{1}{2}}(\Omega)} + \tau \int_\Omega f_\varepsilon(u) |\partial_x I(u)|^2 &= -\int_\Omega g I(u)\, dx \\ &\le \|g\|_{\dot H^{\frac{1}{2}}(\Omega)} \|u\|_{\dot H^{\frac{1}{2}}(\Omega)} \le \frac{1}{2}||g||^2_{\dot H^{\frac{1}{2}}(\Omega)} + \frac{1}{2}||u||^2_{\dot H^{\frac{1}{2}}(\Omega)}.\end{aligned}$$

Finally since G'_ε is smooth with G'_ε and G''_ε bounded and Ω is bounded, we have $G'_\varepsilon(u) \in H^1(\Omega)$. We can thus find $v \in H^2_N(\Omega)$ such that

$$-I(v) + \int_\Omega v(x)\, dx = G'_\varepsilon(u).$$

Equation (38) then implies:

$$-\int_\Omega u G'_\varepsilon(u)\, dx + \tau \int_\Omega f_\varepsilon(u)\, F''_\varepsilon(u)\, \partial_x I(u)\, \partial_x u\, dx = -\int_\Omega g G'_\varepsilon(u)\, dx$$

and so (using the definition of G_ε given in Proposition 5)

$$-\tau \int_\Omega \partial_x I(u)\, \partial_x u\, dx = \int_\Omega G'_\varepsilon(u)(g-u)\, dx$$

Since G_ε is convex ($G''_\varepsilon \ge 0$), we have $G'_\varepsilon(u)(g-u) \le G_\varepsilon(g) - G_\varepsilon(u)$ and we deduce (36) (using Proposition 4). $\square$

4.2 Proof of Proposition 5

In order to construct the solution u^ε of (26), we discretize the problem with respect to t, and construct a piecewise constant function

$$u^\tau(x,t) = u^n(x) \text{ for } t \in (n\tau, (n+1)\tau), n \in \{0, \dots, N+1\},$$

where $\tau = T/N$ and $(u^n)_{n\in\{0,\dots,N+1\}}$ is such that

$$\frac{1}{\tau}(u^{n+1} - u^n) + \partial_x(f_\varepsilon(u^{n+1})\partial_x I(u^{n+1})) = 0\,.$$

The existence of the u^n follows from Proposition 6 by induction on n. We deduce:

Corollary 2 (Discrete in time approximate solution)**.** *For any $N > 0$ and $u_0^\varepsilon \in H^{\frac{1}{2}}(\Omega)$, there exists a function $u^\tau \in L^\infty(0,T; H^{\frac{1}{2}}(\Omega))$ such that*

- *$t \mapsto u^\tau(x,t)$ is constant on $[k\tau, (k+1)\tau)$ for $k \in \{0, \dots, N\}$, $\tau = \frac{T}{N}$,*
- *$u^\tau = u_0$ on $[0,\tau) \times \Omega$,*
- *for all $\varphi \in \mathcal{C}^1(0,T,H^1(\Omega))$,*

$$\iint_{Q_{\tau,T}} \frac{u^\tau - S_\tau u^\tau}{\tau} \varphi \, dx\, dt = \iint_{Q_{\tau,T}} f_\varepsilon(u^\tau)\partial_x I(u^\tau)\partial_x\varphi \, dx\, dt \qquad (40)$$

where $Q_{\tau,T} = (\tau, T) \times \Omega$ and $S_\tau u^\tau(x,t) = u^\tau(t-\tau, x)$.

Moreover, the function u^τ satisfies

$$\int_\Omega u^\tau(x,t)\, dx = \int_\Omega u_0(x)\, dx \qquad a.e. \ t \in (0,T) \qquad (41)$$

and for all $t \in (0,T)$

$$\|u^\tau(t,\cdot)\|^2_{\dot H^{\frac{1}{2}}(\Omega)} + 2\int_0^t \int_\Omega f_\varepsilon(u^\tau)(\partial_x I(u^\tau))^2 \, dx\, dt \le \|u_0\|^2_{\dot H^{\frac{1}{2}}(\Omega)} \qquad (42)$$

and if $\int_\Omega G_\varepsilon(u_0)\, dx < \infty$, then for all $t \in (0,T)$

$$\int_\Omega G_\varepsilon(u^\tau(t,\cdot))\, dx + \int_0^t \|u^\tau\|^2_{\dot H^{\frac{3}{2}}_N(\Omega)} \, ds \le \int_\Omega G_\varepsilon(u_0)\, dx. \qquad (43)$$

It remains to prove that u^τ converges to a solution of (28) as τ goes to zero. This is fairly classical and we detail the proof for the interested reader in Appendix B.

