Singular limit and homogenization for flame propagation in periodic excitable media

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Abstract

This paper is concerned with a class of singular equations modeling the combustion of premixed gas in periodic media. The model involves two parameters: the period of the medium $|L|$ and a singular parameter ε related to the activation energy.

The existence of pulsating travelling fronts for fixed ε and $|L|$ was proved by H. Berestycki and F. Hamel in [BH]. In the present paper, we investigate the behaviour of such solutions when $\varepsilon \leq \underline{\varepsilon}|L| \ll 1$. More precisely, we establish that Pulsating Travelling Fronts behave like Travelling Waves, when the period $|L|$ is small and $\varepsilon \leq \underline{\varepsilon}|L|$. We also study the convergence as ε goes to zero (and $|L|$ is fixed) of the solution toward a solution of a free boundary problem.

1 introduction

In this paper, we focus on front propagation phenomena for a class of one phase free boundary problems describing laminar flames:

$$\begin{cases} u_t + q(x) \cdot \nabla u = \triangle u & \text{in} \quad \Omega(u) = \{u > 0\} \\ |\nabla u|^2 = 2f(x)M & \text{on} \quad \partial\Omega(u)\,. \end{cases} \tag{1}$$

Such an equation naturally arises as the asymptotic limit (ε goes to zero) of the following advection-reaction-diffusion equation:

$$u_t + q(x) \cdot \nabla u(x) = \triangle u - f(x)\beta_\varepsilon(u), \tag{2}$$

where the reaction term is defined by $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$, with $\beta(s)$ a Lipschitz function satisfying:

$$\begin{cases} \beta(s) > 0 \text{ in } (0,1) \text{ and } \beta(s) = 0 \text{ otherwise,} \\ M = \int_0^1 \beta(s)ds. \end{cases} \tag{3}$$

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Note that, in combustion theory, one usually models the evolution of the temperature $T = 1 - u$; the limit $\varepsilon \to 0$ is then referred to as the high activation energy limit.

When the domain is the whole space $\mathbb{R}^n$, or a cylinder $\mathbb{R}^d \times \omega$ (with ω bounded open subset of $\mathbb{R}^{n-d}$), and when the media is homogeneous (i.e. when the reaction term $f(x)$ and the advection term $q(x)$ are constant), it is natural to seek for travelling waves solutions. These are solutions of the form $u(x,t) = \phi(x \cdot e - ct)$ (where e is the direction of propagation and c is the speed), with

$$\begin{aligned} \phi(s) &\longrightarrow 0\,, \qquad && \text{as } s \to -\infty, \\ \phi(s) &\longrightarrow 1\,, \qquad && \text{as } s \to +\infty. \end{aligned}$$

The existence of travelling waves solving (2) in cylindrical domains is proved in [BN] (see [BNS] for the one dimensional case). Moreover, such solutions are proved to be stable (see [BLR], [R1] and [R2]).

The question of the approximation of (1) by (2), though formally simple, is a delicate problem. The elliptic case was first dealt with in [BCN], where travelling waves solutions of (2) were proved to converge (as $\varepsilon \to 0$) to solutions of (1). The situation in the parabolic context appears to be more delicate. The first results were obtained by L. A. Caffarelli and J. L. Vazquez [CV], for the initial value problem of the advection-free model ($q = 0$); the convergence of u^ε as ε goes to zero was studied under suitable assumptions on the initial data.

More recently, fundamental gradient estimates for solutions of (2) have been established in [CK] in a very general setting. The behaviour of the solutions as $\varepsilon \to 0$ is also studied for the two phases parabolic problem in [CLW1], [CLW2], [LVW] and [D], under a nondegeneracy condition on the negative part of u^ε (all those results do not apply in the one phase situation).

In this paper, we are concerned with equation (2) when the advection term $q(x)$ and the reaction term $f(x)$ are no longer constant, but have some periodicity. In this framework, the notion of travelling waves can be replaced by the more general notion of **pulsating travelling fronts**.

The space domain will be assumed to be an infinite cylinder in $\mathbb{R}^n$:

$$\Omega = \mathbb{R}^d \times \omega, \qquad \text{with } \omega \text{ bounded connected subset of } \mathbb{R}^{n-d}. \tag{4}$$

We also assume that the boundary of ω is smooth. In the sequel, we will split the space variable $(x,y) \in \Omega$, with $x \in \mathbb{R}^d$, and $y \in \omega$.

Let $L = \prod_{i=0}^d L_i\mathbb{Z}$ be a lattice in $\mathbb{R}^d$, with $L_i > 0$, and denote $C = \prod_{i=0}^d [0, L_i] \times \omega$, and $|L| = \sup_{i=1..d} L_i$. We assume that $f(x,y)$ and $q(x,y)$ satisfy:

$$\begin{cases} f(x+k,y) = f(x,y), \text{ for all } k \in L. \\ \text{There exist two constants } \lambda \text{ and } \Lambda \text{ such that} \\ \qquad 0 < \lambda \leq f(x,y) \leq \Lambda. \end{cases}$$

and

$$\begin{cases} \operatorname{div}_{x,y} q(x,y) = 0 \text{ in } \overline{\Omega} \\ \displaystyle\int_C q_i(x,y)dx\,dy = 0\,, \quad \forall 1 \le i \le d \\ q \cdot \nu = 0 \text{ on } \partial\Omega \\ q(x+k,y) = q(x,y)\,, \text{ for all } k \in L, \end{cases} \tag{5}$$

In this framework, we can define pulsating travelling fronts as follows:

Definition 1

Let e be any unit direction in $\mathbb{R}^d$. We say that the function $u(x,y,t)$ is a pulsating travelling front, in the direction e, and with effective speed $c \neq 0$, if $(u(x,y,t),c)$ is the classical solution of the following free boundary problem

$$\begin{aligned} u_t + q \cdot \nabla u &= \triangle u && \text{in } \Omega(u) = \{u > 0\} \\ |\nabla u|^2 &= 2f(x,y)M && \text{on } \partial\Omega(u)\backslash\partial\Omega \\ u_\nu &= 0 && \text{on } \partial\Omega \\ u &\to 0 && \text{as } x \cdot e \to -\infty, \\ u &\to 1 && \text{as } x \cdot e \to +\infty, \\ u(x+k,y,t) &= u\left(x,y,t-\tfrac{k\cdot e}{c}\right), && \forall k \in L \end{aligned} \tag{$\mathbf{P}$}$$

where $\nu(x,y)$ denotes the outward unit normal to $\partial\Omega$, and $u_\nu = \frac{\partial u}{\partial \nu}$.

As we said before, the solutions of ($\mathbf{P}$) can be formally obtained as limits of solutions of the following singular perturbation problem:

$$\begin{aligned} u^\varepsilon_t + q \cdot \nabla u^\varepsilon &= \triangle u^\varepsilon - f(x,y)\beta_\varepsilon(u) \\ u^\varepsilon_\nu &= 0 \quad \text{on } \partial\Omega \\ u^\varepsilon &\to 0 \quad \text{as } x \cdot e \to -\infty, \\ u^\varepsilon &\to 1 \quad \text{as } x \cdot e \to +\infty, \\ u^\varepsilon(x+k,y,t) &= u^\varepsilon\left(x,y,t-\tfrac{k\cdot e}{c^\varepsilon}\right), \quad \forall k \in L \end{aligned} \tag{$\mathbf{P}_\varepsilon$}$$

where β_ε is as in (3).

