Anomalous diffusion limit for kinetic equations with degenerate collision frequency

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Abstract

This paper is devoted to hydrodynamic limits for collisional linear kinetic equations. It is a classical result that under certain conditions on the collision operator, the long time/small mean free path asymptotic behavior of the density of particles can be described by diffusion type equations. We are interested in situations in which the degeneracy of the collision frequency for small velocities causes this limit to break down. We show that the appropriate asymptotic analysis leads to an anomalous diffusion regime.

1 Introduction

In this paper we investigate diffusion regimes for the following kinetic equation:

$$\begin{cases} \varepsilon^\alpha \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = L(f^\varepsilon) & \text{for all } (x,v) \in \mathbb{R}^N \times \mathbb{R}^N \text{ and } t>0 \\ f^\varepsilon(x,v,0) = f_0(x,v) & \text{for all } (x,v) \in \mathbb{R}^N \times \mathbb{R}^N \end{cases} \tag{1}$$

when the operator L is a linear collision operator of the form:

$$L(f) := \int_{\mathbb{R}^N} b(v,v')[f(v')M(v) - f(v)M(v')]\, dv'. \tag{2}$$

The thermodynamical equilibrium function $M(v)$ is a given function and it is normalized so that

$$\int_{\mathbb{R}^N} M(v)\, dv = 1.$$

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Typically, one can take M to be a Maxwellian distribution function $M(v) = \frac{1}{(2\pi T)^{N/2}} e^{-\frac{v^2}{2T}}$, (for some temperature $T > 0$) though other distribution functions could be considered without difficulties. The collision kernel $b(v, v')$ in (2) is usually assumed to satisfy

$$b(v, v') = b(v', v) \geq 0 \tag{3}$$

and in this paper, we are going to limit ourself to a very simple case where we have

$$b(v, v') = a(v)a(v') \tag{4}$$

for some function $a(v) \geq 0$. Of course, our result also applies to more general collision kernels, with minor modifications.

When (4) holds, the operator L reads:

$$L(f) = a(v) \int a(v')f(v')\, dv' M(v) - \nu(v)f(v)$$

where

$$\nu(v) := \int b(v, v')M(v')\, dv' = a(v) \int a(v')M(v')\, dv'$$

denotes the collision frequency. We can also write L as follows (and we will use mainly this notation from now on):

$$L(f) = \nu(v)[\rho_\nu M(v) - f(v)] \qquad \text{with} \qquad \rho_\nu := \frac{\displaystyle\int \nu(v)f(v)\, dv}{\displaystyle\int \nu(v)M(v)\, dv} \tag{5}$$

(note that ρ_ν is not the usual density of f, except when $f(x, v, t) = \rho(x, t)M(v)$).

The parameter ε in (1) is the Knudsen number. It is defined as the ratio of the mean free path of the particles to the typical macroscopic length. We are interested in the small mean free path ($\varepsilon \ll 1$) long time ($t \sim \varepsilon^{-\alpha}$) regime, which leads to the scaling in (1). The derivation of hydrodynamic limits for kinetic equations such as (1) was first investigated by E. Wigner [11], A. Bensoussan, J.L. Lions and G. Papanicolaou [5] and E.W. Larsen and J.B. Keller [8] and it has been the topic of many papers since (see in particular C. Bardos, R. Santos and R. Sentis [1] and P. Degond, T. Goudon and F. Poupaud [6] and references therein). It is well known (see [6] for instance) that for $\alpha = 2$, the distribution function f^ε converges, as $\varepsilon \to 0$, to a function $\rho(x, t)M(v)$ with ρ solution of the diffusion equation

$$\partial_t \rho - \mathrm{div}\,(D\nabla\rho) = 0$$

where

$$D = \int v \otimes v \frac{M(v)}{\nu(v)}\, dv. \tag{6}$$

We are interested in a situation in which this limit fails because of the degeneracy of the collision frequency. More precisely, we will assume that $\nu(0) = 0$ and

$$\nu(v) \sim \nu_0 |v|^{N+2+\beta}, \qquad \text{as } |v| \to 0 \quad \text{for some } \beta > 0. \tag{7}$$

The restriction $\beta > 0$ and the fact that $M(0) > 0$ implies that the diffusion matrix D above is infinite, because of the non integrable singularity at $v = 0$. In that case, the usual diffusion limit breaks down, which can be interpreted by saying that the time scale $t \sim \varepsilon^{-2}$ was too long. We will thus show that for an appropriate choice of $\alpha < 2$ (depending on β), the asymptotic regime $\varepsilon \to 0$ in (1) gives rise to anomalous diffusion regimes (the term anomalous refers to the fact that the mean squared displacement of a particle is no longer a linear function of time).