5 Proof of Theorem 1: Case $n \geq 2$

Proposition 5 provides the existence of a solution $u^\varepsilon \in L^\infty(0,T;H^{\frac{1}{2}}(\Omega)) \cap L^2(0,T;H^2_N(\Omega))$ of (28). Our goal is now to pass to the limit $\varepsilon \to 0$. We point out that at this point, the solution u^ε may change sign and that it is only at the limit $\varepsilon \to 0$ that we are able to recover a non-negative solution, using the fact that $n \geq 2$.

Step 1: Compactness result. First, we note that (30) implies

$$||u^\varepsilon(t)||_{H^{\frac{1}{2}}(\Omega)} \leq ||u_0(t)||_{H^{\frac{1}{2}}(\Omega)} \quad \text{for all } \varepsilon > 0. \tag{44}$$

The bound (44) and Sobolev embedding theorems imply that the sequence $(u^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(0,T;L^p(\Omega))$ for all $p<\infty$ and so $f_\varepsilon(u^\varepsilon)$ is bounded in $L^\infty(0,T;L^p(\Omega))$ for all $p<\infty$. Furthermore, (30) also gives that $f_\varepsilon(u^\varepsilon)^{\frac{1}{2}}\partial_x I(u^\varepsilon)$ is bounded in $L^2(0,T;L^2(\Omega))$. We deduce that

$$f_\varepsilon(u^\varepsilon)\partial_x I(u^\varepsilon) \quad \text{is bounded in } L^2(0,T;L^r(\Omega))$$

for all $r \in [1,2)$. Writing

$$\partial_t u^\varepsilon = -\partial_x(f_\varepsilon(u^\varepsilon)\partial_x I(u^\varepsilon)) \qquad \text{in } \mathcal{D}'(\Omega),$$

we deduce that $(\partial_t u^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(0,T;W^{-1,r'}(\Omega))$ for all $r' \in (2,+\infty)$.

Thanks to the following embeddings

$$H^{\frac{1}{2}}(\Omega) \hookrightarrow L^q(\Omega) \to W^{-1,r'}(\Omega)$$

for all $q<\infty$, if follows (using Aubin's lemma) that $(u^\varepsilon)_{\varepsilon>0}$ is relatively compact in $\mathcal{C}^0(0,T,L^q(\Omega))$ for all $q<+\infty$. Hence, we can extract a subsequence, still denoted by u^ε such that

$$u^\varepsilon \longrightarrow u \quad \text{in } \mathcal{C}^0(0,T,L^q(\Omega)) \text{ for all } q<\infty$$

and

$$u^\varepsilon \longrightarrow u \quad \text{almost everywhere in } Q.$$

Step 2: Passing to the limit in Equation (28). We now have to pass to the limit in (28). We fix $\varphi \in \mathcal{D}(Q)$. Since $u^\varepsilon \to u$ in $\mathcal{C}^0(0,T,L^1(\Omega))$, we have

$$\iint_Q u^\varepsilon \partial_t\varphi\,dx\,dt \to \iint_Q u\,\partial_t\varphi\,dx\,dt.$$

Next, we remark that(30) implies

$$\varepsilon\iint_Q (\partial_x I(u^\varepsilon))^2\,dx\,dt \leq \frac{1}{2}||u_0||_{H^{\frac{1}{2}}(\Omega)}.$$

Cauchy-Schwarz inequality thus implies

$$\iint_Q \varepsilon \partial_x I(u^\varepsilon) \partial_x \varphi \, dx \, dt \quad \longrightarrow \quad 0\,.$$

Finally, (30) implies that $(u^\varepsilon)_+^{\frac{n}{2}} \partial_x I(u^\varepsilon)$ is bounded in $L^2(0,T,L^2(\Omega))$. Since u^ε is bounded in $L^\infty(0,T;L^p(\Omega))$ for all $p<\infty$ we deduce that $(u^\varepsilon)_+^n \partial_x I(u^\varepsilon)$ is bounded in $L^2(0,T;L^r(\Omega))$ for all $r \in [1,2)$ and so

$$h^\varepsilon := (u^\varepsilon)_+^n \partial_x I(u^\varepsilon) \rightharpoonup h \quad \text{ in } L^2(0,T;L^r(\Omega))\text{-weak.}$$

Passing to the limit in (28), we get:

$$\iint_Q u \, \partial_t \varphi \, dx \, dt + \iint_Q h \, \partial_x \varphi \, dx \, dt = - \iint_Q u_0 \varphi(0,\cdot) \, dx \, dt$$

for all $\varphi \in \mathcal{D}(\overline{Q})$.