The existence of pulsating travelling fronts for ($\mathbf{P}_\varepsilon$) has been established by H. Berestycki and F. Hamel in [BH]; we recall their result, as well as a couple of other properties of such solutions in Section 2. Given a solution, it seems natural to investigate its behaviour when

- the singular parameter ε goes to zero (convergence to the free boundary problem),
- the period $|L|$ goes to zero (*homogenization* limit).

In this paper, we wish to investigate the behaviour of the pulsating travelling fronts when

$$\varepsilon \le \underline{\varepsilon}|L| \ll 1,$$

for some constant $\underline{\varepsilon}$.

First, in Section 3, we show that the solution of $(\mathbf{P}_\varepsilon)$ oscillates between two travelling waves propagating with the speed c^ε:

Proposition 1 *Let e be a unit vector in $\mathbb{R}^d$ and denote by $(u^\varepsilon, c^\varepsilon)$ the corresponding solution of $(\mathbf{P}_\varepsilon)$.*
For all $L_o > 0$, there exist $\varepsilon_o(L_o) > 0$ such that if $\varepsilon \leq \varepsilon_o$ and $|L| \leq L_o$, then we have

$$\max\left(0, 1 - \frac{1}{\kappa} e^{-\gamma^\varepsilon(e\cdot x - c^\varepsilon(t+M^*))}\right) \leq u^\varepsilon(x,y,t) \leq \max\left(\varepsilon, 1 - \kappa e^{-\gamma^\varepsilon(e\cdot x - c^\varepsilon(t - M_*\varepsilon)}\right), \tag{6}$$

where M^*, M_* and κ are universal constants, and $\gamma^\varepsilon = \gamma(c^\varepsilon) > 0$ is an increasing function of c^ε.
Moreover, when $\Omega = \mathbb{R}^n$ (i.e. $d = n$) and $\varepsilon \leq \underline{\varepsilon}|L|$ we have

$$M^* \leq \rho^*|L|,$$

with ρ^* universal constant, and $\kappa \to 1$ as $|L| \to 0$.

Here and subsequently, a constant is said to be *universal* if it only depends on β, Λ, λ, $|q|_\infty$ and Ω (diam(ω), d and n). In particular, it is important to note that, unless stated otherwise, all the estimates will be uniform with respect to $|L|$, ε and e.

The key point to establish Proposition 1 is Proposition 13, which states that the free boundary remains in a finite neighbourhood of the hyperplane $x \cdot e - ct = 0$. As a consequence of Proposition 1, when $\Omega = \mathbb{R}^n$ and $\varepsilon \leq \underline{\varepsilon}|L|$, the ε-level set of the solution is almost an hyperplane in $\mathbb{R}^n \times \mathbb{R}$ when the period is small, and u^ε converges to an ordinary travelling wave when $|L| \to 0$.

Whether similar result holds for a general domain Ω satisfying (4) is still an open question. However, in Section 4 we obtain partial results in this direction and we establish in particular the following nondegeneracy estimate:

Proposition 2 *There exists L_o, $\underline{\varepsilon}$, ρ_* and R_o such that if $|L| \leq L_o$ and $\varepsilon \leq \underline{\varepsilon}|L|$, then for any (x_o, y_o, t_o) satisfying*

$$u^\varepsilon(x_o, y_o, t_o) = \varepsilon,$$

we have:

$$\sup_{B_r(x_o,y_o)} u^\varepsilon(x,y,t_o) \geq Cr,$$

for all r such that

$$\rho_*|L| \leq r \leq R_o.$$

Finally, the last section of this paper is concerned with the singular limit $\varepsilon \to 0$, when the period of the lattice is fixed. We prove the following theorem, which shows that the limit u of u^ε satisfies the free boundary condition at any regular point of the free boundary.

Theorem 3

Let $e \in S^{d-1}$, and denote by $(u^\varepsilon, c^\varepsilon)$ the corresponding solution of $(\mathbf{P}_\varepsilon)$ (given by Theorem 4). There exists a subsequence $\varepsilon_j \to 0$ such that

$$\begin{cases} c^{\varepsilon_j} \longrightarrow c, \\ u^{\varepsilon_j}(x,y,t) \longrightarrow u(x,y,t), \ \textit{uniformly on compact sets,} \end{cases}$$

with $u \in \mathcal{C}^{1,1/2}(\Omega \times \mathbb{R})$ solution to

$$u_t + q \cdot \nabla u = \triangle u \quad \textit{in } \{u > 0\}, \qquad u_\nu = 0 \textit{ on } \partial\Omega,$$

and satisfying $u\left(x, y, t - \frac{k \cdot e}{c}\right) = u(x + k, y, t)$, $\forall k \in L$.

Moreover, if $(X_o, t_o) = ((x_o, y_o), t_o)$ is a free boundary point ($(X_o, t_o) \in \partial\{u > 0\} \setminus \partial\Omega$) such that

$$\begin{cases} \textit{there exist } Y_o \in \Omega \textit{ and } r > 0 \textit{ such that} \\ X_o \in \partial B_r(Y_o) \textit{ and } B_r(Y_o) \subset \{u(X, t_o) > 0\}, \end{cases}$$

then, u has a linear behaviour in $B_r(X_o) \times (t_o - r^2, t_o)$, and satisfies the free boundary condition:

$$u(X,t) = \sqrt{2Mf(X_o)}\langle X - X_o, \nu\rangle^+ + o(|X - X_o| + |t - t_o|^{1/2}),$$

with $\nu = \frac{Y_o - X_o}{|Y_o - X_o|}$.

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2 Existence and first properties of solutions of $(\mathbf{P}_\varepsilon)$

2.1 Existence and uniqueness.

H. Berestycki and F. Hamel [BH] have shown the existence and uniqueness of a solution of $(\mathbf{P}_\varepsilon)$ (for fixed $\varepsilon > 0$). More precisely, Theorem 1.13 in [BH] gives:

Theorem 4

Let e be a unit vector in $\mathbb{R}^d$, then we have:

1. *there exists a classical solution $(c,u)=(c^{\varepsilon,e},u^{\varepsilon,e}(x,y,t))$ of $(\mathbf{P}_\varepsilon)$ (by classical solution, we mean $u\in\mathcal{C}^{1,\alpha}(\Omega\times\mathbb{R})$, with continuous second derivatives in space).*

2. *The speed c is unique and positive. The solution $u(x,y,t)$ is unique up to a translation with respect to t.*

3. *The function $u(x,y,\frac{x\cdot e-s}{c})$ is $L-$periodic with respect to x, and increasing with respect to s (in particular, $u(x,y,t)$ is decreasing with respect to t).*

The key point of the proof is the following change of variables:

$$\phi(s,x,y)=u(x,y,\frac{x\cdot e-s}{c}). \tag{7}$$

Then, $\phi(s,x,y)$ is L-periodic with respect to x and satisfies

$$\begin{array}{ll}\phi(s,x,y)\longrightarrow 0, & \text{as } x\to-\infty,\\ \phi(s,x,y)\longrightarrow 1, & \text{as } x\to+\infty.\end{array}$$

Moreover, ϕ solves the following degenerate nonlinear elliptic equation

$$\triangle_{x,y}\phi+|e|^2\phi_{ss}+2e\cdot\nabla_{x,y}\phi_s+c\phi_s-\phi_s e\cdot q-q\cdot\nabla_{x,y}\phi=f(x,y)\beta_\varepsilon(\phi).$$

2.2 Hölder estimates in space and time.

It was proved by L. Caffarelli and C. Koenig [CK] that solutions to (2) have local Lipschitz regularity in space. We deduce:

Lemma 5 (Uniform gradient estimate)