In previous works, we investigated similar anomalous diffusion regimes, that were due to the large velocity ($|v| \to \infty$) behavior of the equilibrium function M and the collision frequency ν. In particular it is shown, in [10, 9, 4], that if ν is bounded and non-degenerate and $M(v)$ is a heavy tail function satisfying

$$M(v) \sim |v|^{-N-\alpha} \quad \text{as } |v| \to \infty, \qquad \text{for some } \alpha \in (0, 2)$$

(the restriction $\alpha < 2$ implies that the diffusion matrix D above is infinite because of the behavior of the integrand for $|v| \to \infty$) then the solution of (1) converges to $\rho(x,t)M(v)$ with ρ solution of a fractional diffusion equation of order α.

The goal of this paper is to show that a similar phenomenon can arise when the equilibrium function is, for instance, the usual Maxwellian distribution, but the collision frequency satisfies (7).

In order to make the computations a little bit simpler, we will assume that there exists $\delta > 0$, $\beta > 0$ and $\nu_0 > 0$ such that

$$\begin{aligned} \nu(v) &= \nu_0 |v|^{N+2+\beta}, && \text{for } |v| \le \delta \\ M(v) &= M_0 > 0, && \text{for } |v| \le \delta. \end{aligned} \tag{8}$$

Neither of these conditions are really necessary, and it is easy to check that they could be replaced by

$$\lim_{|v| \to 0} \frac{\nu(v)}{|v|^{N+2+\beta}} = \nu_0 > 0, \qquad \lim_{|v| \to 0} M(v) = M_0 > 0.$$

Next, as usual with diffusion limits, it is very important to assume that the flux associated to $M(v)$ vanishes:

$$\int_{\mathbb{R}^N} v M(v)\, dv = 0 \tag{9}$$

(this is usually a consequence of the fact that $M(-v) = M(v)$). Finally, we need the following technical assumptions (which clearly holds for a wide range of functions $\nu(v)$ if $M(v)$ is a Maxwellian distribution function):

$$\int_{|v|\geq\delta} \frac{|v|^2}{\nu(v)} M(v)\,dv < \infty, \qquad \int_{\mathbb{R}^N} \nu(v)^2 M(v)\,dv < \infty \tag{10}$$

(the first inequality - in which we recognize the integrant of the diffusion matrix (6) - asserts that the only problem in the break down of the usual diffusion limit is due to the behavior of ν and M for small velocity. The second inequality will be used to show that ρ^ε_ν is bounded in L^2).

We can now state our main result:

Theorem 1.1. *Assume that (8), (9) and (10) hold and let*

$$\alpha = \frac{\beta + 2N + 2}{\beta + N + 1} = 2 - \frac{\beta}{\beta + N + 1}. \tag{11}$$

Then the solution f^ε of (1) converges weakly in $L^2_{\nu M^{-1}}(\mathbb{R}^N \times \mathbb{R}^N \times (0,T))$ to a function $\rho(x,t)M(v)$ where ρ solves

$$\begin{cases} \partial_t \rho + \kappa(-\Delta)^{\alpha/2}\rho = 0 \\ \rho(x,0) = \rho_0 \end{cases} \tag{12}$$

with

$$\kappa = \frac{1}{c_{N,\alpha}} \frac{1}{\beta + N + 1} \frac{M_0}{\nu_0^{\alpha-1}} \int_0^\infty z^\alpha e^{-z}\,dz.$$

The constant $c_{N,\alpha}$ is the constant that appears in the definition of the fractional Laplace operator as a singular integral:

$$(-\Delta)^{\alpha/2} u = c_{N,\alpha} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(x+y)}{|y|^{N+\alpha}}\,dy$$

Note that when $\beta > 0$, α defined by (11) satisfies

$$1 < \alpha < 2.$$

The critical case $\beta = 0$, which would give $\alpha = 2$ is also interesting. The standard diffusion limit (with time scale ε^2) fails in that case, and following [10], it can be checked that one needs to consider an anomalous diffusion scaling in (1) ($\varepsilon^2 \ln(\varepsilon^{-1})$ instead of ε^α in (1)) but that the limiting density ρ will solve a standard diffusion equation.

It is also worth noticing that unlike the anomalous diffusion regime due to power tail equilibrium functions, the degenerate collision frequency framework cannot lead to fractional diffusion equation of order less than one (there were no such restriction in [10, 9, 4]).

The main interest of this paper is thus to show that anomalous diffusion is not necessarily due to fast particles and that it does not require the thermodynamical equilibrium to have infinite moments. In particular, it can occur as asymptotic model for kinetic equations in which the velocity is bounded (relativistic models) or when the variable v is replaced by a wave vector k lying in a torus $\mathbb{T}^N$ (as in the modeling of transport in semiconductor devices).