Step 3: Equation for the flux h. In order to get (7), it only remains to show that

$$h = u_+^n \partial_x I(u)$$

in the following sense:

$$\iint_Q h \, \phi \, dx \, dt = - \iint_Q n u_+^{n-1} \partial_x u \; I(u) \; \phi \, dx \, dt - \iint_Q u_+^n I(u) \; \partial_x \phi \, dx \, dt \tag{45}$$

for all test function ϕ such that $\phi|_{\partial\Omega} = 0$; that is

$$h = \partial_x (u_+^n I(u)) - n u_+^{n-1} (\partial_x u) I(u) \quad \text{ in } \mathcal{D}'(\Omega).$$

For that we note that since

$$\int_\Omega G_\varepsilon(u_0) \, dx \le C,$$

Inequality (31) implies that $(u^\varepsilon)_\varepsilon$ is bounded in $L^2(0,T;H^{\frac{3}{2}}(\Omega))$. Recall that $(\partial_t u^\varepsilon)_\varepsilon$ is bounded in $L^2(0,T,W^{-1,r'}(\Omega))$ for all $r' \in (2,+\infty)$. Aubin's lemma then implies that u^ε is relatively compact in $L^2(0,T;H^s(\Omega))$ for $s<3/2$. In particular, we can assume that

$$I(u^\varepsilon) \longrightarrow I(u) \quad \text{ in } L^2(0,T;L^2(\Omega))$$

and

$$\partial_x u^\varepsilon \longrightarrow \partial_x u \quad \text{ in } L^2(0,T;L^p(\Omega)), \text{ for all } p<\infty.$$

Writing

$$\begin{aligned} \iint_Q h^\varepsilon \phi &= \iint_Q (u^\varepsilon)_+^n \partial_x I(u^\varepsilon) \; \phi \, dx \, dt \\ &= - \iint_Q n (u^\varepsilon)_+^{n-1} \partial_x u^\varepsilon \; I(u^\varepsilon) \; \phi \, dx \, dt - \iint_Q (u^\varepsilon)_+^n \; I(u^\varepsilon) \; \partial_x \phi \, dx \, dt, \end{aligned}$$

we see that those estimates, together with the fact that u^ε converges to u in $L^\infty(0,T;L^p(\Omega))$ for all $p<\infty$, are enough to pass to the limit and get (45).

Step 4: Properties of u. It is readily seen that u satisfies the conservation of mass (10) (by passing to the limit in (29)), and the lower semicontinuity of the norm implies the entropy inequality (12).

Next, Inequality (30) implies that $g^\varepsilon = (u_+^\varepsilon)^{\frac{n}{2}} \partial_x I(u^\varepsilon)$ converges weakly in $L^2((0,T)\times\Omega)$ to a function g, and the lower semicontinuity of the norm implies (11). Proceeding as above we can easily show that

$$g = \partial_x(u_+^{\frac{n}{2}} I(u)) - \frac{n}{2} u_+^{\frac{n}{2}-1} \partial_x u \, I(u) \quad \text{ in } \mathcal{D}'(\Omega).$$

Step 5: non-negative solutions. It remains to prove that u is non-negative. This will be a consequence of (31) and the fact that $n \geq 2$. Indeed, we recall that for all t we have

$$\int_\Omega G_\varepsilon(u^\varepsilon(t)\, dx \leq \int_\Omega G_\varepsilon(u_0)\, dx \tag{46}$$

where is such that $G_\varepsilon''(s) = \frac{1}{(s_+)^n + \varepsilon}$. As noted in the introduction, we can take

$$G_\varepsilon(s) = \int_1^s \int_1^r G_\varepsilon''(t)\, dt\, dr$$

which is a nonnegative convex function for all ε. Noticing that we can also write $G_\varepsilon(s) = \int_s^1 \int_r^1 G_\varepsilon''(t)\, dt\, dr$ when $s \leq 1$, it is readily seen that $G_\varepsilon(s)$ is decreasing with respect to ε (so $G_\varepsilon(s) \leq G_0(s)$ for all $\varepsilon > 0$). Hence

$$\int_\Omega G_\varepsilon(u_0)\, dx \leq \int_\Omega G_0(u_0)\, dx < +\infty\,.$$

We deduce (using (46)):

$$\limsup_{\varepsilon\to 0} \int_\Omega G_\varepsilon(u^\varepsilon(t))\, dx < +\infty\,. \tag{47}$$

Next, we recall that $u^\varepsilon(\cdot,t)$ converges strongly in $L^p(\Omega)$ to $u(\cdot,t)$. We can thus assume that it also converges almost everywhere. Egorov's theorem then implies the existence of a set $A_\eta \subset \Omega$ such that $u^\varepsilon(\cdot,t) \to u(\cdot,t)$ uniformly in A_η and $|\Omega \setminus A_\eta| < \eta$. For some $\delta > 0$, we now consider

$$C_{\eta,\delta} = A_\eta \cap \{u(\cdot,t) \leq -2\delta\}.$$

For every $\eta,\ \delta > 0$ there exists $\varepsilon_0(\eta,\delta)$ such that if $\varepsilon \leq \varepsilon_0(\eta,\delta)$, then $u^\varepsilon(\cdot,t) \leq -\delta$ in $C_{\eta,\delta}$.