Let u^ε be the solution of $(\mathbf{P}_\varepsilon)$ given by Theorem 4. There exists a universal constant C_0 such that

$$|\nabla_{x,y}u^\varepsilon(x,y,t)|\leq C_0\,,\qquad\forall\,(x,y,t)\in\Omega\times\mathbb{R}.$$

Proof: Since $0\leq u^\varepsilon\leq 1$, interior estimate is given by [CK]. We only have to check that the result also holds in the neighbourhood of a boundary point $(x_o,y_o)\in\partial\Omega$. Consider a local transformation of the y variable which straightens out the boundary so that it becomes $y\cdot e_n=0$, and, nearby, Ω lies in $y\cdot e_n>0$. We also assume that the normal vector ν becomes e_n, in such a way that the Neumann boundary condition allows us to extend u by making it even in y_n. The result of [CK] then applies, and gives Lemma 5. □

Moreover, as usual with parabolic type equations, Lipschitz regularity in space implies $\frac{1}{2}$-Hölder regularity in time. Therefore we have (see D. Danielli [D] for detail):

Proposition 6 *The solution u^ε given by Theorem 4 is uniformly bounded in the hölder space $\mathcal{C}^{1,1/2}(\Omega \times \mathbb{R})$: There exists a universal constant N_0 such that*

$$|u^\varepsilon(X,t) - u^\varepsilon(X',t')| \leq N_0 \left(|X - X'| + |t - t'|^{1/2} \right),$$

for all $(X,t), (X',t') \in (\Omega \times \mathbb{R})^2$.

2.3 A upper bound for the effective speed.

For later purpose, we now want to show that the speed of propagation $c^\varepsilon(e)$ is bounded by above by a universal constant:

Lemma 7 *For all $e \in S^{d-1}$, and for all $\varepsilon \leq 1/2$, we have*

$$c^\varepsilon(e) \leq c_{max} = 4\sqrt{2\Lambda K} + |q|_\infty,$$

with $K = \sup \beta(s)$ (note that this bound is far from being sharp).

Proof: We construct a subsolution of $(\mathbf{P}_\varepsilon)$ which moves at speed $4\sqrt{2\Lambda K} + |q|_\infty$: We start with the following subsolution of $(\mathbf{P})$:

$$v(x,t) = \frac{2\sqrt{2\Lambda K}}{\nu} \Big(1 - \exp(-\nu(x \cdot e - (\nu + |q|_\infty)t)) \Big).$$

It is easy to check that v is a subsolution of

$$u_t + q(x) \cdot \nabla u = \triangle u,$$

and satisfies $|\nabla_x v| \geq \sqrt{2\Lambda K}$ in $\{0 \leq v \leq 1/2\}$. By regularizing $v(x,t)$ in the level set $\{0 < v(x,t) < \varepsilon\}$ we can construct a smooth function v^ε satisfying

$$v^\varepsilon_t + q(x) \cdot \nabla v^\varepsilon \leq \triangle v^\varepsilon - f(x,y)\beta_\varepsilon(v^\varepsilon),$$

such that

$$\begin{cases} v^\varepsilon(x,t) = v(x,t) & \text{in } \{v^\varepsilon > \varepsilon\}, \\ v^\varepsilon(x,t) = 0 & \text{in } \{x \cdot e \leq (\nu + |q|_\infty)t - 1\}. \end{cases}$$

(We can proceed in a similar way as described in Appendix A for supersolutions; see also the proof of Lemma 29).

Next, we observe that when $\nu \geq 4\sqrt{2\Lambda K}$, we have $v \to \frac{2\sqrt{2\Lambda K}}{\nu} \leq \frac{1}{2}$ as $x \cdot e \to \infty$. Hence, there exists R such that

$$u^\varepsilon(x,y,0) \geq v^\varepsilon(x + Re, 0), \quad \forall (x,y) \in \Omega. \tag{8}$$

We now have to prove that (8) holds for all $t \geq 0$. Since we shall make this reasoning a couple of times in this paper, let us write it properly. First, we introduce

$$z(x,y,t) = u^\varepsilon(x,y,t) - v^\varepsilon(x + Re, 0),$$

and denote by t_* the first time for which z vanishes at some point:

$$t_* = \sup\{t \mid z(x,y,t) \geq 0 \ \ \forall (x,y) \in \Omega\}.$$

We can always assume $t_* > 0$ (if not, we translate v some more, by choosing a larger R), and we have:

$$\begin{aligned} &\partial_t z - \triangle z + q \cdot \nabla z \geq -\frac{N_\beta f(x,y)}{\varepsilon^2} z, \quad \text{in } \Omega \times [0,t_*], \\ &z_\nu = 0 \quad \text{on } \partial\Omega, \\ &z \longrightarrow 1/2, \quad \text{as } x \cdot e \to +\infty, \end{aligned} \tag{9}$$

where N_β is the Lipschitz constant of β. Assume now that $t_* < \infty$.

Then, there exists a sequence (x_n, y_n, t_n) in $\Omega \times [t_*, t_*+1]$ such that

$$t_n \longrightarrow t_*, \quad \text{and } z(x_n, y_n, t_n) \leq 0.$$

Since ω is bounded, we can assume that $y_n \to y_*$, and we have to consider two situations:

Case I: $(x_n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}^d$.

Then, we can assume $x_n \to x_*$, and by continuity of z we deduce: $z(x_*, y_*, t_*) \leq 0$. The strong maximum principle (and the Hopf principle if $y_* \in \partial\omega$) implies that $z = 0$ in $\Omega \times [0, t_*]$, which contradicts $z \to 1/2$ as $x \cdot e \to +\infty$.

Case II: $(x_n)_{n\in\mathbb{N}}$ is unbounded.

First of all, we note that, since $v^\varepsilon(x,t) = 0$ in $\{x \cdot e \leq (\nu + |q|_\infty)t - 1\}$, we have $x_n \cdot e \geq (\nu + |q|_\infty)(t_*) - 1$. Moreover, since $z \to 1/2$ as $x \cdot e \to +\infty$, there exists a constant C such that $x_n \cdot e \leq C$.

Therefore, for all n, there exists $k_n \in L$ such that:

$$\tilde{x}_n = x_n - k_n \in B_{R_*}(0), \quad |k_n \cdot e| \leq C,$$

with R_* such that $B_{R_*}(0)$ contains a cell of the lattice L. We set $u_n(x,y,t) = u(x + k_n, y, t) = u(x, y, \frac{t - k_n \cdot e}{c})$ and $v_n(x,t) = v(x + k_n, t) = v(x + (k_n \cdot e)e, t)$. Since $k_n \cdot e$ is bounded, we can assume that u_n and v_n converge to $\tilde{u}$ and $\tilde{v}$. Using the periodicity of $q(x,y)$ and $f(x,y)$ with respect to L, we are back to case I, with $\tilde{z} = \tilde{u} - \tilde{v}$.

As a conclusion, we have $t_* = \infty$ and therefore (8) holds for all $t \geq 0$. It follows that the ε-level set of u^ε cannot move faster than $\nu + |q|_\infty$, and the proof is complete. □

2.4 A lower bound for the effective speed

Finally, we end this preliminary section by proving that $c^\varepsilon(e)$ is also bounded by below:

Lemma 8 *There exists a universal constant c_{min} (independent of ε and $|L|$) such that if $\varepsilon \leq \underline{\varepsilon}|L|$, we have:*

$$c^{\varepsilon} \geq c_{min}$$

Proof: The proof of Lemma 8 is very different from the proof of the previous Lemma. As a matter of fact, the advection term $q \cdot \nabla_x$ prevents us from constructing a supersolution to $(\mathbf{P}_{\varepsilon})$ that propagates with a positive speed.