Such an equation arises in the modeling of weak turbulence for chains of harmonic oscillators: In [2], Basile, Olla and Spohn show that the density of energy distribution $f(x,k,t)$ for a chain of harmonic oscillators with Hamiltonian dynamics perturbed by stochastic terms, satisfies a linear phonon Boltzmann equation of the form

$$\partial_t f + v(k) \cdot \partial_x f = L(f) \tag{13}$$

where

$$L(f) = \int \sigma(k,k')[f(k') - f(k)]\, dk'.$$

Here, the Fourier mode k belongs to $\mathbb{T}^1$ and the velocity $v(k) = \omega'(k)$ is the derivative of the dispersion relation of the lattice. It satisfies (see [2]) $v(k) \sim 1$ as $k \to 0$. Furthermore, the collision frequency is degenerate for small k and satisfies $\nu(k) \sim |k|^2$ as $k \to 0$. Our framework then applies (with minor modifications) and shows that for an appropriate scaling, the long time behavior of f can be described by an anomalous diffusion equation of order $\alpha = 3/2$. We thus recover, by a very different approach, a result that Jara, Komorowski and Olla [7] obtained using a purely probabilistic approach.

The general method developed in this paper is similar to that of [9]: Derivation of a priori estimates, multiplication of (1) by the solution of an appropriate auxiliary equation and passage to the limit. However, the classical a priori estimates for (1) (which are used in [9]) do not hold for degenerate collision frequencies and so new estimates are needed (see Lemma 2.2).

Dedication: This paper is the first of a series of projects that we were discussing with Naoufel Ben Abdallah when he passed away on July 5th 2010. Naoufel was a talented mathematician and a wonderful collaborator. He was also a very generous person, deeply devoted to his family and to his friends. By his constant support he played a determinant role in our (young) careers. We did our best to complete this work without him and hope that this final version meets his very high standard.

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2 Proof of Theorem 1.1

2.1 A priori estimates

It is a classical result that operators of the form (2) satisfy

$$-\int_{\mathbb{R}^N} L(f)\frac{f}{M}\,dv \geq 0.$$

Under some conditions on the collision kernel (see for instance [10]) one can usually also show that

$$-\int_{\mathbb{R}^N} L(f)\frac{f}{M}\,dv \geq c||f-\rho M||^2_{L^2_{M^{-1}}}$$

where $\rho = \int_{\mathbb{R}^N} f\,dv$. When the collision frequency is degenerate, however, such an estimate does not seem to hold, but a simple computation yields the following lemma:

Lemma 2.1. *Assume that $b(v,v') = a(v)a(v')$. Then*

$$-\int L(f)\frac{f}{M}\,dv = \int \nu(v)\frac{|f-\rho_\nu M|^2}{M}\,dv$$

where

$$\rho_\nu = \frac{\displaystyle\int \nu(v)f(v)\,dv}{\displaystyle\int \nu(v)M(v)\,dv}.$$

Note that a similar result holds for more general collision kernel that do not satisfy (4). Indeed, we have:

Lemma 2.2. *Assume that L is given by (2) where $b(v,v')$ satisfies (3) and*

$$\sup_{v\in\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\nu(v)\nu(v')^2}{b(v,v')}\,M(v')\,dv' \leq C < \infty. \tag{14}$$

Then,

$$-\int L(f)\frac{f}{M}\,dv \geq \frac{\left(\int \nu M\,dv\right)^2}{2C}\int \nu(v)\frac{|f-\rho_\nu M|^2}{M}\,dv$$

for all f.

This lemma is the only technical lemma that is needed to generalize Theorem 1.1 to more general collision operators (still given by (2)) and it is also of independent interest. We thus give its proof in Appendix A, even though it is not required for the proof of Theorem 1.1.

Multiplying (1) by $f^\varepsilon M^{-1}$ and integrating with respect to x and v, we deduce the following equalities:

$$\begin{aligned}\varepsilon^\alpha \frac{d}{dt}\int_{\mathbb{R}^{2N}} \frac{(f^\varepsilon)^2}{2} M^{-1}\,dvdx &= \int_{\mathbb{R}^{2N}} L(f^\varepsilon)\, f^\varepsilon\, M^{-1}\,dvdx \\ &= -\int_{\mathbb{R}^{2N}} \nu(v)|f^\varepsilon - \rho^\varepsilon_\nu M|^2 M^{-1}\,dvdx \quad (15)\end{aligned}$$

We thus introduce the spaces $L^2_{M^{-1}}$ and $L^2_{\nu M^{-1}}$ equipped with the norms:

$$||f||_{L^2_{M^{-1}}} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(x,v)|^2 M(v)^{-1}\,dv\,dx$$

and

$$||f||_{L^2_{\nu M^{-1}}} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(x,v)|^2 \nu(v) M(v)^{-1}\,dv\,dx.$$

Integrating (15) with respect to t, we deduce the following proposition:

Proposition 2.3. *The solution f^ε of (1) satisfies the following estimates:*

$$\sup_t ||f^\varepsilon(t)||_{L^2_{M^{-1}}} \leq ||f_0||_{L^2_{M^{-1}}} \qquad (16)$$

$$||g^\varepsilon||_{L^2(0,\infty;L^2_{\nu M^{-1}})} \leq \varepsilon^{\alpha/2}||f_0||_{L^2_{M^{-1}}} \qquad (17)$$

where $g^\varepsilon = f^\varepsilon - \rho^\varepsilon_\nu M$ with $\rho^\varepsilon_\nu = \frac{\int \nu f^\varepsilon\,dv}{\int \nu M\,dv}$.

