FLAME PROPAGATION IN ONE-DIMENSIONAL STATIONARY ERGODIC MEDIA

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Abstract. This paper investigates front propagation in random media for a free boundary problem arising in combustion theory. We show the existence of asymptotic travelling waves solutions with effective speed depending only on the essential infimum of the combustion rate. This result generalizes a previous result of the same authors in the periodic case.

1. Introduction

This paper is devoted to the description of fronts propagation in random media. We consider the following singular reaction-diffusion equation:

$$\partial_t u = u_{xx} - f\left(\frac{x}{\varepsilon}, \omega\right)\beta_\delta(u) \quad \text{ in } \mathbb{R}\times[0,T], \tag{1.1}$$

where the reaction term is given by $\beta_\delta(s) = \frac{1}{\delta}\beta(\frac{s}{\delta})$, with $\beta(s)$ a Lipschitz function satisfying:

$$\begin{cases} \beta(s) > 0 \text{ if } x \in (0,1),\ \beta(s) = 0 \text{ otherwise,} \\ \int_0^1 \beta(s)ds = 1 \end{cases}$$

(β_δ is an approximation of the Dirac measure). This equation arises in combustion theory and models the propagation of deflagration flame in premixed gas. It is known as the ignition temperature model.

The results presented in this paper could be immediately generalized to the following n-dimensional equation

$$\begin{aligned} &\partial_t u - \Delta u + q(y)\cdot\nabla_y u = -f\left(\tfrac{x}{\varepsilon},\omega\right)\beta_\delta(u) && \text{in } Q\times[0,T] \\ &\partial_\nu u = 0 && \text{on } \partial Q\times[0,T] \end{aligned}$$

which models the combustion of a premixed gas in a cylinder $Q = \mathbb{R}\times S \subset \mathbb{R}^n$ with S a smooth subset of $\mathbb{R}^{n-1}$ (where we denote by (x, y) the variable in Q with $x \in \mathbb{R}$, $y \in S$).

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It is a well known fact that when δ goes to zero, Equation (1.1) formally gives rise to the following free boundary equation (see [BCN]):

$$\begin{cases} \partial_t u - u_{xx} = 0 & \text{in } \{u > 0\} \\ |\nabla u|^2 = 2f\,(x/\varepsilon, \omega) & \text{on } \partial\{u > 0\}. \end{cases} \tag{1.2}$$

This limit is usually refer to as the high activation energy limit.

The function f arising in (1.1) and (1.2) is related to the combustion rate; it is independent of the space variable when the media is perfectly homogeneous. In this paper, we assume that heterogeneities arise in the premixed gas in a random manner. More precisely, $(\Omega, \mathcal{F}, \mathcal{P})$ is a given probability space and for each $\omega \in \Omega$, the function $z \mapsto f(z, \omega)$ is a positive continuous function. Moreover, we assume that there exist positive constants $\underline{\Lambda}$ and $\overline{\Lambda}$ such that

$$\underline{\Lambda} \leq f(z, \omega) \leq \overline{\Lambda} \quad \text{ for all } z \in \mathbb{R}.$$

In the periodic setting (i.e. when $f(z + k) = f(z)$ for all $k \in a\mathbb{Z}$), fronts propagation is described by pulsating travelling fronts, that are solutions of (1.1) or (1.2) satisfying

$$\begin{array}{ll} u(x,t) \longrightarrow 0 & \text{as } x \to -\infty, \\ u(x,t) \longrightarrow 1 & \text{as } x \to +\infty, \\ u(x+k,t) = u\left(x, t - \dfrac{k}{c}\right), & \forall k \in a\mathbb{Z} \end{array} \tag{1.3}$$

for some $c > 0$ (c is referred to as the effective speed of propagation). Pulsating traveling fronts for (1.1) were studied in [BH] by H. Berestycki and F. Hamel for fixed δ. In [CLM1] and [CLM2], we investigated their asymptotic behavior when ε and δ go to zero. Several regimes arise depending on the respective rates of convergence of the parameters ε (related to the size of the inhomogeneities in the gas) and δ (related to the width of the flame). When $\varepsilon \ll \delta$, it is relatively easy to show that the asymptotic behavior is simply described by the average of the combustion rate f (this regime correspond to the homogenization of the non-linear diffusion equation (1.1)). The most interesting regimes are thus when $\delta = \tau\varepsilon$ ($\tau > 0$) or $\delta \ll \varepsilon$. When $\delta \ll \varepsilon$, which amounts to studying the homogenization of the free boundary equation (1.2), we showed (see [CLM2]) that pulsating travelling fronts converge to

$$u_0(x,t) = \left(1 - e^{-\gamma(x - \gamma t)}\right)_+$$

as ε goes to zero, where the effective speed of propagation γ is given by

$$\gamma = \sqrt{2 \inf f}.$$

It is this surprising fact (only the infimum of the combustion rate determines the asymptotic behavior) that we wish to generalize here in the case of random inhomogeneities.