But this implies that $C_{\eta,\delta}$ has measure zero. Indeed, if not, then for $\varepsilon \leq \varepsilon_0(\eta,\delta)$ we have

$$G_\varepsilon(u^\varepsilon(x,t)) \geq G_\varepsilon(-\delta) \longrightarrow G_0(-\delta) = +\infty \text{ for all } x \in C_{\eta,\delta}$$

(we use here the assumption $n \geq 2$) and by Fatou lemma, we get

$$\liminf_{\varepsilon\to 0} \int_{C_{\eta,\delta}} G_\varepsilon(u^\varepsilon(x,t))\, dx \geq \int_{C_{\eta,\delta}} \liminf_{\varepsilon\to 0} G_\varepsilon(u^\varepsilon(x,t))\, dx = +\infty$$

which contradicts (47).

We deduce that for all $\delta > 0$ and all $\eta > 0$ we have

$$|\{u(\cdot,t) \le -2\delta\}| \le |C_{\eta,\delta}| + |\Omega \setminus A_\eta| \le \eta$$

and so $|\{u(\cdot,t) \le -2\delta\}| = 0$ for all $\delta > 0$. We can conclude that

$$\{u(\cdot,t) < 0\} = \bigcup_{n\ge 1} \left\{ u(\cdot,t) < -\frac{1}{n} \right\}$$

has measure zero and so $u(x,t) \ge$ a.e. $x \in \Omega$ and for all $t > 0$.

6 Proof of Theorem 1: Case $n \in [1,2)$

When $n \in [1,2)$ the entropy inequality cannot be used to prove that $\lim_{\varepsilon\to 0} u^\varepsilon$ is non-negative. We thus proceed as in Bertozzi and Pugh [14]: Introducing

$$f_\delta(s) = \frac{s^{3+n}}{\delta s^n + s^3}$$

and

$$u_0^\delta(x) = u_0(x) + \delta.$$

For $n < 2$, we have $f_\delta(s) \sim s^3/\delta$ as $s \to 0$, and so the corresponding entropy G_δ, defined by

$$G_\delta(s) = \int_1^s \int_1^r \frac{1}{f_\delta(t)}\, dt\, dr = \int_1^s \int_1^r \frac{\delta}{t^3} + \frac{1}{t^n}\, dt\, dr.$$

satisfies

$$G_\delta(s) = \delta \left(\frac{1}{2s} + \frac{s}{2} - 1 \right) + G_0(s)$$

where $G_0(s)$ is bounded in the neighborhood of $s = 0$. It is thus readily seen that there exists C such that

$$\int_\Omega G_\varepsilon(u_0^\delta(x))\, dx < C.$$

Furthermore, we have $G_\delta(0) = +\infty$, so the proof developed in the previous section (regularizing the equation by introducing $f_{\delta,\varepsilon}(s) = f_\delta(s) + \varepsilon$) implies the existence of a non-negative solution u^δ of

$$\partial_t u^\delta + \partial_x (f_\delta(u^\delta) \partial_x I(u^\delta)) = 0$$

satisfying the usual inequalities (mass conservation, energy and entropy inequality).

Proceeding as in the previous section, we can now show that the sequence $(u^\delta)_{\delta>0}$ is relatively compact in $\mathcal{C}^0(0,T,L^q(\Omega))$ for all $q < +\infty$ and in $L^2(0,T;H^s(\Omega))$

for $s < 3/2$. In particular, we can extract a subsequence, still denoted u^δ, such that

$$\begin{aligned}
u^\delta &\longrightarrow u && \text{in } \mathcal{C}^0(0,T,L^q(\Omega)) \text{ for all } q < +\infty \\
u^\delta &\longrightarrow u && \text{almost everywhere in } Q \\
I(u^\delta) &\longrightarrow I(u) && \text{in } L^2(0,T;L^2(\Omega)) \\
\partial_x u^\delta &\longrightarrow \partial_x u && \text{in } L^2(0,T;L^p(\Omega)),\ \text{for all } p < +\infty.
\end{aligned}$$

Furthermore, since $u^\delta \geq 0$ for all $\delta > 0$, we have

$$u \geq 0 \quad \text{ a.e. } (x,t) \in \Omega \times (0,T).$$

In order to pass to the limit in the equation, we mainly need to check that $f_\delta(u^\delta)$ (respectively $f'_\delta(u^\delta)$) converges to u^n (respectively nu^{n-1}) in $L^p(\Omega)$. This is a direct consequence of the convergence almost everywhere of u^δ, Lebesgue dominated convergence theorem and the fact that

$$f_\delta(s) \leq s^n \text{ for all } s \geq 0$$

and

$$f'_\delta(s) = \frac{3\delta s^{2+2n} + n s^{5+n}}{(\varepsilon s^n + s^3)^2} \leq (n+2)s^{n-1} \text{ for all } s \geq 0$$

(note that we need $n \geq 1$ to complete this computation).