Instead, the proof relies on the following equality obtained in [BH] by integrating (2) over $G \times \mathbb{R}$, for any subset G of Ω of the form $G = \cup_i C(x_i) \times \omega$ (where $C(x_i)$ denotes a cell of the lattice L centered in x_i):

$$\int_{\mathbb{R}} \int_G f(x,y)\beta_{\varepsilon}(u^{\varepsilon})dx\, dy\, dt \;= |G|. \tag{10}$$

Before going any further, let us introduce some notation that we shall use later on: $B_R(X_o)$ denotes the Euclidean ball in $\mathbb{R}^n$ centered in $X_o = (x_o, y_o) \in \mathbb{R}^d \times \omega$, with radius R, and we denote by $Q_R(X_o,t_o)$ the parabolic neighbourhood:

$$Q_R(X_o,t_o) = B_R(X_o) \times]t_o - R^2, t_o[.$$

We denote by $\mathcal{L}^n$ (resp. $\mathcal{H}^{n-1}$) the Lebesgue measure in $\mathbb{R}^n$ (resp. the $n-1$-Hausdorff measure), and we introduce the density (for $A \subset \Omega$ and $B \subset \Omega \times \mathbb{R}$):

$$\begin{aligned} \Theta^n(A \cap B_R) &= \frac{\mathcal{L}^n(A \cap B_R)}{\mathcal{L}^n(B_R)}. \\ \Theta^{n+1}(B \cap Q_R) &= \frac{\mathcal{L}^{n+1}(B \cap Q_R)}{\mathcal{L}^{n+1}(Q_R)}. \end{aligned} \tag{11}$$

We also introduce K a positive constant such that $\beta(s) \geq K$ for all $s \in [\frac{\varepsilon}{4}, \frac{3\varepsilon}{4}]$; we therefore have:

$$\beta_{\varepsilon}(s) \geq \frac{K}{\varepsilon} \qquad \text{for all } s \in [\frac{\varepsilon}{4}, \frac{3\varepsilon}{4}]. \tag{12}$$

Let us now go back to the proof of Lemma 8. We start with the following observation: The quantity

$$\Theta^n \left(\left\{ u^{\varepsilon}(x,y,t) \geq \frac{3\varepsilon}{4} \right\} \cap B_{20|L|}(0) \right)$$

is null for large positive t and is equal to 1 for large negative t; Therefore, there exists a time t_o for which

$$\Theta^n \left(\left\{ u^{\varepsilon}(x,y,t_o) \geq \frac{3\varepsilon}{4} \right\} \cap B_{20|L|}(0) \right) = \frac{1}{4}. \tag{13}$$

Next, we show that for such a t_o we have:

$$\Theta^n\left(\left\{u^\varepsilon(x,y,t)\geq\frac{3\varepsilon}{4}\right\}\cap B_{20|L|}(0)\right)\quad\begin{array}{l}\geq\dfrac{1}{4},\\ \leq\dfrac{1}{2}\quad\text{for all } t\in[t_o-|L|/c^\varepsilon,t_o],\end{array}\tag{14}$$

and

$$\Theta^n\left(\left\{u^\varepsilon(x,y,t)\leq\frac{\varepsilon}{4}\right\}\cap B_{20|L|}(x_o,y_o)\right)\quad\begin{array}{l}\leq\dfrac{3}{4},\\ \geq\dfrac{1}{4},\quad\text{for all } t\in[t_o-|L|/c^\varepsilon,t_o].\end{array}\tag{15}$$

We postpone the proof of (14) and (15) to the end of this section and conclude the proof of Lemma 8:

Using the coarea formula (see [F]) we can write

$$\begin{aligned}&\int_{t_o-|L|/c}^{t_o}\int_{B_{20|L|}}f(x,y)\beta_\varepsilon(u^\varepsilon)\,dx\,dy\,dt\\&\qquad\geq\frac{\lambda}{N_o}\int_{t_o-|L|/c}^{t_o}\int_{\mathbb{R}}\beta_\varepsilon(z)\mathcal{H}^{n-1}(\{u^\varepsilon(t)=z\}\cap B_{20|L|})dzdt.\end{aligned}\tag{16}$$

For all $z\in[\frac{\varepsilon}{4},\frac{3\varepsilon}{4}]$ and $t\in[t_o-20|L|,t_o]$, we have

$$\begin{aligned}\mathcal{L}^n(\{u^\varepsilon(t)\geq z\}\cap B_{20|L|})\geq\frac{1}{4}\mathcal{L}^n(B_{20|L|})\\ \mathcal{L}^n(\{u^\varepsilon(t)\leq z\}\cap B_{20|L|})\geq\frac{1}{4}\mathcal{L}^n(B_{20|L|}),\end{aligned}$$

therefore, Lemma 31 (i) (which is a consequence of the isoperimetric inequality) gives (for all $z\in[\frac{\varepsilon}{4},\frac{3\varepsilon}{4}]$):

$$\begin{aligned}\mathcal{H}^{n-1}(\{u^\varepsilon(t)=z\}\cap B_{20|L|})\quad&\geq\kappa\left(\mathcal{L}^n(\{u^\varepsilon(t)\geq z\}\cap B_{20|L|})\right)^{(n-1)/(n)}\\&\geq\kappa(\mathcal{L}^n(B_{20|L|}))^{(n-1)/(n)}.\end{aligned}$$

Together with (16), it yields

$$\int_{t_o-|L|/c^\varepsilon}^{t_o}\int_{B_{20|L|}}f(x,y)\beta_\varepsilon(u^\varepsilon)\,dx\,dy\,dt\geq\frac{K}{\varepsilon}\frac{\varepsilon}{2}\frac{|L|}{c^\varepsilon}\left(\mathcal{L}^n(B_{20|L|})\right)^{(n-1)/n},$$

or

$$\int_{\mathbb{R}}\int_{B_{20|L|}}f(x,y)\beta_\varepsilon(u^\varepsilon)\,dx\,dy\,dt\geq\frac{\widetilde{K}}{c^\varepsilon}|L|^n.$$

The lemma follows from (10). □

Proof of (14) and (15): The first inequality in (14) follows from (13) and the monotonicity of u with respect to t. The second inequality is a consequence of the periodicity of u: with $k_o \in L$ such that $k_o \cdot e \geq |L|$ and $k_o \leq 2|L|$ we have, for all $t \in [t_o - |L|/c^\varepsilon, t_o]$:

$$\begin{aligned}\left\{u^\varepsilon(x,y,t) \geq \frac{3\varepsilon}{4}\right\} \cap B_{20|L|}(0) \quad &\subset \left\{u^\varepsilon(x,y,t_o - \frac{k_o \cdot e}{c^\varepsilon}) \geq \frac{3\varepsilon}{4}\right\} \cap B_{20|L|}(0) \\ &\subset \left\{u^\varepsilon(x,y,t_o) \geq \frac{3\varepsilon}{4}\right\} \cap B_{20|L|}(k_o,0).\end{aligned}$$

Since

$$\mathcal{L}^n\big(B_{20|L|}(k_o,0) \setminus B_{20|L|}(0,0)\big) \leq \frac{1}{4}\mathcal{L}^n\big(B_{20|L|}(0)\big),$$

we get (14).