In the present paper, we do not assume that the inhomogeneities arise in a periodic manner. Instead, following previous works on such topic by various authors, we assume that the process $f(z,\omega)$ is stationary ergodic. More precisely, we assume the existence, for every $z \in \mathbb{R}$, of a measure preserving transformation $\tau_z : \Omega \to \Omega$ such that:

$$f(z+z',\omega) = f(z,\tau_{z'}\omega) \text{ for all } z,z' \in \mathbb{R} \text{ and } \omega \in \Omega.$$

In particular, this implies that the distribution of the random variable $f(z,\cdot) : \Omega \to \mathbb{R}$ is independent of z. This property is referred to as stationarity; it is the most general extension of the notions of periodicity and almost periodicity for a function to have some self-averaging behavior. Furthermore, we assume that the underlying transformation τ_z is ergodic, that is if $A \subset \Omega$ is such that $\tau_z A = A$ for all $z \in \mathbb{R}$, then $P(A) = 0$ or 1. We say that f is stationary ergodic.

In this framework, it is not difficult to show the following lemma, the proof of which we present in appendix for the sake of completeness:

Lemma 1.1. *There exists a constant $\alpha_0 \geq \underline{\Lambda}$ (independent on ω) such that for all bounded subset A of $\mathbb{R}$ with positive measure, we have*

$$\lim_{t\to+\infty} \inf_{z\in tA} f(z,\omega) = \alpha_0 \qquad a.e.\ \omega \in \Omega$$

In particular, it implies

$$\lim_{\varepsilon\to 0} \inf_{x\in A} f(\frac{x}{\varepsilon},\omega) = \alpha_0 \qquad \text{a.e. } \omega \in \Omega. \tag{1.4}$$

Naturally, we expect α_0 to play the role of $\inf(f)$ in determining the effective speed of propagation when ε goes to zero with $\delta \ll \varepsilon$.

For reasons that will be made clear in Section 2.2, we introduce, for $\tau > 0$, the function

$$\varphi^\tau(x,\omega) = \sup_{y\in(x-\tau,x+\tau)} f(y,\omega).$$

It is readily seen that φ^τ is also stationary ergodic, so applying Lemma 1.1, we deduce the existence of a constant α^τ such that

$$\lim_{t\to+\infty} \inf_{x\in tA} \varphi(x,\omega) = \alpha^\tau \qquad \text{a.e. } \omega \in \Omega,$$

or

$$\lim_{\varepsilon\to 0} \inf_{x\in A} \sup_{y\in(x-\tau\varepsilon,x+\tau\varepsilon)} f(\frac{y}{\varepsilon},\omega) = \alpha^\tau \qquad \text{a.e. } \omega \in \Omega. \tag{1.5}$$

Note that if $f(\cdot,\omega)$ is uniformly continuous on $\mathbb{R}$, then we have

$$\alpha^\tau \longrightarrow \alpha_0 \qquad \text{as } \tau \to 0.$$

Our main result is the following:

Theorem 1.2. *Let*

$$\gamma_0 = \sqrt{2\alpha_0} \qquad \textit{and } \gamma^\tau = \sqrt{2\alpha^\tau},$$

if we denote by $u^{\varepsilon,\delta}(x,t)$ the solution to

$$\begin{cases} \partial_t u = u_{xx} - f(x/\varepsilon,\omega)\beta_\delta(u) \\ u(x,0) = \left(1 - e^{-\gamma_0 x}\right)_+ \end{cases} \tag{1.6}$$

Then, for every $\lambda > 0$ and $\sigma > 0$, there exists $\varepsilon_0(\sigma, \lambda) > 0$ such that for every $\varepsilon < \varepsilon_0$ and every $\delta < \tau\varepsilon$, the function $u^{\varepsilon,\delta}(x,t)$ satisfies

$$\frac{1}{1+\lambda}\left(1 - e^{-\gamma^\tau(1+\lambda)(x-\gamma^\tau(1+\lambda)t - C_{\tau,\lambda})}\right)_+ \leq u^{\varepsilon,\delta}(x,t) \leq \left(1 - e^{-\gamma_0(x-\gamma_0 t)}\right)_+$$

with probability $1-\sigma$ (i.e. for almost all ω in a set of measure $1-\sigma$), where the constant $C_{\tau,\lambda}$ is given by

$$C_{\tau,\lambda} = -\frac{(1+\lambda)\gamma^\tau - \gamma_0}{\gamma_0\gamma^\tau}\ln\lambda + \mathcal{O}((1+\lambda)\gamma^\tau - \gamma_0)$$

In particular, if f is uniformly continuous with respect to x, then

$$\lim_{\lambda\to 0}\lim_{\tau\to 0} C_{\tau,\lambda} = 0$$

and so when ε goes to zero with $\delta \ll \varepsilon$, the function $u^{\varepsilon,\delta}(x,t)$ converges to

$$u(x,t) = \left(1 - e^{-\gamma_0(x-\gamma_0 t)}\right)_+$$

uniformly in $(x,t) \in \mathbb{R} \times [0,T]$ and almost surely in ω.

The proof of Theorem 1.2 relies on the maximum principle and the use of barriers (sub- and super-solutions) that propagate with speed close to γ_0 and γ^τ. The construction of the sub- and supersolution is detailed in Section 2. We start by buiding the barriers for the free boundary equation (1.2), and then for the non-linear equation (1.1). Theorem 1.2 is then proved in Section 3.

The main ingredient in the proof (the construction of the subsolution) relies on observations from numerical simulations that were presented in [CLM2]. Those simulations showed that (in the periodic case) the free boundary moves by jump, staying for a long time on each minimum of f and then travelling quickly through all other values of f. As we will see in Section 2, this is the idea that led us to the construction of the sub-solution.