This complete the proof of Theorem 1.

7 Proof of Theorem 2

In this section, we prove Theorem 2. We recall that $n > 3$ and we consider the sequence u^ε of solution of the regularized equation (26) introduced in the proof of Theorem 1.

We recall that inequality (31) implies that $(u^\varepsilon)_\varepsilon$ is bounded in $L^2(0,T;H^{\frac{3}{2}}(\Omega))$. Since $(\partial_t u^\varepsilon)_\varepsilon$ is bounded in $L^2(0,T,W^{-1,r'}(\Omega))$ for all $r' \in (2,+\infty)$, Aubin's lemma implies that u^ε converges strongly in $L^2(0,T;\mathcal{C}^\alpha(\Omega))$ for all $\alpha < 1$. We can thus find a subsequence such that $u^\varepsilon(t)$ converges strongly in $\mathcal{C}^\alpha(\Omega)$ for almost every t (that is for all $t \in P$, where $|(0,T) \setminus P| = 0$).

Next, we note that for $t \in P$, u is actually strictly positive. Indeed, if $u(x_0,t_0) = 0$, then for any $\alpha < 1$, there is a constant C_α such that

$$u(x,t) \leq C|x - x_0|^\alpha$$

We deduce

$$\int G(u(x,t_0))\,dx \geq \int \frac{1}{(C_\alpha |x-x_0|^\alpha)^{n-2}}\,dx$$

Given $n > 3$ we can choose $\alpha < 1$ such that $\alpha(n-2) > 1$. We deduce

$$\int G(u(x,t_0))\,dx = \infty$$

which contradicts (31).

We deduce that there exists $\delta > 0$ (depending on t) such that for ε small enough

$$u^\varepsilon(\cdot,t) \geq \delta \text{ in } \Omega.$$

Next, we note that after removing another set of measure zero to P, we can always assume that

$$\liminf_{\varepsilon\to 0} \int f_\varepsilon(u^\varepsilon)|\partial_x I(u^\varepsilon)|^2\,dx < \infty \qquad \text{for all } t \in P.$$

Indeed, if we denote

$$A_k = \{t \in P\,;\, \liminf_{\varepsilon\to 0} \int f_\varepsilon(u^\varepsilon)|\partial_x I(u^\varepsilon)|^2\,dx \geq k\}$$

we have (using Fatou's lemma):

$$\begin{aligned}
C &\geq \liminf_{\varepsilon\to 0} \int_0^T \int f_\varepsilon(u^\varepsilon)|\partial_x I(u^\varepsilon)|^2\,dx\,dt \\
&\geq \liminf_{\varepsilon\to 0} \int_{A_k} \int f_\varepsilon(u^\varepsilon)|\partial_x I(u^\varepsilon)|^2\,dx\,dt \\
&\geq \int_{A_k} \liminf_{\varepsilon\to 0} \int f_\varepsilon(u^\varepsilon)|\partial_x I(u^\varepsilon)|^2\,dx\,dt \\
&\geq k|A_k|
\end{aligned}$$

We deduce $|A_k| \leq C/k$ and thus

$$|\{t \in P\,;\, \liminf_{\varepsilon\to 0} \int f_\varepsilon(u^\varepsilon)|\partial_x I(u^\varepsilon)|^2\,dx = \infty\}| = 0.$$

It follows that for $t \in P$, we have

$$\liminf_{\varepsilon\to 0} |\partial_x I(u^\varepsilon)|^2\,dx < \infty$$

and so

$$u^\varepsilon(\cdot,t) \rightharpoonup u(\cdot,t) \qquad \text{in } H^2_N(\Omega)\text{-weak}.$$

In particular we can pass to the limit in the flux $J_\varepsilon = f_\varepsilon(u^\varepsilon)\partial_x I(u^\varepsilon)$ and write

$$\lim_{\varepsilon\to 0} J_\varepsilon = J = f(u)\partial_x I(u) \qquad \text{in } L^1(\Omega), \text{ a.e. } t \in (0,T)\ .$$

Furthermore, we note that we recover the boundary condition in the classical sense:

$$u_x(x,t) = 0 \qquad \text{for } x \in \partial\Omega \text{ and a.e. } t \in (0,T).$$

A Proof of Proposition 7

We denote
$$V = H^2_N(\Omega).$$
For any $u \in V$ the functional $A(u)$ is clearly linear on V and since V is continuously embedded in $L^\infty(\Omega)$, we have

$$|A(u)(v)| \leq \left[||u||_{H^{\frac{1}{2}}(\Omega)} + \tau(\varepsilon + ||u||^3_V)||u||_V \right] ||v||_V \, . \tag{48}$$

(Note that we used Proposition 4 to get this inequality). The non-linear operator A is thus well-defined as a map from V to V'. Moreover, it is bounded.