Note that Cauchy-Schwarz inequality also gives:

$$\begin{aligned}\rho^\varepsilon_\nu(t,x) &= \frac{1}{\int \nu M\,dv}\int_{\mathbb{R}^N} \nu \frac{f^\varepsilon}{M^{1/2}} M^{1/2}\,dv \\ &\leq \frac{\left(\int \nu^2 M\,dv\right)^{1/2}}{\int \nu M\,dv}\left(\int_{\mathbb{R}^N} \frac{(f^\varepsilon)^2}{M}\,dv\right)^{1/2},\end{aligned}$$

and so (using (10))

$$\sup_{t\geq 0}\int_{\mathbb{R}^N} \rho^\varepsilon_\nu(t,.)^2\,dx \leq C\sup_{t\geq 0}\|f^\varepsilon\|^2_{L^2_{M^{-1}}} \leq C\|f_0\|^2_{L^2_{M^{-1}}}. \qquad (18)$$

We deduce

Corollary 2.4. *The sequence f^ε converges weakly in $L^\infty(0,T;L^2_{M^{-1}}(\mathbb{R}^N\times\mathbb{R}^N))$ to a function $\rho(x,t)F(x,v)$ where ρ is the weak limit of ρ^ε_ν in $L^\infty(0,T;L^2(\mathbb{R}^N))$.*

2.2 An auxiliary equation

The main tool in the proof of Theorem 1.1 is the introduction of an appropriate auxiliary equation. Proceeding as in [9], we consider $\chi^\varepsilon(x,v,t)$ solution of

$$\nu(v)\chi^\varepsilon(v) - \varepsilon v\cdot\nabla_x\chi^\varepsilon = \nu(v)\varphi(x,t), \qquad (19)$$

where $\varphi(x,t)$ is a test function in $\mathcal{D}(\mathbb{R}^N \times [0,\infty))$. A simple computation shows that χ^ε is given by the following formula:

$$\chi^\varepsilon = \int_0^\infty e^{-\nu(v)z}\nu(v)\varphi(x+\varepsilon vz, t)\,dz,$$

and we will make repeated use of the following formula:

$$\chi^\varepsilon - \varphi = \int_0^\infty e^{-\nu(v)z}\nu(v)[\varphi(x+\varepsilon vz,t) - \varphi(x,t)]\,dz. \tag{20}$$

In the next section, we will use χ^ε as a test function in (1). We thus point out that χ^ε is smooth and bounded in L^∞. Moreover, we have

$$\begin{aligned} |\chi^\varepsilon - \varphi| &= \left| \int_0^\infty e^{-\nu(v)z}\nu(v)\big[\varphi(x+\varepsilon vz,t) - \varphi(x,t)\big]\,dz \right| \\ &\le ||D\varphi||_{L^\infty}\varepsilon|v|, \end{aligned}$$

and thus

$$\chi^\varepsilon(x,v,t) \longrightarrow \varphi(x,t) \qquad \text{as } \varepsilon \to 0$$

uniformly with respect to x and t. However, this convergence is not uniform with respect to v, so we will need the following lemma:

Lemma 2.5. *Let $\varphi \in \mathcal{D}(\mathbb{R}^N \times [0,\infty))$, and define χ^ε by (20). Then*

$$\chi^\varepsilon \longrightarrow \varphi \qquad \text{strongly in } L^\infty(0,\infty; L^2_M(\mathbb{R}^N \times \mathbb{R}^N))$$

and

$$\partial_t\chi^\varepsilon \longrightarrow \partial_t\varphi \qquad \text{strongly in } L^\infty(0,\infty; L^2_M(\mathbb{R}^N \times \mathbb{R}^N))$$

Proof. Using (20) we get

$$\begin{aligned} ||\chi^\varepsilon - \varphi||^2_{L^2_M} &= \int_{\mathbb{R}^{2N}} M(v) \left| \int_0^\infty e^{-\nu(v)z}\nu(v)[\varphi(x+\varepsilon vz) - \varphi(x)]\,dz \right|^2 dx\,dv \\ &\le \int_{\mathbb{R}^{2N}} M(v) \int_0^\infty e^{-\nu(v)z}\nu(v)[\varphi(x+\varepsilon vz) - \varphi(x)]^2\,dz\,dx\,dv \\ &\le \int_{\mathbb{R}^N}\int_0^\infty M(v)e^{-\nu(v)z}\nu(v)||\varphi(\cdot+\varepsilon vz) - \varphi||^2_{L^2(\mathbb{R}^N)}\,dz\,dv. \end{aligned}$$

It is well know that

$$\lim_{\varepsilon\to 0} ||\varphi(\cdot+\varepsilon vz) - \varphi||^2_{L^2(\mathbb{R}^N)} = 0 \qquad \text{for all } v \text{ and } z.$$

Furthermore,

$$M(v)e^{-\nu(v)z}\nu(v)||\varphi(\cdot+\varepsilon vz) - \varphi||^2_{L^2(\mathbb{R}^N)} \le 2||\varphi||^2_{L^2(\mathbb{R}^N)}M(v)e^{-\nu(v)z}\nu(v)$$

Since $M(v)e^{-\nu(v)z}\nu(v)$ is in $L^1(\mathbb{R}^N \times (0,\infty))$ (in fact, its integral is equal to 1), Lebesgue dominated convergence theorem gives the result.