Remark 1.3. *It is possible to show that the limit when ε goes to zero with $\delta = \tau\varepsilon$ gives rise to a travelling wave travelling with a speed depending on τ. However, there is no explicit formula for γ^τ, and the proof is very technical. We thus chose to restrict ourself to the limiting case $\varepsilon \to 0$ with $\delta \ll \varepsilon$.*

Note that, in this paper, we restrict ourself to one-dimensional fronts. The full n-dimensional result, as in [CLM2] will be developed in a forthcoming paper.

2. Barrier

2.1. **The free boundary problem.** In this section, we construct barriers for the free boundary equation:

$$u_t - u_{xx} = 0 \qquad \text{in } \{u > 0\} \tag{2.1}$$

$$|u_x|^2 = 2f(x/\varepsilon, \omega) \qquad \text{on } \partial\{u > 0\}. \tag{2.2}$$

Note that the function

$$\varphi_\gamma(x,t) = \left(1 - e^{-\gamma(x-\gamma t)}\right)$$

is clearly a solution to (2.1) for all $\gamma \in \mathbb{R}^+$. The goal of this section is to show that for suitable γ, it is possible to modify φ_γ is a neighborhood of the free boundary $\partial\{\varphi_\gamma > 0\}$ to construct a super- or a sub-solution of (2.2).

We define

$$\gamma_\eta^\pm = \sqrt{2\alpha_0(1-(\pm\eta))},$$

which satisfy

$$\gamma_\eta^+ < \gamma_0 \quad \text{and} \quad \gamma_\eta^- > \gamma_0,$$

and we establish the following result:

Proposition 2.1. *For all $T > 0$, for all $\eta > 0$ and for almost all $\omega \in \Omega$, there exists $\varepsilon_0(\omega)$ such that if*

$$\varepsilon < \varepsilon_0(\omega),$$

then there exists a function $h^+_{\eta,\varepsilon}(x,t,\omega)$ (respectively a function $h^-_{\eta,\varepsilon}(x,t,\omega)$) supersolution of (2.1)-(2.2) (respectively subsolution), propagating with speed γ_η^+ (respectively γ_η^-).

Moreover, we have

$$\begin{aligned} h^-_{\eta,\varepsilon}(x,0,\omega) &< \left(1 - e^{-\gamma_\eta^- x}\right)_+ \\ h^+_{\eta,\varepsilon}(x,0,\omega) &> \left(1 - e^{-\gamma_\eta^+ x}\right)_+, \end{aligned} \tag{2.3}$$

and

$$h^\pm_{\eta,\varepsilon} \longrightarrow \left(1 - e^{-\gamma_\eta^\pm(x-\gamma_\eta^\pm t)}\right)_+ \qquad \text{as } \varepsilon \to 0 \tag{2.4}$$

Proof. Super-solution: The construction of the super-solution is the most straightforward. As a matter of fact, the function

$$h_\eta^+(x,t) = \left(1 - e^{-\gamma_\eta^+(x-\gamma_\eta^+ t)}\right)_+,$$

satisfies

$$\begin{cases} \partial_t h_\eta^+ - \partial_{xx} h_\eta^+ = 0, & \text{in } \{h_\eta^+ > 0\}, \\ |\partial_x h_\eta^+|^2 = 2\alpha_0(1-\eta), & \text{on } \partial\{h_\eta^+ > 0\}, \end{cases}$$

with

$$\partial\{h_\eta^+ > 0\} = \{(x,t)\,;\, x = \gamma_\eta^+ t\}.$$

So it is a solution of (2.1) and it will be a super-solution of (2.2) for some $\omega \in \Omega$ as soon as $f(\gamma_\eta^+ t/\varepsilon, \omega) > \alpha_0(1-\eta)$.

By definition of α_0 (see (1.4)), we know that for almost every ω in Ω, there exists $\varepsilon_0(\omega)$ such that

$$\inf_{[0,T]} f\left(\gamma_\eta^+ t/\varepsilon, \omega\right) > \alpha_0(1-\eta) \quad \text{for all } \varepsilon \le \varepsilon_0(\omega).$$

Therefore, for all $\varepsilon < \varepsilon_0(\omega)$, $h_\eta^+(x,t)$ is a super-solution for (2.1)-(2.2). Finally, it is readily seen that (2.3) and (2.4) hold.

Sub-solution: The construction of a sub-solution is a bit more technical: Let

$$\gamma = \gamma_\eta^- = \sqrt{2\alpha_0(1+\eta)}.$$

Then the function

$$w(x,t) = \left(1 - e^{-\gamma(x-\gamma t)}\right)_+$$

is a solution of (2.1) and satisfies $|w_x| = \gamma$, along its free boundary $\{x = \gamma t\}$. In particular, w is a subsolution of (2.2) on the set $\partial\{w(x,t) > 0\} \cap \{f(x/\varepsilon) \le \alpha\}$, where we denote

$$\alpha = \alpha_0(1+\eta).$$

To obtain a subsolution of (2.2), we construct $h_{\eta,\varepsilon}^-(x,t,\omega)$ whose free boundary remains in the set $\{f(x/\varepsilon,\omega) < \alpha\}$.