In order to establish the second inequality in (15) (the first one is an immediate consequence of (14)), we need the following lemma:

Lemma 9 *For any parabolic neighbourhood Q_R such that $B_R \subset \Omega$, we have*

$$\Theta^{n+1}(\{\frac{\varepsilon}{4} \leq u^\varepsilon(x,t) \leq \frac{\varepsilon}{2}\} \cap Q_R) \leq C\frac{\varepsilon}{R}.$$

Applying Lemma 9, we see that the set $\{u^\varepsilon \leq \varepsilon/4\}$ has density at least $1/2$ in $Q_{20|L|}(x_o, y_o, 0)$, for some $\underline{\varepsilon}$ small enough. Using again the translation property, we easily deduce the second inequality in (14). □

Proof of Lemma 9: We integrate equation (2) on $Q_R(x_o, y_o, t_o)$, and get:

$$\begin{aligned}\int_{B_R} u^\varepsilon(x,t_o) - u^\varepsilon(x,t_o - R^2)\,dx\,dy - \int_{t_o-R^2}^{t_o}\int_{\partial B_R} \nabla_x u^\varepsilon \cdot \nu\,d\sigma(x,y)\,dt \\ + \int_{Q_R} f(x,y)\beta_\varepsilon(u^\varepsilon)\,dx\,dy\,dt = 0.\end{aligned}$$

Since $|u^\varepsilon(t_o) - u^\varepsilon(t_o - R^2)| \leq N_o R$ and $|\nabla_x u^\varepsilon| \leq N_o$ (see Proposition 6), it follows:

$$\int_{Q_R} f(x,y)\beta_\varepsilon(u^\varepsilon)\,dx\,dy\,dt \leq CR^{n+1}.$$

On the other side, (12) gives

$$\int_{Q_R} f(x,y)\beta_\varepsilon(u^\varepsilon)\,dx\,dy\,dt \geq \lambda\frac{K}{\varepsilon}\mathcal{L}^{n+1}(\{\frac{\varepsilon}{4} \leq u^\varepsilon(x,y,t) \leq \frac{\varepsilon}{2}\} \cap Q_R),$$

and the result follows. □

3 Global behaviour of the pulsating travelling fronts

This section is devoted to the proof of Proposition 1. To begin with, we establish two key properties satisfied by the pulsating travelling fronts: The Birkoff property and a weak nondegeneracy property. We then show that the oscillations of each level set at a given time are uniformly bounded, and deduce that pulsating travelling fronts remain in a finite (and uniform) neighbourhood of a standard travelling wave propagating with the speed c^ε.

Throughout this section, $(u^\varepsilon, c^\varepsilon)$ denotes the solution of $(\mathbf{P}_\varepsilon)$ given by Theorem 4.

3.1 Birkoff's property

The Birkoff property is a simple geometric property that will allow us to control the global behaviour of the level sets of u. Similar Property was first used by L. Caffarelli and R. de La Llave [CL] (Theorem 8.1) for minimal surfaces.

For any vector $m \in \mathbb{R}^d$, we denote by T_m the translation operator in the direction m:

$$T_m(D) = \{(x+m, y) | (x, y) \in D\},$$

where D is a subset of Ω. We also define

$$\Omega_\eta(t) = \{(x, y, t) \in \Omega \times \mathbb{R} | u^\varepsilon(x, y, t) \geq \eta\}.$$

Then, we have:

Lemma 10 (Birkoff's Property)
For all $\eta > 0$ and $t \in \mathbb{R}$, we have

$$\begin{array}{lll} T_m(\partial\Omega_\eta(t)) & \subset \Omega_\eta(t) & \text{for all } m \in L \text{ such that } m \cdot e \leq 0, \\ T_m(\partial\Omega_\eta(t)) & \subset \Omega_\eta(t) & \text{for all } m \in L \text{ such that } m \cdot e \geq 0. \end{array}$$

Proof: This is an immediate consequence of Theorem 4 (3):
For $m \in L$ such that $m \cdot e > 0$, we easily check that

$$u^\varepsilon(x+m, y, t) = u^\varepsilon(x, y, t - \frac{m \cdot e}{c}) \geq u^\varepsilon(x, y, t)\,,$$

and the lemma follows. □

A first consequence is the following lemma, which tells us that the homogenization limit of u^ε does not depends on the transverse variable $x - (x \cdot e)e$:

Lemma 11 *For all $m \in \mathbb{R}^d$ such that $m \cdot e = 0$, we have*

$$|u^\varepsilon(x+m, y, t) - u^\varepsilon(x, y, t)| \leq N_o |L|.$$

Proof: Let k_0, k_1 be such that

$$k_i \in L, \quad |k_i - m| \leq |L|, \quad k_0 \cdot e \geq 0, \quad k_1 \cdot e \leq 0.$$

Then thanks to the Birkoff property we have

$$\begin{cases} u^\varepsilon(x + k_0, y, t) \geq u^\varepsilon(x, y, t) \\ u^\varepsilon(x + k_0, y, t) \leq u^\varepsilon(x, y, t), \end{cases}$$

and therefore

$$\begin{aligned} & u^\varepsilon(x + m, y, t) - u^\varepsilon(x + k_0, y, t) \\ & \qquad \leq u^\varepsilon(x + m, y, t) - u^\varepsilon(x, y, t) \leq \\ & \qquad\qquad u^\varepsilon(x + m, y, t) - u^\varepsilon(x + k_1, y, t). \end{aligned}$$

The lemma follows from the gradient estimate. □

3.2 A weak nondegeneracy property

The derivation of a nondegeneracy property is always the key point in the study of the limit $\varepsilon \to 0$. However, the nondegeneracy property that we derive in this section only describes the behaviour of u^ε at distance larger than $\rho_o|L|$ from the most left end point of the free boundary.

We assume first that u^ε is such that:

$$(0, 0) \in \partial\Omega_\varepsilon(0) \quad \text{and} \quad \partial\Omega_\varepsilon(0) \subset \{e \cdot x \geq 0\}, \tag{17}$$

and we denote by $B_r(0)$ the ball in $\mathbb{R}^d$ with radius r and centered in $x = 0$. Then we have:

Lemma 12

There exist ε_o and ρ_o universal such that, for all $\varepsilon \leq \varepsilon_o$ and $R_o > \rho_o|L|$, there exists a positive constant $C_o(R_o)$ such that

$$\sup_{(x,y) \in B_r(0) \times \omega} u^\varepsilon(x, y, s) \geq C(R_o) r\,, \quad \forall s \leq 0,$$

for all $\rho_o|L| < r < R_o$.

Proof: We are going to prove that

$$\sup_{(x,y,t) \in \Omega \times \mathbb{R}^-,\, x \cdot e - ct \leq r/2} u^\varepsilon(x, y, t) \geq C_o(R_o) r\,. \tag{18}$$

Let us first check that (18) gives the Lemma: let (x, y, t) be such that

$$x \cdot e - c^\varepsilon t \leq r/2 \quad \text{and} \quad u^\varepsilon(x, y, t) \geq Cr,$$

and let $k \in L$ be such that

$$-k \cdot e \leq c^\varepsilon t \leq -k \cdot e + |L|\,.$$

Then, we have

$$u^\varepsilon(x+k,y,0) = u^\varepsilon(x,y-\frac{k\cdot e}{c^\varepsilon}) \geq u^\varepsilon(x,y,t) \geq Cr.$$

Set $\overline{x} = x+k$. In order to conclude, we need to find a vector $l \in L$ such that $\overline{x}+l \in B_r$ and $l\cdot e \geq 0$. This is possible if the set $\{r/2 \leq x\cdot e \leq r\,, |x| \leq r\}$ contains a cell of the lattice L, which holds for $r \geq \rho_o|L|$. We deduce

$$u^\varepsilon(\overline{x}+l,y,0) = u^\varepsilon(\overline{x},y,-\frac{l\cdot e}{c^\varepsilon}) \geq Cr\,,$$

which gives Lemma 12.