Next, we remark that we have

$$A(u)(u) \geq -\int_\Omega u\, I(u)\, dx + \left(\int_\Omega u\, dx \right)^2 + \varepsilon \int_\Omega |\partial_x I(u)|^2\, dx.$$

We deduce from Proposition 4 that

$$A(u)(u) \geq \tau\varepsilon ||u|^2_{H^2_N(\Omega)}. \tag{49}$$

In particular, we have

$$\frac{A(u)(u)}{\|u\|_V} \to +\infty \quad \text{ as } \|u\|_V \to +\infty.$$

The operator A is thus coercive. Proposition 7 will now be a consequence of classical theorems if we prove that A is a pseudo-monotone operator. Since we already know that A is bounded, it remains to prove the following lemma:

Lemma 2 (A is pseudo-monotone)**.** *Let u_j be a sequence of functions in V such that $u_j \rightharpoonup u$ weakly in V. Then*

$$\liminf_j \; A(u_j)(u_j - v) \geq A(u)(u - v).$$

Before we prove this lemma, let us notice that for $g \in H^{\frac{1}{2}}(\Omega)$, the application

$$T_g : v \mapsto -\int_\Omega g I(v)\, dx + \left(\int_\Omega g\, dx \right) \left(\int_\Omega v\, dx \right)$$

belongs to V'. Hence, using Theorem 2.7 (page 180) of [31], we deduce that for all $g \in H^{\frac{1}{2}}(\Omega)$, there exists a function $u \in V$ such that $A(u) = T_g$ in V', which completes the proof of Proposition 7.

It remains to prove Lemma 2.

Proof of Lemma 2. We first write

$$
\begin{aligned}
A(u_j)(u_j - v) &= -\int_\Omega u_j I(u_j - v)\,dx + \left(\int_\Omega u_j\,dx\right)\left(\int_\Omega (u_j - v)\,dx\right) \\
&\quad + \tau \int_\Omega f_\varepsilon(u_j)\partial_x I(u_j)\partial_x I(u_j - v)\,dx \\
&= \|u_j\|^2_{H^{\frac{1}{2}}(\Omega)} - \langle u_j, v\rangle_{H^{\frac{1}{2}}} \\
&\quad + \tau \int_\Omega f_\varepsilon(u_j)(\partial_x I(u_j))^2 - \tau \int_\Omega f_\varepsilon(u_j)\partial_x(Iu_j)\partial_x(Iv) \qquad (50)
\end{aligned}
$$

where

$$
\langle u, v\rangle_{H^{\frac{1}{2}}} = -\int_\Omega uI(v)\,dx + \left(\int_\Omega u\,dx\right)\left(\int_\Omega v\,dx\right).
$$

We need to check that we can pass to the limit in each of those terms.

Since u_j converges weakly in V we immediately get

$$
\liminf_{j\to+\infty} \|u_j\|^2_{H^{\frac{1}{2}}(\Omega)} \geq \|u\|^2_{H^{\frac{1}{2}}(\Omega)}
$$

and

$$
\lim_{j\to+\infty} \langle u_j, v\rangle_{H^{\frac{1}{2}}} = -\langle u, v\rangle_{H^{\frac{1}{2}}}.
$$

Since u_j is bounded in $H^2_N(\Omega)$, it is compact in $L^\infty(\Omega)$, and so $f_\varepsilon(u_j)$ converges to $f_\varepsilon(u)$ strongly in $L^\infty(\Omega)$. We thus write

$$
\begin{aligned}
\int_\Omega f_\varepsilon(u_j)(\partial_x I(u_j))^2 &= \int_\Omega \big(f_\varepsilon(u_j) - f_\varepsilon(u)\big)(\partial_x I(u_j))^2 + \int_\Omega f_\varepsilon(u)(\partial_x I(u_j))^2 \\
&\geq -\|f_\varepsilon(u_j) - f_\varepsilon(u)\|_{L^\infty(\Omega)}\|u_j\|^2_V + \int_\Omega f_\varepsilon(u)(\partial_x I(u_j))^2.
\end{aligned}
$$

The first term goes to zero and we have

$$
\sqrt{f_\varepsilon(u)}\partial_x I(u_j) \rightharpoonup \sqrt{f_\varepsilon(u)}\partial_x I(u) \text{ in } L^2(\Omega).
$$

Again, the lower semicontinuity of the L^2-norm gives

$$
\lim_{j\to\infty} \tau \int_\Omega f_\varepsilon(u)(\partial_x I(u_j))^2 \geq \int_\Omega f_\varepsilon(u)(\partial_x I(u))^2.
$$

Finally, we have

$$
\begin{aligned}
f_\varepsilon(u_j)\partial_x I(v) &\to f_\varepsilon(u)\partial_x I(v) && \text{in } L^2(\Omega) \text{ strong,} \\
\partial_x I(u_j) &\rightharpoonup \partial_x I(u) && \text{in } L^2(\Omega) \text{ weak}
\end{aligned}
$$

which gives the convergence of the last term in (50) and completes the proof of the lemma. $\square$

B Proof of Proposition 5

The proof of Proposition 5 is divided in three steps.