To that purpose, we introduce

$$\ell^\varepsilon(x,\omega) = \sup\{y \le x\,;\, f(y/\varepsilon,\omega) < \alpha_0(1+\eta/2)\}.$$

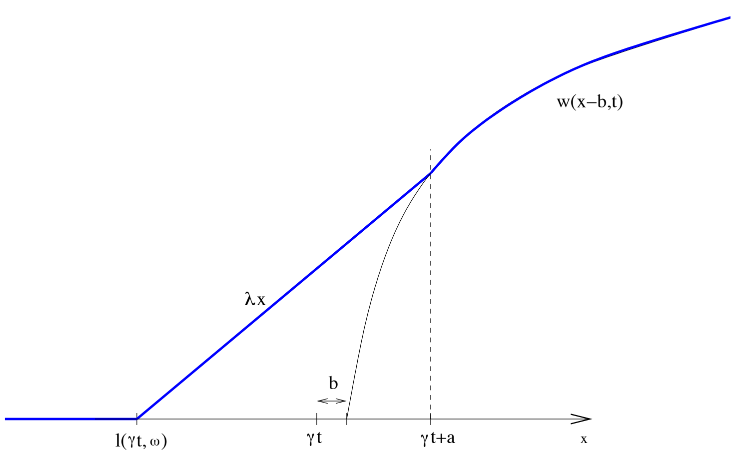
For every x and ω, $\ell^\varepsilon(x,\omega)$ denotes the closest point to x, on the left of x at which the combustion rate f is less that α.

Let a and b be some (small) parameter to be made precise later, we define $h_{\eta,\varepsilon}^-(x,t,\omega)$ as follows:

$$h_{\eta,\varepsilon}^-(x,t,\omega) = \begin{cases} 0 & x < \ell^\varepsilon(\gamma t,\omega) \\ \lambda^\varepsilon(t,\omega)(x - \ell^\varepsilon(\gamma t,\omega)) & \ell^\varepsilon(\gamma t,\omega) < x < \gamma t + a \\ w(x-b,t) & x > \gamma t + a, \end{cases} \tag{2.5}$$

where $\lambda^\varepsilon(t,\omega) > 0$ is such that $h_{\eta,\varepsilon}^-$ is continuous with respect to x at $x = \gamma t + a$, i.e.

$$\lambda^\varepsilon(t,\omega) = \frac{w(\gamma t + a - b, t)}{a + \gamma t - \ell^\varepsilon(\gamma t,\omega)} = \frac{1 - e^{-\gamma(a-b)}}{a + \gamma t - \ell^\varepsilon(\gamma t,\omega)}.$$



Note that $\lambda^\varepsilon(t,\omega)$ is a decreasing function of t on any time interval on which $\ell^\varepsilon(\gamma t,\omega)$ is constant, and that

$$\frac{1-e^{-\gamma(a-b)}}{a+\gamma t-\ell^\varepsilon(\gamma t,\omega)} \le \lambda^\varepsilon(t,\omega) \le \frac{1-e^{-\gamma(a-b)}}{a}.$$

In particular, $\partial_t h^-_{\eta,\varepsilon} - \partial_{xx} h^-_{\eta,\varepsilon} \le 0$ in the linear part of $h^-_{\eta,\varepsilon}$. Thus, to show that $h^-_{\eta,\varepsilon}$ is indeed a sub-solution of (2.1)-(2.2), we only need to show that

$$\lambda^\varepsilon(t,\omega) > \sqrt{2f(\ell^\varepsilon(\gamma t)/\varepsilon,\omega)}$$

(to have a subsolution of the free boundary condition), and that $\partial_{xx} h^-_{\eta,\varepsilon} > 0$ at $x = \gamma t + a$. This will follow from the following lemma:

Lemma 2.2. *For all $\eta > 0$, there exist some constants a_0 and c_0 such that if*

$$a \le a_0, \quad b \le c_0\, a, \quad \text{and} \quad \mu \le c_0\, a,$$

then

$$\frac{1-e^{-\gamma(a-b)}}{a+\mu} > \sqrt{2\alpha_0(1+\eta/2)} \tag{2.6}$$

$$\frac{1-e^{-\gamma(a-b)}}{a} < \gamma e^{-\gamma(a-b)} \tag{2.7}$$

Proof. Let $b = ca$ and $\mu = ca$. Then, when a tends to zero, (2.6) becomes

$$\sqrt{2\alpha_0(1+\eta)}\frac{1-c}{1+c} > \sqrt{2\alpha_0(1+\eta/2)},$$

which is equivalent to

$$\frac{1-c}{1+c} > \sqrt{\frac{1+\eta/2}{1+\eta}}.$$

This inequality is obviously satisfied for small c. Similarly, (2.7) becomes

$$\gamma(1-c) < \gamma$$

which holds for all c. Hence the lemma, with c_0 and a_0 small enough. $\square$

So, for any $a < a_0$ and $b < c_0\, a$, and as long as $\gamma t - \ell^\varepsilon(\gamma t,\omega) \le c_0\, a$, (2.6) implies that

$$\lambda^\varepsilon(t,\omega) > \sqrt{2\alpha_0(1+\eta/2)},$$

which means that $h^-_{\eta,\varepsilon}$ is a subsolution for the free boundary condition (2.2). Moreover, under the same conditions, (2.7) gives

$$\lambda^\varepsilon(t,\omega) < \partial_x h^-_{\eta,\varepsilon}(\gamma t + a - b, t, \omega)$$

which guarantees that

$$\partial_{xx} h^-_{\eta,\varepsilon} \ge 0 \text{ along } x = a + \gamma t,$$

and yields

$$\partial_t h^-_{\eta,\varepsilon} - \partial_{xx} h^-_{\eta,\varepsilon} \le 0, \qquad \text{in } \{h^-_{\eta,\varepsilon} > 0\}.$$

(Note that $\partial_t h^-_{\eta,\varepsilon} \le 0$ along the discontinuity jump for $\ell^\varepsilon(\gamma t,\omega)$).