A similar proof holds for $\partial_t\chi^\varepsilon$ (t is just a parameter here). $\square$

2.3 Weak formulation and passage to the limit

Now, we use χ^ε as a test function in (1): Multiplying (1) by χ^ε and integrating with respect to x, v, t, we get:

$$-\varepsilon^\alpha \int_0^\infty \int_{\mathbb{R}^{2N}} f^\varepsilon \partial_t \chi^\varepsilon \, dx\, dv\, dt - \varepsilon^\alpha \int_{\mathbb{R}^{2N}} f_0(x,v)\chi^\varepsilon(x,v,0)\, dx\, dv$$
$$= \int_0^\infty \int_{\mathbb{R}^{2N}} \nu \rho_\nu^\varepsilon M \chi^\varepsilon - \nu f^\varepsilon \chi^\varepsilon + f^\varepsilon \varepsilon v \cdot \nabla_x \chi^\varepsilon \, dx\, dv\, dt$$

which, using the auxiliary equation (19), yields:

$$\begin{aligned}
- \int_0^\infty \int_{\mathbb{R}^{2N}} & f^\varepsilon \partial_t \chi^\varepsilon \, dx\, dv\, dt - \int_{\mathbb{R}^{2N}} f_0(x,v)\chi^\varepsilon(x,v,0)\, dx\, dv \\
&= \varepsilon^{-\alpha} \int_0^\infty \int_{\mathbb{R}^{2N}} \nu \rho_\nu^\varepsilon M \chi^\varepsilon - \nu f^\varepsilon \varphi(x,t)\, dx\, dv\, dt \\
&= \varepsilon^{-\alpha} \int_0^\infty \int_{\mathbb{R}^{2N}} \nu \rho_\nu^\varepsilon M \chi^\varepsilon \, dx\, dv\, dt - \int \rho_\nu^\varepsilon \left(\int \nu M \, dv \right) \varphi(x,t)\, dx\, dt.
\end{aligned}$$

We deduce:

$$\begin{aligned}
- \int_0^\infty \int_{\mathbb{R}^{2N}} & f^\varepsilon \partial_t \chi^\varepsilon \, dx\, dv\, dt - \int_{\mathbb{R}^{2N}} f_0(x,v)\chi^\varepsilon(x,v,0)\, dx\, dv \\
&= \varepsilon^{-\alpha} \int_0^\infty \int_{\mathbb{R}^N} \rho_\nu^\varepsilon \int_{\mathbb{R}^N} \nu(v) M(v) \left[\chi^\varepsilon(x,v,t) - \varphi(x,t) \right] \, dv\, dx\, dt
\end{aligned}$$

which we write as

$$\begin{aligned}
- \int_0^\infty \int_{\mathbb{R}^{2N}} & f^\varepsilon \partial_t \chi^\varepsilon \, dx\, dv\, dt - \int_{\mathbb{R}^{2N}} f_0(x,v)\chi^\varepsilon(x,v,0)\, dx\, dv \\
&= \int_0^\infty \int_{\mathbb{R}^N} \rho_\nu^\varepsilon \mathcal{L}^\varepsilon(\varphi) dx\, dt \qquad (21)
\end{aligned}$$

with

$$\mathcal{L}^\varepsilon(\varphi) = \varepsilon^{-\alpha} \int_{\mathbb{R}^N} \nu(v) M(v) \left[\chi^\varepsilon(x,v,t) - \varphi(x,t) \right] \, dv.$$

The rest of the proof consists in passing to the limit $\varepsilon \to 0$ in (21). We immediately check that the left hand side converges to

$$- \int_0^\infty \int_{\mathbb{R}^N} \rho\, \partial_t \varphi \, dx\, dt - \int_{\mathbb{R}^N} \rho_0(x) \varphi(x,0)\, dx$$

(this is an immediate consequence of Lemma 2.5 and Corollary 2.4). Passing to the limit in the right hand side of (21) is the most interesting part of the proof since the nonlocal operator should now appear in the limit of $\mathcal{L}^\varepsilon$ (it is also the part of the proof that differs the most from [9]). More precisely, we have to prove:

Proposition 2.6. *Assume that the conditions of Theorem 1.1 hold and that χ^ε is defined by (19). Then*

$$\mathcal{L}^\varepsilon(\varphi) := \varepsilon^{-\alpha} \int_{\mathbb{R}^N} \nu(v) M(v) \left[\chi^\varepsilon(x,v,t) - \varphi(x,t)\right] dv \tag{22}$$

converges strongly in $L^2(\mathbb{R}^N \times (0,\infty))$ as ε goes to zero to

$$-\kappa(-\Delta)^{\alpha/2}(\varphi) = \kappa\, c_{N,\alpha}\, PV \int_{\mathbb{R}^N} \frac{\varphi(x+y) - \varphi(x)}{|y|^{N+\alpha}}\, dy$$

with κ given in Theorem 1.1 (note that we also have uniform convergence with respect to x and t).