Let us therefore assume that (18) does not hold, that is

$$\sup_{(x,y,t)\in\Omega\times\mathbb{R}^-,\, x\cdot e - c^\varepsilon t\leq r} u^\varepsilon(x,y,t) \leq \zeta r\,, \quad \forall y\in\Omega\,, \tag{19}$$

for ζ small, which will be chosen later. The contradiction will come from the following remark:

Remark 1 *There exists $k_o \in L$ such that $|L|/2 \leq k_o\cdot e \leq 2|L|$, and for such a point, we have:*

$$u^\varepsilon(k_o,0,t) > \varepsilon \quad \textit{for all } t < \tfrac{|L|}{2c^\varepsilon}.$$

We shall get a contradiction by constructing a supersolution h^ε of $(\mathbf{P}_\varepsilon)$, which moves faster than $4c$ and therefore reaches the point k_o before time $t_o = \frac{|L|}{2c^\varepsilon}$. For ζ small enough, we will see that h^ε is greater than u^ε along the boundary $\{x\cdot e - c^\varepsilon t = r\}$, and for all x such that $x\cdot e - c^\varepsilon T \leq r$ for some $T\leq 0$ very large. The maximum principle and the previous remark will lead to a contradiction.

We first define the following supersolution of $(\mathbf{P})$:

$$h(x,t) = \frac{\sqrt{A}}{\nu+|q|_\infty}\Big[1-\exp\big(-(\nu+|q|_\infty)(x\cdot e-\nu t)\big)\Big], \tag{20}$$

with A to be determined later. It is easy to check that $h(x,t)$ satisfies:

$$h_t - \triangle h + q(x)\cdot\nabla h \geq 0, \quad \text{ on } \Omega\times\mathbb{R},$$

and

$$|\nabla_x h(x,t)|^2 = A, \quad \text{ along the F.B. } \partial\{h>0\}.$$

Therefore, h is a supersolution of $(\mathbf{P})$, as soon as $A \leq 2\lambda M$. Moreover, it is easy to check that if $\nu = 4c^\varepsilon$, the $\varepsilon-$level set of h^ε will reach the point k_o defined in remark 1 before time t_o. (Note that, in view of Lemma 7, we have $\nu \leq 4c_{max}$ universal constant).

Next, we construct h^ε supersolution of $(\mathbf{P}_\varepsilon)$ such that $h^\varepsilon = h$ in $\{h\geq\varepsilon\}$: Let a, $b\in[0,1]$ and $K>0$ be such that

$$\beta(s) \geq K \quad \forall s\in[a,b],$$

then, we prove in Appendix A that for $\varepsilon \le \varepsilon_o$ (ε_o only depending on a, b, K and λ) we can construct h^ε supersolution of $(\mathbf{P}_\varepsilon)$ such that:

$$\begin{array}{ll} h^\varepsilon(x,t) = h(x,t) & \text{in } \{h(x,t) \ge b\varepsilon\}, \\ h^\varepsilon(x,t) \ge a\varepsilon\,, & \forall (x,y,t) \in \Omega \times \mathbb{R}\,. \end{array}$$

Moreover, it is easy to check that if $\nu = 4c^\varepsilon$, the $\varepsilon-$level set of h^ε will reach the point k_o defined in remark 1 before time t_o. (Note that, in view of Lemma 7, we have $\nu \le 4c_{max}$ universal constant).

Now, we want to check that for ζ small enough, one has $h^\varepsilon \ge u^\varepsilon$ in $\{x \cdot e - c^\varepsilon t \le r\}$.

First of all, along the boundary $x \cdot e - c^\varepsilon t = r$, and for $t \le t_o$, we have $x \cdot e - \nu t \ge r + (c^\varepsilon - \nu) t_o \ge r + \frac{c^\varepsilon - \nu}{2c^\varepsilon}|L|$, and therefore

$$h^\varepsilon(x,t) \ge \frac{\sqrt{A}}{\nu + |q|_\infty}\Big[1 - \exp\big(-(\nu + |q|_\infty)(r + \frac{c^\varepsilon - \nu}{2c^\varepsilon}|L|)\big)\Big] \ge C_o(R_0) r,$$

as soon as $3|L| \le r \le R_o$. Next, we want to see that for a large negative T we have

$$h^\varepsilon(x,T) \ge u^\varepsilon(x,y,T)\,, \qquad \text{for all } x \text{ such that } x \cdot e - c^\varepsilon T \le r.$$

To that purpose, let us introduce

$$\begin{array}{rl} \phi^\varepsilon(s,x,y) & = u^\varepsilon(x, y, \frac{x\cdot e - s}{c^\varepsilon}), \\ \psi^\varepsilon(s,x) & = h(x, \frac{x \cdot e - s}{c^\varepsilon}) \\ & = \dfrac{\sqrt{A}}{\nu + |q|_\infty}\Big[1 - \exp\big(-(\nu + |q|_\infty)((1 - \dfrac{\nu}{c^\varepsilon})x \cdot e + \dfrac{\nu}{c^\varepsilon}s)\big)\Big]\,. \end{array}$$

Since ϕ^ε is L-periodic with respect to x, and $\lim_{s \to -\infty} \phi^\varepsilon(s,x,y) = 0$, there exists $B \ge 0$ such that

$$\phi^\varepsilon(s,x,y) \le a\varepsilon \qquad \forall s \le -B\,.$$

Moreover, for $|T|$ large, we have:

$$\psi^\varepsilon(s,x) \ge \varepsilon\,, \qquad \forall s \ge -B\,,\ x \cdot e \le r + c^\varepsilon T\,.$$

It follows that

$$\begin{array}{ll} h^\varepsilon(x,T) \ge a\varepsilon \ge u^\varepsilon(x,y,T)\,, & x \cdot e - c^\varepsilon T \le -B\,, \\ h^\varepsilon(x,T) \ge \varepsilon \ge u^\varepsilon(x,y,T)\,, & -B \le x \cdot e - c^\varepsilon T \le 0\,, \\ h^\varepsilon(x,T) \ge Cr \ge u^\varepsilon(x,y,T)\,, & 0 \le x \cdot e - c^\varepsilon T \le r\,. \end{array}$$

Hence $z(x,y,t) = h^\varepsilon(x,t) - u^\varepsilon(x,y,t)$ is positive along the parabolic boundary $\partial\{(x,y,t)\ s.t.\ t \in [T,t_o]\,,\ x \cdot e - c^\varepsilon t \le r\} \setminus \partial\Omega$ (and satisfies $z_\nu = 0$ along $\partial\Omega$), and using the maximum principle and Hopf Lemma, we deduce in the same way as in the proof of Lemma 7, that:

$$h^\varepsilon(x,t) \ge u^\varepsilon(x,y,t)\,, \qquad \text{in } \{(x,y,t)\ ;\ t \in [T,t_o]\,,\ x \cdot e - c^\varepsilon t \le r\}.$$

which contradicts

$$h^\varepsilon(k_o,t_o) < \varepsilon \le u^\varepsilon(k_o,0,t_o)\,,$$

with $k_o \in L$ as in Remark 1. $\square$

Remark 2 *When (17) is not satisfied, since the equation ($\mathbf{P}_\varepsilon$) is invariant under discrete translations with respect to the x variable, we may always assume that $\Omega_\varepsilon(u^\varepsilon) \subset \{e\cdot x \ge c^\varepsilon t\}$ and that there is a point $(x_o,y_o) \in \partial\Omega_{\varepsilon,0}(u^\varepsilon) \cap \{0 \le e\cdot x \le \sqrt{d}L\}$. It is readily seen that a small modification in the proof of Lemma 12 then yields:*

$$\sup_{(x,y)\in B_r(x_o)\times\omega} u^\varepsilon(x,y,s) \ge C(R_o)r\,, \quad \forall s \le 0\,.$$