Therefore, the proof of Proposition 2.1 will be complete if we show that for ε small enough, we have

$$\mu^\varepsilon(\omega) = \sup_{t\in[0,T]} \gamma t - \ell^\varepsilon(\gamma t,\omega) \leq c_0 a.$$

This is a consequence of the following lemma, which says that

$$\lim_{\varepsilon\to 0} \mu^\varepsilon(\omega) = 0 \qquad \text{a.e. } \omega \in \Omega. \tag{2.8}$$

Lemma 2.3. *For every r, s in $\mathbb{R}$, for every $\mu > 0$ and for almost every $\omega \in \Omega$, there exists $\varepsilon_0(\omega)$ such that for all $\varepsilon \leq \varepsilon_0$ we have*

$$x - \ell^\varepsilon(x,\omega) \leq \mu \qquad \forall x \in [r,s]$$

Proof. Let n be a positive integer greater than $2(s-r)/\mu$. We divide $[r,s]$ into n interval of width $(s-r)/n \leq \mu/2$. For each of these intervals $I_k = [r + k\frac{s-r}{n}, r + (k+1)\frac{s-r}{n}]$, there exists $\varepsilon_k(\omega)$ such that if $\varepsilon < \varepsilon_k$, then

$$\inf_{x\in I_k} f(\frac{x}{\varepsilon},\omega) < \alpha_0(1+\eta/2).$$

Taking the infimum of the $\varepsilon_k(\omega)$ for $k \leq n$, we deduce the existence of $\varepsilon_0(\omega)$ such that if $\varepsilon \leq \varepsilon_0(\omega)$, then for all k, there exists $x_k \in I_k$ for which

$$f(x_k/\varepsilon,\omega) < \alpha_0(1+\eta/2).$$

Since $|I_k| \leq \mu/2$, the lemma follows easily. □

Thanks to this lemma, we see that with $a = \mu^\varepsilon/c_0$ and $b = \mu^\varepsilon$ in (2.5), the function $h^-_{\eta,\varepsilon}$ is a subsolution for (2.1)-(2.2) for all ε such that $\mu^\varepsilon/c_0 \leq a_0$. Moreover, using (2.8), it is readily seen that $h^-_{\eta,\varepsilon}$ satisfies

$$h^-_{\eta,\varepsilon}(x,t) \longrightarrow \left(1 - e^{\gamma^-_\eta(x-\gamma^-_\eta t)}\right)_+ \qquad \text{as } \varepsilon \to 0,$$

and that up to a translation of μ^ε to the right, we can always assume that (2.3) holds.

2.2. **Barrier for the nonlinear equation (1.1).**

Barriers for the nonlinear equation (1.1) can be built by bending the functions constructed in the previous section in a neighborhood of their free boundary. The detailed construction is given in Appendix A and it is enough for us to know that this can be achieved if we have

$$\partial_x h^+_{\eta,\varepsilon}(x,t) < \sqrt{2f(x/\varepsilon,\omega)} \quad \text{whenever} \quad 0 \leq h^+_{\eta,\varepsilon}(x,t) \leq \delta \tag{2.9}$$

and

$$\partial_x h^-_{\eta,\varepsilon}(x,t) > \sqrt{2f(x/\varepsilon,\omega)} \quad \text{whenever} \quad 0 \leq h^-_{\eta,\varepsilon}(x,t) \leq \delta \tag{2.10}$$

Inequality (2.9) is satisfied without further restriction since $\partial_x h_{\eta,\varepsilon} \leq \sqrt{2\alpha_0(1-\eta)}$ where α_0 denotes the infinum of f. But we need to modify slightly the definition of $h^-_{\eta,\varepsilon}$ for (2.10) to hold:

Proceeding as in the previous section, we construct $h^-(x,t)$ sub-solution of (2.1), (2.2) propagating with speed $\sqrt{2\alpha^\tau(1+\eta)}$, where α^τ is given by (1.5). Such a function will satisfy (2.10) as soon as $\delta \leq \alpha\tau\varepsilon$. In other words, we have the following result:

Proposition 2.4. *For all $T>0$, for all $\eta>0$ and for almost all $\omega\in\Omega$, there exists $\varepsilon_0(\omega)$ such that if*

$$\varepsilon < \varepsilon_0(\omega), \qquad \textit{and } \delta \leq \tau\varepsilon$$

then there exists a function $h^+_{\eta,\varepsilon,\delta}(x,t)$ (respectively a function $h^-_{\eta,\varepsilon,\delta}(x,t)$) super-solution of (1.1) (respectively sub-solution), propagating with speed

$$\gamma^+_{\eta,\tau} = \sqrt{2M\alpha_0(1-\eta)} \qquad \textit{(respectively } \gamma^-_{\eta,\tau} = \sqrt{2M\alpha^\tau(1+\eta)}\textit{)}$$