Proposition 2.6 and the weak convergence of ρ^ε allow us to pass to the limit in the right hand side of (21). We deduce:

$$-\int \rho\, \partial_t \varphi\, dx\, dt - \int \rho_0(x) \varphi(x,0)\, dx = -\int \rho \kappa (-\Delta)^{\alpha/2} \varphi\, dx\, dt$$

which is the weak formulation of (12). This completes the proof of Theorem 1.1, and it only remains to prove Proposition 2.6.

Proof of Proposition 2.6. We split the integral in (22) into two parts by writing

$$\mathcal{L}^\varepsilon(\varphi) = I_1^\varepsilon(x,t) + I_2^\varepsilon(x,t)$$

with

$$\begin{aligned} I_1^\varepsilon(x,t) &= \varepsilon^{-\alpha} \int_{|v|\geq\delta} \nu(v) M(v) \left[\chi^\varepsilon(x,v,t) - \varphi(x,t)\right] dv \\ I_2^\varepsilon(x,t) &= \varepsilon^{-\alpha} \int_{|v|\leq\delta} \nu(v) M(v) \left[\chi^\varepsilon(x,v,t) - \varphi(x,t)\right] dv. \end{aligned}$$

We first show that only the small values of $|v|$ matter in (22), by showing that $I_1^\varepsilon(x,t)$ goes to zero: Using formula (20) and integrations by parts, we can write

$$\begin{aligned} \nu(v)[\chi^\varepsilon(x,v,t) - \varphi(x,t)] &= \int_0^\infty \nu(v)^2 e^{-\nu(v)z} [\varphi(x+\varepsilon v z, t) - \varphi(x,t)]\, dz \\ &= \int_0^\infty \nu(v) e^{-\nu(v)z} \varepsilon v \cdot \nabla_x \varphi(x+\varepsilon v z, t)\, dz \\ &= \varepsilon v \cdot \nabla_x \varphi(x) + \varepsilon^2 \int_0^\infty e^{-\nu(v)z} v^T \cdot D_x^2 \varphi(x+\varepsilon v z, t) \cdot v\, dz. \end{aligned}$$

Using the null flux condition (9) (and the fact that $M = M_0$ for $|v| \leq \delta$), we deduce

$$I_1^\varepsilon(x,t) = \varepsilon^{2-\alpha} \int_{|v|\geq\delta} \int_0^\infty M(v) e^{-\nu(v)z} v^T \cdot D_x^2 \varphi(x+\varepsilon v z, t) \cdot v\, dz.\, dv$$

and so

$$
\begin{aligned}
I_1^\varepsilon(x,t) \quad &\leq \quad \varepsilon^{2-\alpha}\left(\int_{|v|\geq\delta}\int_0^\infty M(v)|v|^2 e^{-\nu(v)z}\,dz\,dv\right)^{1/2} \\
&\qquad \times\left(\int_{|v|\geq\delta}\int_0^\infty M(v)|v|^2 e^{-\nu(v)z}|D_x^2\varphi(x+\varepsilon vz,t)|^2\,dz\,dv\right)^{1/2}.
\end{aligned}
$$

We deduce

$$
\begin{aligned}
||I_1^\varepsilon(t)||_{L^2(\mathbb{R}^N)} \quad &\leq \quad \varepsilon^{2-\alpha}||D_x^2\varphi(t)||_{L^2(\mathbb{R}^N)}\left(\int_{|v|\geq\delta}\int_0^\infty M(v)|v|^2 e^{-\nu(v)z}\,dz\,dv\right) \\
&\leq \quad \left(\int_{|v|\geq\delta} M(v)\frac{|v|^2}{\nu(v)}\,dv\right)||D_x^2\varphi(t)||_{L^2(\mathbb{R}^N)}\varepsilon^{2-\alpha}
\end{aligned}
$$

which goes to 0 as $\varepsilon\to 0$ (note that we use (10) here).