Step 1: *a priori* estimates. We summarize the a priori estimates in the following lemma:

Lemma 3 (*A priori* estimates)**.** *The solution u^τ constructed in Corollary 2 satisfies*

$$\|u^\tau\|_{L^\infty(0,T,H^{\frac{1}{2}}(\Omega))} \leq \|u_0^\varepsilon\|_{H^{\frac{1}{2}}(\Omega)}, \tag{51}$$

$$\sqrt{\varepsilon}\|\partial_x I(u^\tau)\|_{L^2(Q)} \leq C, \tag{52}$$

$$\left\|\frac{u^\tau - S_\tau u^\tau}{\tau}\right\|_{L^2(\tau,T,W^{-1,r'}(\Omega))} \leq C, \tag{53}$$

for all $r' \in (2,+\infty)$ where C does not depend on $\tau > 0$ (but does depend on r').

Proof. Estimate (51) and (52) are direct consequences of (41) and (42).

Next, we note that

$$\frac{u^\tau - S_\tau u^\tau}{\tau} = \partial_x\Big(-f_\varepsilon(u^\tau)\partial_x I(u^\tau)\Big).$$

The bound (51) and Sobolev embedding theorems imply that the sequence $(u^\tau)_{\tau>0}$ is bounded in $L^\infty(0,T,L^p(\Omega))$ for all $p<\infty$ and so $f_\varepsilon(u^\tau)$ is bounded in $L^\infty(0,T,L^p(\Omega))$ for all $p<\infty$. Since $\partial_x I(u^\tau)$ is bounded in $L^2(Q)$, we deduce that $f_\varepsilon(u^\tau)\partial_x I(u^\tau)$ is bounded in $L^2(\tau,T,L^r(\Omega))$ for all $r\in[1,2)$. It follows that

$$\partial_x\left(f_\varepsilon(u^\tau)\partial_x I(u^\tau)\right) \text{ is bounded in } L^2(\tau,T,W^{-1,r'}(\Omega))$$

for all $r'\in(2,\infty)$. □

Step 2: Compactness result. Thanks to the following imbeddings

$$H^{\frac{1}{2}}(\Omega) \hookrightarrow L^q(\Omega) \to W^{-1,r'}(\Omega)$$

for all $q<\infty$, we can use Aubin's lemma to obtain that $(u^\tau)_\tau$ is relatively compact in $\mathcal{C}^0(0,T,L^q(\Omega))$ for all $q<\infty$.

Remark that $(\partial_x I(u^\tau))_\tau$ is bounded in $L^2(Q)$ and $(u^\tau)_\tau$ is bounded in $L^\infty(0,T;L^1(\Omega))$. It follows that $(u^\tau)_\tau$ is bounded in $L^2(0,T,H^2_N(\Omega))$. Since

$$H^2_N(\Omega) \hookrightarrow H^{\frac{3}{2}}_N(\Omega) \to W^{-1,r'}(\Omega)$$

we deduce that $(u^\tau)_\tau$ is relatively compact in $L^2(0,T;H^{\frac{3}{2}}_N(\Omega))$. Up to a subsequence, we can thus assume that as $\tau\to 0$, we have the following convergences:

- $u^\tau \to u^\varepsilon \in L^\infty(0,T,H^{\frac{1}{2}}(\Omega))$ almost everywhere in Q;
- $u^\tau \to u^\varepsilon$ in $L^2(0,T,H^1(\Omega))$ strong;
- $\partial_x I(u^\tau) \rightharpoonup \partial_x I(u^\varepsilon)$ in $L^2(Q)$ weak.

Step 3: Derivation of Equation (28)**.** We want to pass to the limit in (40).

We fix $\varphi \in \mathcal{C}^1_c([0,T), H^1(\Omega))$. Then

$$\iint_{Q_\tau} \frac{u^\tau - S_\tau u^\tau}{\tau}\varphi = \iint_Q u^\tau(x,t)\frac{\varphi(x,t)-\varphi(t+\tau,x)}{\tau}$$
$$- \frac{1}{\tau}\int_0^\tau \int_\Omega u^\tau(x,t)\varphi(x,t)\,dx + \frac{1}{\tau}\int_{T-\tau}^T \int_\Omega u^\tau(x,t)\varphi(x,t)\,dx.$$

We deduce:

$$\iint_{Q_\tau} \frac{u^\tau - S_\tau u^\tau}{\tau}\varphi \to -\iint_Q u^\varepsilon(\partial_t\varphi) - \int_\Omega u^\varepsilon(0,x)\varphi(0,x)\,dx + 0.$$