Moreover, we have

$$\begin{array}{rcl} h^-_{\eta,\varepsilon,\delta}(x,0) & < & \left(1-e^{-\gamma^-_\eta x}\right)_+ \\ h^+_{\eta,\varepsilon,\delta}(x,0) & > & \left(1-e^{-\gamma^+_\eta x}\right)_+ \end{array} \tag{2.11}$$

for η small enough, and

$$h^\pm_{\eta,\varepsilon,\delta}(x,t) \longrightarrow \left(1-e^{-\gamma^\pm_{\eta,\tau}(x-\gamma^\pm_{\eta,\tau}t)}\right)_+ \qquad \textit{as } \varepsilon,\delta\to 0 \tag{2.12}$$

3. Proof of Theorem 1.2

Using the barriers constructed in the previous sections, we can now prove Theorem 1.2: Let $u^{\varepsilon,\delta}(x,t,\omega)$ be a solution to

$$\left\{ \begin{array}{ll} \partial_t u = u_{xx} - f(x/\varepsilon,\omega)\beta_\delta(u) & \text{in } \mathbb{R}\times(0,+\infty) \\ u(x,0) = \left(1-e^{-\gamma_0 x}\right)_+ & \text{in } \mathbb{R}\,. \end{array} \right. \tag{3.1}$$

We recall (see [CK] for details) that $u^{\varepsilon,\delta}$ is uniformly bounded in $\mathcal{C}^{1,1/2}$, that is:

$$|u^{\varepsilon,\delta}(t,x) - u^{\varepsilon,\delta}(t,x')| \leq C(|x-x'| + |t-t'|^{1/2}).$$

In particular, $u^{\varepsilon,\delta}$ converges uniformly on every compact subset of $\mathbb{R}\times[0,T]$ along some subsequences.

We will use the super-solution and sub-solution constructed in Section 2 (Proposition 2.4). More precisely, for any positive ε, δ, η and $\lambda^\pm$, we take

$$h^+ = h^+_{\eta,(1-\lambda^+)\varepsilon,(1-\lambda^+)\delta}$$

and

$$h^- = h^-_{\eta,(1+\lambda^-)\varepsilon,(1+\lambda^-)\delta}\,.$$

Then, for almost every $\omega \in \Omega$, there exists $\varepsilon_0(\omega) > 0$ such that for any $\varepsilon < \varepsilon_0(\omega)$ and for every $\lambda^\pm > 0$, the functions

$$h^+_\lambda(x,t) = \frac{1}{1-\lambda^+} h^+((1-\lambda^+)x, (1-\lambda^+)^2 t)$$
$$h^-_\lambda(x,t) = \frac{1}{1+\lambda^-} h^-((1+\lambda^-)x, (1+\lambda^-)^2 t)\,.$$

are respectively super- and sub-solution of the first equation in (3.1), propagating with speed

$$\gamma^+ = (1-\lambda^+)\sqrt{2M\alpha_0(1-\eta)} = (1-\lambda^+)\gamma^+_{\eta,\tau}$$

and

$$\gamma^- = (1+\lambda^-)\sqrt{2M\alpha^\tau(1+\eta)} = (1+\lambda^-)\gamma^-_{\eta,\tau}.$$

Note that we cannot find $\overline{\varepsilon}$ such that $\varepsilon_0(\omega) > \overline{\varepsilon}$ a.e. $\omega \in \Omega$. However, it is a classical result that for every σ, there exists a set $A \subset \Omega$ with measure $|A| = 1-\sigma$, and $\overline{\varepsilon} > 0$ such that

$$\varepsilon_0(\omega) > \overline{\varepsilon} \qquad \text{a.e. } \omega \in A.$$

Moreover, inequality (2.11) gives

$$h^+_\lambda(x,0) \geq \frac{1}{1-\lambda^+}\left(1-e^{-\gamma^+ x}\right)_+ \qquad \text{for all } x \in \mathbb{R}\ .$$

Therefore, a simple computation shows that

$$h^+_\lambda(x+M^+,0) \geq \left(1-e^{-\gamma_0 x}\right)_+ \quad \text{for all } x \in \mathbb{R}$$

with

$$M^+ = -\frac{\gamma_0-\gamma^+}{\gamma_0\gamma^+}\ln\lambda^+ + \mathcal{O}(\gamma_0-\gamma^+)$$

Similarly, we have

$$h^-_\lambda(x-M^-,0) \leq \left(1-e^{-\gamma_0 x}\right)_+ \quad \text{for all } x \in \mathbb{R}$$

with

$$M^- = -\frac{\gamma^- -\gamma_0}{\gamma_0\gamma^-}\ln\lambda^- + \mathcal{O}(\gamma^- -\gamma_0)$$

It follows that for every $\varepsilon < \overline{\varepsilon}$ and for almost every $\omega \in A$, $h^+_\lambda(x+M^+,t)$ and $h^-_\lambda(x-M^-,t)$ are respectively super- and sub-solution of (3.1) for all $t \in [0,T]$. The maximum principle thus yields

$$h^-_\lambda(x-M^-,t) \leq u^{\varepsilon,\delta}(x,t) \leq h^+_\lambda(x+M^+,t)$$

for all $x \in \mathbb{R}$ and $t \in [0,T]$.