So it only remains to show that $I_2^\varepsilon(x,t)$ converges strongly in $L^2(\mathbb{R}^N\times(0,\infty))$ to $-\kappa(-\Delta)^{\alpha/2}(\varphi)$. We have:

$$
I_2^\varepsilon(x,t) \quad = \quad \varepsilon^{-\alpha}\int_{|v|\leq\delta}\int_0^\infty e^{-\nu(v)z}\nu(v)^2 M(v)[\varphi(x+\varepsilon vz,t)-\varphi(x,t)]\,dz\,dv.
$$

Integrating by parts (with respect to z) and performing the change of variable $s=\nu(v)z$, we can write

$$
\begin{aligned}
I_2^\varepsilon(x,t) \quad &= \quad \varepsilon^{-\alpha}\int_{|v|\leq\delta}\int_0^\infty e^{-\nu(v)z}\nu(v)M(v)\varepsilon v\cdot\nabla_x\varphi(x+\varepsilon vz,t)\,dz\,dv \\
&= \quad \varepsilon^{-\alpha}\int_{|v|\leq\delta}\int_0^\infty e^{-s}M(v)\varepsilon v\cdot\nabla_x\varphi(x+\varepsilon\frac{v}{\nu(v)}s,t)\,ds\,dv.
\end{aligned}
$$

We now use Condition (8) and the change of variable $w=\varepsilon\frac{v}{\nu_0|v|^{N+2+\beta}}$. We denote $\gamma=N+1+\beta$, so that

$$
w=\varepsilon\frac{v}{\nu_0|v|^{\gamma+1}}\,,\qquad |w|=\frac{\varepsilon}{\nu_0}|v|^{-\gamma}\,,\qquad v=\left(\frac{\varepsilon}{\nu_0|w|^{\gamma+1}}\right)^{\frac{1}{\gamma}}w
$$

and

$$
dv=\frac{1}{\gamma}\left(\frac{\varepsilon}{\nu_0|w|^{\gamma+1}}\right)^{\frac{N}{\gamma}}dw.
$$

We deduce (note that $\alpha = \frac{\gamma+N+1}{\gamma}$):

$$
\begin{aligned}
&I_2^\varepsilon(x,t) \\
&\quad = M_0\varepsilon^{-\alpha}\int_{|v|\le\delta}\int_0^\infty e^{-s}\varepsilon v\cdot\nabla_x\varphi(x+\varepsilon\frac{v}{\nu_0|v|^{\gamma+1}}s,t)\,ds\,dv \\
&\quad = M_0\varepsilon^{-\alpha}\int_{|w|\ge\frac{\varepsilon}{\nu_0}\delta^{-\gamma}}\int_0^\infty e^{-s}\left(\frac{\varepsilon^{\gamma+1}}{\nu_0|w|^{\gamma+1}}\right)^{\frac{1}{\gamma}} w\cdot\nabla_x\varphi(x+ws,t)\,ds\,\frac{1}{\gamma}\left(\frac{\varepsilon}{\nu_0|w|^{\gamma+1}}\right)^{\frac{N}{\gamma}}dw \\
&\quad = \frac{\varepsilon^{-\alpha}\varepsilon^{\frac{\gamma+1+N}{\gamma}}}{\gamma}\frac{M_0}{\nu_0^{(N+1)/\gamma}}\int_{|w|\ge\frac{\varepsilon}{\nu_0}\delta^{-\gamma}}\int_0^\infty e^{-s}\frac{1}{|w|^{N+\frac{\gamma+1+N}{\gamma}}}w\cdot\nabla_x\varphi(x+ws,t)\,ds\,dw \\
&\quad = \frac{1}{\gamma}\frac{M_0}{\nu_0^{\alpha-1}}\int_{|w|\ge\frac{\varepsilon}{\nu_0}\delta^{-\gamma}}\int_0^\infty e^{-s}\frac{1}{|w|^{N+\alpha}}w\cdot\nabla_x\varphi(x+ws,t)\,ds\,dw
\end{aligned}
\tag{23}
$$

The integral above can also be written as

$$
J^\varepsilon = \int_{|w|\ge c(\delta)\varepsilon}\int_0^\infty e^{-s}\frac{\varphi(x+ws,t)-\varphi(x,t)}{|w|^{N+\alpha}}\,ds\,dw. \tag{24}
$$

The definition of the Cauchy principal value implies that it converges (pointwise) to

$$
\begin{aligned}
J^0 &= \mathrm{PV}\int_{\mathbb{R}^N}\int_0^\infty e^{-s}\frac{\varphi(x+ws,t)-\varphi(x,t)}{|w|^{N+\alpha}}\,ds\,dw \\
&= \left(\int_0^\infty e^{-s}s^\alpha\,ds\right)\mathrm{PV}\int_{\mathbb{R}^N}\frac{\varphi(x+y,t)-\varphi(x,t)}{|y|^{N+\alpha}}\,dy
\end{aligned}
$$

for all (x,t), which is what we wanted. In order to show that we have convergence in $L^2(\mathbb{R}^N)$, we recall that we can also write

$$
\begin{aligned}
J^0 &= \int_{|w|\ge 1}\int_0^\infty e^{-s}\frac{\varphi(x+ws,t)-\varphi(x,t)}{|w|^{N+\alpha}}\,ds\,dw \\
&\quad + \int_{|w|\le 1}\int_0^\infty \frac{e^{-s}}{|w|^{N+\alpha}}[\varphi(x+ws,t)-\varphi(x,t)-sw\cdot\nabla\varphi(x)]\,ds\,dw
\end{aligned}
$$

with all integrals being defined in the classical sense (no principal value). Proceeding similarly, we can rewrite (24) as follows:

$$
\begin{aligned}
J^\varepsilon &= \int_{|w|\ge 1}\int_0^\infty e^{-s}\frac{\varphi(x+ws,t)-\varphi(x,t)}{|w|^{N+\alpha}}\,ds\,dw \\
&\quad + \int_{c(\delta)\varepsilon\le|w|\le 1}\int_0^\infty \frac{e^{-s}}{|w|^{N+\alpha}}[\varphi(x+ws,t)-\varphi(x,t)-sw\cdot\nabla\varphi(x,t)]\,ds\,dw
\end{aligned}
$$

and so

$$
J^\varepsilon - J^0 = \int_{|w|\le c(\delta)\varepsilon}\int_0^\infty \frac{e^{-s}}{|w|^{N+\alpha}}[\varphi(x+ws,t)-\varphi(x,t)-sw\cdot\nabla\varphi(x,t)]\,ds\,dw.
$$

Finally, integrating by parts (twice) with respect to s, we deduce:

$$J^\varepsilon - J^0 = \int_{|w|\le c(\delta)\varepsilon} \int_0^\infty \frac{e^{-s}}{|w|^{N+\alpha}} w^T \cdot D^2\varphi(x+ws,t)\cdot w\, ds\, dw$$

and so using the fact that

$$\int_{|w|\le c(\delta)\varepsilon} \int_0^\infty \frac{e^{-s}}{|w|^{N+\alpha-2}}\, ds\, dw \le C\varepsilon^{2-\alpha}$$

we deduce

$$\left(\int_{\mathbb{R}^N} |J^\varepsilon - J^0|^2\, dx\right)^{1/2} \le C||D^2\varphi(\cdot,t)||_{L^2(\mathbb{R}^N)}\, \varepsilon^{2-\alpha}.$$

Since $\alpha < 2$, this implies that I_2^ε converges strongly in $L^2(\mathbb{R}^N \times (0,\infty))$ to $-\kappa(-\Delta)^{\alpha/2}(\varphi)$.

Writing

$$\mathcal{L}^\varepsilon(\varphi) = I_1^\varepsilon(x,t) + I_2^\varepsilon(x,t),$$

this completes the proof of Proposition 2.6 □

A Proof of Lemma 2.2

We stress out the fact that a similar estimate (with ρ instead of ρ_ν) was derived in [10] under the condition:

$$\int_{\mathbb{R}^N} M(v')\, \frac{\nu(v)}{b(v,v')}\, dv' \le C \quad \text{for all} \quad v\in\mathbb{R}^N,$$

which, in the simplest case $b(v,v') = a(v)a(v')$, is equivalent to

$$\int_{\mathbb{R}^N} \frac{M(v')}{a(v')}\, dv' \int a(v')M(v')\, dv' \le C.$$

Unfortunately, this condition is clearly incompatible with the degeneracy of the collision frequency (8). By contrast, condition (14) reduces, when $b(v,v') = a(v)a(v')$, to

$$\int_{\mathbb{R}^N} a(v)M(v)\, dv < \infty$$

and is thus clearly satisfied.

Proof of Lemma 2.2. A simple computation using (3) yields

$$-\int_{\mathbb{R}^N} L(f)\frac{f}{F}\, dv = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} b(v,v')M\, M' \left[\frac{f'}{M'} - \frac{f}{M}\right]^2 dv\, dv', \tag{25}$$

where $M = M(v)$ and $M' = M(v')$. Next, integrating (with respect to v') the identity

$$f\,\nu'\,M' - f'\,\nu'\,M = \nu' M\,M' \left[\frac{f}{M} - \frac{f'}{M'}\right]$$

we get

$$f \int \nu M\,dv - \rho_\nu M \int \nu M\,dv = \int_{\mathbb{R}^N} \nu'\,M\,M' \left[\frac{f}{M} - \frac{f'}{M'}\right] dv'$$

or

$$g \int \nu M\,dv = \int_{\mathbb{R}^N} \nu'\,M\,M' \left[\frac{f}{M} - \frac{f'}{M'}\right] dv'$$

where $g = f - \rho_\nu M$. The Cauchy-Schwarz inequality implies

$$\begin{aligned} & g^2 \left(\int \nu M\,dv\right)^2 \\ & \quad \leq \left(\int_{\mathbb{R}^N} b(v,v')\,M\,M' \left[\frac{f}{M} - \frac{f'}{M'}\right]^2 dv'\right) \left(\int_{\mathbb{R}^N} \frac{\nu'^2}{b(v,v')}\,M\,M'\,dv'\right), \end{aligned}$$

so that

$$\begin{aligned} & \int_{\mathbb{R}^N} \frac{\nu g^2}{M}\,dv \left(\int \nu M\,dv\right)^2 \\ & \quad \leq \left(\sup_{v\in\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\nu\nu'^2}{b(v,v')} M' dv'\right) \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} bMM' \left[\frac{f}{M} - \frac{f'}{M'}\right]^2 dv'\,dv\right) \end{aligned}$$

and (25) gives the result.
□

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