It remains to pass to the limit in the non-linear term. Let $\eta > 0$. Since $u^\tau \to u^\varepsilon$ almost everywhere in Q, Egorov's theorem yields the existence of a set $A_\eta \subset Q$ such that $|Q \setminus A_\eta| \le \eta$ and

$$u^\tau \to u^\varepsilon \text{ uniformly in } A_\eta\,.$$

In particular,

$$\sqrt{f_\varepsilon(u^\tau)}\partial_x\varphi \to \sqrt{f_\varepsilon(u^\varepsilon)}\partial_x\varphi \text{ in } L^2(A_\eta)$$

and

$$\sqrt{f_\varepsilon(u^\tau)}\partial_x I(u^\tau) \rightharpoonup \sqrt{f_\varepsilon(u^\varepsilon)}\partial_x I(u^\varepsilon) \text{ in } L^2(A_\eta)\,. \tag{54}$$

Hence

$$\int_{A_\eta} f_\varepsilon(u^\tau)\partial_x I(u^\tau)\partial_x\varphi \to \int_{A_\eta} f_\varepsilon(u^\varepsilon)\partial_x I(u^\varepsilon)\partial_x\varphi$$

as τ goes to zero.

Finally, we look at what happens on $Q \setminus A_\eta$. Choose p_1, p_2, p_3 such that $\sum_i p_i^{-1} = 1$ and write

$$\int_{Q\setminus A_\eta} |f_\varepsilon(u^\tau)\partial_x I(u^\tau)\partial_x\varphi|$$
$$\le \|\partial_x\varphi\|_{L^\infty(0,T,L^{p_1}(\Omega))}\int_0^T \|f_\varepsilon(u^\tau)\partial_x I(u^\tau)\|_{L^{p_2}(\Omega)}\|\mathbf{1}_{Q\setminus A_\eta}\|_{L^{p_3}(\Omega)}$$
$$\le \|\partial_x\varphi\|_{L^\infty(0,T,L^{p_1}(\Omega))}\|f_\varepsilon(u^\tau)\partial_x I(u^\tau)\|_{L^2(0,T,L^{p_2}(\Omega))}\|\mathbf{1}_{Q\setminus A_\eta}\|_{L^2(0,T,L^{p_3}(\Omega))}\,.$$

We now choose $p_2 \in [1,2)$ (and so $p_1 > 2$ and $p_3 > 2$).

$$\int_{Q\setminus A_\eta} |f_\varepsilon(u^\tau)\partial_x I(u^\tau)\partial_x\varphi| \le C(\varphi)\|\mathbf{1}_{Q\setminus A_\eta}\|_{L^{p_3}(Q)} \le C(\varphi)\eta^{\frac{1}{p_3}}\,.$$

Since η is arbitrary, the proof is complete.

Step 4: Inequalities. Since $u^\tau \to u^\varepsilon$ in $L^\infty(0,T,L^1(\Omega))$, mass conservation equation (29) follows from (41).

Estimate (30) follows from (42). Indeed, since $u^\tau \to u^\varepsilon$ almost everywhere, Proposition 4 and Fatou's lemma imply that for almost $t \in (0,T)$

$$\|u^\varepsilon(t)\|^2_{\dot{H}^{\frac{1}{2}}(\Omega)} \leq \liminf_{\tau\to 0} \|u^\tau(t)\|^2_{\dot{H}^{\frac{1}{2}}(\Omega)}.$$

Thanks to (54), we also have

$$\int_0^T \int_\Omega f_\varepsilon(u^\varepsilon)|\partial_x I(u^\varepsilon)|^2\,dx\,dt \leq \liminf_{\tau\to 0} \int_0^T \int_\Omega f_\varepsilon(u^\tau)(\partial_x I(u^\tau))^2\,dx\,dt + \int_0^T \int_{\Omega\setminus A_\eta} f_\varepsilon(u^\varepsilon)|\partial_x I(u^\varepsilon)|^2\,dx\,dt.$$

Letting $\eta \to 0$ permits to conclude.

To derive (31) we note that $G_\varepsilon(u^\tau) \to F_\varepsilon(u^\varepsilon)$ almost everywhere. So Fatou's Lemma implies for almost every $t \in (0,T)$

$$\int_\Omega G_\varepsilon(u^\varepsilon(x,t))\,dx \leq \liminf_{\tau\to 0} \int_\Omega G_\varepsilon(u^\tau(x,t))\,dx \leq \int_\Omega G_\varepsilon(u_0)\,dx.$$

Finally, since $(u^\tau)_\tau$ is relatively compact in $L^2(0,T;H^{\frac{3}{2}}_N(\Omega))$, we have

$$\int_0^t ||u^\varepsilon(s)||^2_{\dot{H}^{\frac{3}{2}}}\,ds = \lim_{\tau\to 0} \int_0^t ||u^\tau(s)||^2_{\dot{H}^{\frac{3}{2}}}\,ds$$

and so (31) follows from (43).

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