Using (2.12), we get

$$\frac{1}{1+\lambda^-}\left(1-e^{-\gamma^-(x-M^- -\gamma^- t)}\right)_+ \leq \lim_{\substack{\varepsilon\to 0\\ \delta\leq\tau\varepsilon}} u^{\varepsilon,\delta} \leq \frac{1}{1-\lambda^+}\left(1-e^{-\gamma^+(x+M^+ -\gamma^+ t)}\right)_+ .$$

Since this holds for every $\eta > 0$, we can let η go to zero, and we deduce

$$\begin{aligned}\frac{1}{1+\lambda^-}&\left(1-e^{-\gamma^\tau(1+\lambda^-)(x-M^- -\gamma^\tau(1+\lambda^-)t)}\right)_+\\ &\leq \lim_{\substack{\varepsilon\to 0\\ \delta\leq\tau\varepsilon}} u^{\varepsilon,\delta} \leq \frac{1}{1-\lambda^+}\left(1-e^{-\gamma_0(1-\lambda^+)(x+M^+ -\gamma_0(1-\lambda^+)t)}\right)_+\end{aligned} \tag{3.2}$$

for all $\varepsilon \leq \overline{\varepsilon}$ and for almost all $\omega \in A$.

Next, we note that when η goes to zero, we have $\gamma^+ \to (1-\lambda^+)\gamma_0$, and so

$$M^+ = -\frac{\lambda^+}{(1+\lambda^+)\gamma_0}\ln\lambda^+ + \mathcal{O}(\lambda^+) \longrightarrow 0 \quad \text{as} \quad \lambda^+ \to 0.$$

We deduce (letting λ^+ go to zero):

$$\frac{1}{1+\lambda^-}\left(1-e^{-\gamma^\tau(1+\lambda^-)(x-M^- -\gamma^\tau(1+\lambda^-)t)}\right)_+ \leq \lim_{\substack{\varepsilon\to 0\\ \delta\leq\tau\varepsilon}} u^{\varepsilon,\delta} \leq \left(1-e^{-\gamma_0(x-\gamma_0 t)}\right)_+ .$$

Finally, we note that we have

$$\lim_{\lambda^-\to 0} M^- = +\infty,$$

so we cannot take the limit $\lambda^- \to 0$. However, since $\alpha^\tau \to \alpha_0$ as $\tau \to 0$, we get

$$\lim_{\tau\to 0} M^- = -\frac{\lambda^-}{(1+\lambda^-)\gamma_0}\ln\lambda^- + \mathcal{O}(\lambda^-),$$

which goes to zero as λ^- goes to zero. Thus, taking successively the limits $\tau \to 0$ and $\lambda^- \to 0$, we deduce

$$\lim_{\substack{\varepsilon\to 0\\ \delta\ll\varepsilon}} u^{\varepsilon,\delta} = \left(1-e^{-\gamma_0(x-\gamma_0 t)}\right)_+ \qquad \text{a.e. } \omega\in\Omega.$$

which completes the proof of Theorem 1.2. □

Appendix A. Regularization of the sub- and super-solutions

We consider the nonlinear equation

$$\partial_t u - \Delta u = -f(x)\beta_\delta(u), \tag{A.1}$$

and the free boundary problem

$$\begin{array}{ll} \partial_t u - \Delta u = 0 & \text{in } \{u>0\} \\ |\nabla u|^2 = 2f(x) & \text{on } \partial\{u>0\}. \end{array} \tag{A.2}$$

We define $\Gamma^\eta_\delta(s)$ as follows:

$$\Gamma_\delta(s) = a\delta \qquad \text{for } 0 \le s \le a\delta$$

and Γ_δ solution to

$$\left\{ \begin{array}{ll} \Gamma''_\delta(s) = \chi_\delta(\Gamma_\delta(s)) & \text{for } s \ge a\delta, \\ \Gamma_\delta(a\delta) = a\delta & \\ \Gamma'_\delta(a\delta) = 0, & \end{array} \right.$$

with

$$\chi_\delta(u) = \left\{ \begin{array}{ll} \frac{1+\eta}{2}\beta_\delta(u) & \text{if } a\delta \le u \le \delta \\ 0 & \text{otherwise.} \end{array} \right.$$

The constant $\eta > 0$ will be chosen later, and a is such that

$$\int_a^1 \beta(u)du = \frac{1}{1+\eta}.$$

Similarly, we define $\Psi^\eta_\delta(s)$ as follows:

$$\left\{ \begin{array}{l} \Psi''_\delta(s) = \frac{1-\eta}{2}\beta_\delta(\Psi_\delta(s)), \\ \Psi_\delta(0) = 0 \\ \Psi'_\delta(0) = b, \end{array} \right.$$

where b is such that if u_o satisfies $\Psi(u_o) = \delta$, then $\Psi'(u_o) = 1$.

Thanks to the choice of a and b, it is easy to check that

$$\Gamma_\delta(t) = t - \mathcal{O}(\delta) \text{ when } t \ge \delta$$

and

$$\Psi_\delta(t) = t - \mathcal{O}(\delta) \text{ when } t \ge 2\delta.$$

In particular, we have

$$\Gamma_\delta(t) \xrightarrow{\delta\to 0} t, \qquad \Psi_\delta(t) \xrightarrow{\delta\to 0} t.$$

Then we have the following lemma:

Lemma A.1. *Assume that $f(x)$ is continuous. Then the following holds:*
(i) Let u be a classical supersolution of the free boundary problem (A.2) such that there exists τ and η such that

$$|\nabla u|^2 \le \frac{2f(x)}{1+\eta} \qquad \text{in } \{0<u<\tau\},$$

then $\Gamma^\eta_\delta(u)$ is a supersolution of (A.1) for $\delta < \tau$.