3.3 Oscillations of the Free Boundary.

The next proposition states that the ε-level set remains in a finite neighbourhood of the hyperplane $x = c^\varepsilon t$:

Proposition 13 *For all $L_o > 0$, there exist $\varepsilon_o(L_o) > 0$ and $M^*(L_o) \ge 0$ depending only on L_o such that, if $\varepsilon \le \varepsilon_o$ and $|L| \le L_o$, the solution of ($\mathbf{P}_\varepsilon$) satisfies:*

$$\{u^\varepsilon(x,y,t) = \varepsilon\} \subset \{(x,y,t) \mid\ c^\varepsilon t \le e\cdot x \le c^\varepsilon t + M^*\}\,,$$

after a suitable translation in time.

Moreover, when $\Omega = \mathbb{R}^n$ and $\varepsilon \le \mu_o|L|$, we have $M^* \le \rho^*|L|$ with ρ^* depending only on L_o.

Proof: After translation with respect to x, we may assume that $\Omega_\varepsilon(t) \subset \{e\cdot x \ge c^\varepsilon t\}$ and that there exists a point $(x_o,y_o) \in \partial\Omega_\varepsilon(0)\cap\{0 \le e\cdot x \le \sqrt{d}L\}$. Applying Lemma 12 (see Remark 2) with $R_o > \rho_o L_o$, we find $(\tilde{x},\tilde{y}) \in B_{R_o}(x_o)\times\omega$ and $C_o = C(R_o)$ such that

$$\sup_{B_{R_o}(0)\times\omega} u^\varepsilon = u(\tilde{x},\tilde{y},0) \ge C_o.$$

Define $R_* = \max\{\text{diam}(\omega), pL_o\}$, with p such that the ball $B_{pL_o}(0)$ contains the cell $\Pi_{i=1}^d]-L_i,L_i[$. If we prove that for some t_* we have

$$B_{R_*}(\tilde{x},\tilde{y})\cap\Omega \subset \Omega_\varepsilon(t_*), \tag{21}$$

for some t_* under control, then, we are done. As a matter of fact, the Birkoff property (Lemma 10) implies

$$\cup_{m\in L,\, e\cdot m\ge 0} T_m\left(B_{p|L|}(\tilde{x})\times\omega\right) \subset \Omega_\varepsilon(t_*),$$

and since $\cup_{m\in L,\, e\cdot m\geq 0} T_m(B_{p|L|}(\tilde{x})\times\omega)$ contains a half plane $\{(x,y)\in\mathbb{R}^d\times\omega \mid x\cdot e\geq \widetilde{M}\}$, we deduce:

$$\partial\Omega_\varepsilon(t_*)\subset\{(x,y)|\, 0\leq e\cdot x\leq \widetilde{M}\}.$$

and the proposition follows from the propagation property, with $M^*\sim\widetilde{M}+2(2R_*^2)c_{max}+\sqrt{d}L_o$.

Let us therefore assume that for some large negative t_* (21) does not hold, and therefore that we have (using the monotonicity of u^ε):

$$u^\varepsilon(x_1,y_1,t)\leq\varepsilon \quad\text{and}\quad \sup_{B_{R_*}(\tilde{x},\tilde{y})} u^\varepsilon(x,y,t)\geq C_o \qquad \forall t\in[t_*,0]. \tag{22}$$

With $\varepsilon_o=C_o/4$, if $\varepsilon\leq\varepsilon_o$, the gradient estimate (Lemma 5) gives:

$$u^\varepsilon(x,y,t)\leq\frac{1}{2}C_o,\quad \text{in } B_{\frac{C_o}{4N_o}}(x_1,y_1)\qquad\forall t\in[t_o,0],$$

and therefore (since u^ε is a subsolution of the heat equation):

$$\sup_{B_{R_*}(\tilde{x},\tilde{y})} u^\varepsilon(x,y,t_*+(2R_*)^2)\leq \sup_{B_{2R_*}(\tilde{x},\tilde{y})} u^\varepsilon(x,y,t_*)-\ell_o,$$

with $\ell_o=\ell\frac{C_o}{2}\left(\frac{C_o}{4N_o}\right)^n$ depending only on L_o.

By iterating, we get:

$$\begin{aligned}\sup_{B_{R_*}(\tilde{x},\tilde{y})} u^\varepsilon\,(x,y,t_k) &\leq \sup_{B_{2^nR_*}(\tilde{x},\tilde{y})} u^\varepsilon(x,y,t_*)-k\ell_o\\ &\leq 1-k\ell_o,\end{aligned}$$

for all k such that $t_k=t_*+(2R_*)^2\sum_{j=0}^k 2^{2j}$ is nonpositive (here, we use the fact that $u^\varepsilon\leq 1$). Therefore, choosing k_* such that $1-k_*\ell_o\leq C_o$ and $t_*=-(2R_*)^2\sum_{j=0}^{k_*}2^{2j}$ in (22), we obtain:

$$\begin{aligned}C_o &\leq \sup_{B_{R_*}(\tilde{x},\tilde{y})} u^\varepsilon\,(x,y,0)\\ &\leq \sup_{B_{R_*}(\tilde{x},\tilde{y})} u^\varepsilon\left(x,y,t_*+(2R)^2\sum_{j=0}^k 2^{-2j}\right)\\ &\leq 1-k\ell_o,\end{aligned}$$

for all $k\leq k_*$ which gives a contradiction. Hence (21) holds. □

Remark 3 *When $d=n$ ($\Omega=\mathbb{R}^n$), we can derive a better estimate: Lemma 12 gives*

$$\sup_{B_{\rho|L|}(0)\times\omega} u^\varepsilon=u(\tilde{x},0)\geq C\rho_o|L|.$$

Moreover, we observe that, if $\tilde{t}$ is such that

$$u^\varepsilon(\tilde{x}, \tilde{t}) \geq p(N_o + 1)|L|,$$

then, the gradient estimate (Lemma 5) yields

$$u^\varepsilon(x, \tilde{t}) > \varepsilon, \quad \text{in } B_{p|L|}(\tilde{x}),$$

as soon as $\varepsilon \leq |L|$.

Now, if we rescale $\tilde{u}(x,t) = \frac{1}{|L|} u(|L|x, |L|^2 t)$, it is easy to check that, proceeding as in the proof of Proposition 13, there exists a $t_* \sim |L|^2$ such that

$$B_{p|L|}(\tilde{x}) \subset \Omega_\varepsilon(t_*).$$

And we conclude in the same way as before, by mean of a Birkoff argument. It follows that one can take $M^* = \rho^*|L|$, with ρ^* universal constant. In particular, the free boundary behaves like a hyperplane as the period of the lattice goes to zero.

Proposition 13 and Remark 3 also allow us to improve our nondegeneracy property when $d = n$:

Corollary 14

When $d = n$, there exists a positive constant $C_o(R_o)$ such that, for any $x_o \in \partial\Omega_\varepsilon(t_o)$, we have:

$$\sup_{B_r(x_o)} u^\varepsilon(x,t) \geq C_o(R_o) r \tag{23}$$

for $0 < \varepsilon < \varepsilon_o$ and $0 < \rho_o|L| < r < R_o$.