(ii) Let v be a classical subsolution of the free boundary problem (A.2) such that there exists τ and η such that

$$|\nabla u|^2 \geq \frac{2f(x)}{1-\eta} \quad \text{in } \{0 < u < \tau\},$$

then $\Psi^\eta_\delta(u)$ is a subsolution for (A.1) for $\delta < \tau$.

Proof. We only prove (i), leaving (ii) to the reader. We have:

$$\begin{aligned}
\partial_t \Gamma_\delta(u) - \Delta \Gamma_\delta(u) &= \Gamma'_\delta(u)(\partial_t u - \Delta u) - \Gamma''_\delta(u)|\nabla u|^2 \\
&\geq -\frac{1+\eta}{2}\beta_\delta(\Gamma_\delta(u))|\nabla u|^2 1_{\{a\delta \leq \Gamma_\delta(u) \leq \delta\}} \\
&\geq -f(x)\beta_\delta(\Gamma_\delta(u))1_{\{a\delta \leq \Gamma_\delta(u) \leq \delta\}} \\
&\geq -f(x)\beta_\delta(\Gamma_\delta(u)),
\end{aligned}$$

which gives the result. □

Appendix B. Proof of Lemma 1.1

First, we take $I = [-1, 1]$. Then, for every $\omega \in \Omega$, the quantity

$$\inf_{tI} f(z, \omega)$$

is decreasing with respect to t and bounded below by λ. So it converges to some number $\alpha(\omega)$. Next, the stationarity property of f yields

$$\inf_{tI} f(z, \tau_a \omega) = \inf_{tI} f(z + a, \omega) = \inf_{[a-t, a+t]} f(z, \omega) \geq \inf_{[-(t+|a|), t+|a|]} f(z, \omega).$$

So taking the limit $t \to +\infty$ yields

$$\alpha(\tau_a \omega) \geq \alpha(\omega) \quad \text{for all } a \in \mathbb{R}.$$

and so

$$\alpha(\omega) = \alpha(\tau_{-a}\tau_a \omega) \geq \alpha(\tau_a \omega) \quad \text{for all } a \in \mathbb{R}.$$

It follows that α is invariant by translation τ_a:

$$\alpha(\tau_a \omega) = \alpha(\omega) \quad \text{for all } a \in \mathbb{R}.$$

The assumption that the transformation τ_a is ergodic implies that α must be constant a.e.: there exists a constant α_0 such that

$$\alpha_0(\omega) = \alpha_0 \quad \text{a.e. } \omega \in \Omega.$$

So Lemma 1.1 holds for $I = [-1, 1]$, or any interval of the form $[-b, b]$. We also deduce

$$E(\inf_{[-t,t]} f(z, \cdot)) \longrightarrow \alpha_0, \quad \text{as } t \to \infty \tag{B.1}$$

with the expectation $E(\inf_{[-b,b]} f(z, \cdot))$ being invariant by translation with respect to z thanks to the stationarity of f.

Now, if I is any bounded interval of $\mathbb{R}$, we have $I \subset [-b,b]$ for b large enough, and so

$$\liminf_{t\to\infty} \big(\inf_{z\in tI} f(z,\omega)\big) \geq \alpha_0.$$

Morever, thanks to (B.1) it is readily seen that

$$E(\inf_{tI} f(z,\cdot)) \longrightarrow \alpha_o \quad \text{ as } t \to \infty$$

and so

$$\lim_{t\to\infty} \big(\inf_{z\in tI} f(z,\omega)\big) = \alpha_0 \quad \text{a.e. } \omega\in\Omega.$$

Finally, if A is a bounded open set of $\mathbb{R}$, we write that $A = \cup_k I_k \subset [-b,b]$ where the I_k are the connexe components of A. Clearly, we have

$$\inf_{t[-b,b]} f(z,\omega) \leq \inf_{tA} f(z,\omega) \leq \inf_{tI_k} f(z,\omega),$$

and the result follows. □

References

[BCN] Berestycki, H.; Caffarelli, L.A.; Nirenberg, L. Uniform estimates for regularization of free-boundary problems. *Lectures Notes in Pure and Appl. Math.* **122**, Marcel Dekker, (1990), 567-619.

[BH] H. Berestycki, F. Hamel, *Front propagation in periodic excitable media*, Comm. Pure Appl. Math. **55** (2002), 949-1032.

[CK] L.A. Caffarelli, C.E. Kenig, *Gradient estimates for variable coefficient parabolic equations and singular perturbation problems*, American Journal of Mathematics **120** (1998), 391-439.

[CLM1] L.A. Caffarelli, K.-A. Lee, A. Mellet, *Singular limit and Homogenization for flame propagation in periodic excitable media*, Arch. Ration. Mech. Anal. 172 (2004), no. 2, 153–190.

[CLM2] L.A. Caffarelli, K.-A. Lee, A. Mellet, *Homogenization and flame propagation in periodic excitable media: The asymptotic speed of propagation*, Comm. Pure Appl. Math. **59** (2006), no. 4, 501-525

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