Proof: Thanks to Remark 3, we note that x_o is at distance at most $\rho^*|L|$ from the most left point of the free boundary. Therefore, a small modification of proof of Lemma 12 will give us the result: we choose ζ small enough in (19), such that the free boundary of h reaches any point at distance $(\rho^* + 2)|L|$ before time $\frac{|L|}{2c}$. $\square$

3.4 Comparison with Travelling Waves.

We can now establish our main result announced in the introduction:

Proposition 15 *For all $L_o > 0$, there exist $\varepsilon_o(L_o) > 0$ and $M^*(L_o) \geq 0$ only depending on L_o such that, if $\varepsilon \leq \varepsilon_o$ and $|L| \leq L_o$, there exists an increasing function $c \mapsto \gamma(c) > 0$, defined for $c > 0$ such that:*

$$\begin{aligned}\max\left(0, 1 - \frac{1}{\kappa} e^{-\gamma(c^\varepsilon)(e\cdot x - c^\varepsilon(t+M^*))}\right) \\ \leq u^\varepsilon(x,y,t) \leq \\ \max\left(\varepsilon, 1 - \kappa e^{-\gamma(c^\varepsilon)(e\cdot x - c^\varepsilon(t - M_*\varepsilon)}\right),\end{aligned} \tag{24}$$

where κ and M_* only depend on Λ, M and $|q|_\infty$, and M^* is as in Proposition 13. Moreover, $\gamma(c)$ goes to zero as c goes to zero.

Proof: We want to construct a barrier for $(\mathbf{P})$ of the form

$$h(x,y,t) = 1 - \psi(x,y)e^{-\gamma(x.e-ct)}\,,$$

with $\psi(x,y)$ L-periodic with respect to x, and $\gamma > 0$. Then, h is a solution of $h_t + q\cdot\nabla h = \triangle h$ if and only if ψ solves:

$$\triangle\psi - (q+\gamma e)\cdot\nabla_x\psi + (\gamma^2 - \gamma(c - e.q))\psi = 0 \tag{25}$$

The existence of such a ψ follows from the following lemma (see [BH]):

Lemma 16

(i) For all γ there exist a unique real number $\mu(\gamma) > 0$ and a unique function (up to a multiplicative constant) $\psi(x,y) > 0$ such that

$$\triangle\psi - (q+\gamma e)\cdot\nabla_x\psi + (\gamma^2 - \gamma(c - e.q))\psi = \mu(\gamma)\psi\,.$$

(ii) Moreover, we have $\mu(\gamma) = c\gamma + \varphi(\gamma)$ where the function $\varphi : \mathbb{R} \mapsto \mathbb{R}$ is concave and satisfies $\varphi(0) = \varphi'(0) = 0$.

It is easy to deduce that there exists a unique $\gamma > 0$ such that $\mu(\gamma) = 0$. The corresponding eigen function ψ furnishes a solution of (25). Moreover, ψ being defined up to a multiplicative constant, we can choose $\sup\psi(x,y) = 1$. Finally, we remark that γ is uniformly bounded (lemma 7). Therefore, equation (25) and the Harnack inequality give us the existence of a universal κ such that

$$0 < \kappa \le \psi(x) \le 1\,.$$

We now define

$$\begin{aligned} h^+(x,y,t) &= 1 - \psi(x,y)e^{-\gamma(x.e-ct)} \ge 1 - e^{-\gamma(x.e-ct)} \\ h^-(x,y,t) &= 1 - \frac{\psi(x,y)}{\kappa}e^{-\gamma(x.e-ct)} \le 1 - e^{-\gamma(x.e-ct)} \end{aligned}$$

The next step is to prove that, for some constant M_*, we have:

$$h^-(x,y,t+M^*) \le u^\varepsilon(x,y,t) \le h^+(x,y,t-M_*\varepsilon)\,, \quad \forall x,t\,. \tag{26}$$

To that purpose, we define

$$h^\pm_\eta(x,y,t) = (1\pm\eta)h^\pm(x,y,t)\,.$$

and

$$\begin{aligned} \phi^\varepsilon(s,x,y) &= u^\varepsilon\left(x,y,\frac{x\cdot e - s}{c}\right) \\ \psi^\pm_\eta(s,x,y) &= h^\pm_\eta\left(x,y,\frac{x\cdot e - s}{c}\right). \end{aligned}$$

Since $\phi^\varepsilon(t,x,y) \longrightarrow 1$ when $x\cdot e \to +\infty$ uniformly with respect to y, for any $\eta > 0$, there exists T such that

$$\psi_\eta^-(s-S,x,y) \leq \phi^\varepsilon(s,x,y) \leq \psi_\eta^+(s+S,x,y)\,, \quad \forall (x,y)\in\Omega\,,\ s\in\mathbb{R}\,,$$

and therefore, there exists T such that

$$h_\eta^-(x,y,t+T) \leq u^\varepsilon(x,y,t) \leq h_\eta^+(x,y,t-T)\,, \quad \forall (x,y)\in\Omega\,,\ t\in\mathbb{R}\,. \qquad (27)$$

Let us now introduce

$$\begin{array}{ll} T^* & = \inf\{T \text{ s.t. } h_\eta^-(x,y,t+T) \leq u^\varepsilon(x,y,t)\} \\ T_* & = \inf\{T \text{ s.t. } h_\eta^+(x,y,t-T) \geq u^\varepsilon(x,y,t)\}. \end{array}$$

We pretend that $T^* \leq M^*$ and $T_* \leq M_*\varepsilon$ for some universal M_*. Let us check this fact for T^* only:

Assume $T^* > M^*$. Then, there exist sequences $T_n < T^*$, s_n, x_n, y_n such that:

$$T_n \longrightarrow T^*\,, \quad \text{and} \quad \psi_\eta^-(s_n,x_n,y_n) \geq \phi^\varepsilon(s_n,x_n,y_n)\,.$$

Since $\lim_{s\to+\infty}\phi^\varepsilon = 1 > 1-\eta = \lim_{s\to+\infty}\psi_\eta^-$, the sequence $(s_n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Moreover, using the periodicity of ϕ^ε with respect to x, we can assume that $x_n\in C$. Passing to the limit $n\to\infty$, it follows that there exists (t_o,x_o,y_o) such that

$$h_\eta^-(x_o,y_o,t_o+T^*) = u^\varepsilon(x_o,y_o,t_o).$$

The function $z(x,y,t) = u^\varepsilon(x,y,t) - h_\eta^-(x,y,t+T^*)$ is nonnegative and vanishes at (x_o,y_o,t_o). In $\{u^\varepsilon(x_o,y_o,t_o) > \varepsilon\}$, z is solution of a parabolic type equation. Therefore, the maximum principle implies

$$(x_o,y_o,t_o) \in \partial\{(x,y,t)\in\mathbb{R}\times\Omega\,;\, u^\varepsilon(x_o,y_o,t_o) > \varepsilon\}$$

Finally, the Hopf principle tells us that we cannot have $(x,y)\in\partial\Omega$. It follows that $u^\varepsilon(x_o,y_o,t_o) = \varepsilon$, and Proposition 13 gives the result. □

3.5 Homogenization limit when $\Omega = \mathbb{R}^n$

For the sake of simplicity, we assume that $L = \delta\mathbb{Z}^n$ (and therefore $|L| = \delta$), and we consider two sequences $(\delta_k)_{k\in\mathbb{N}}$ and $(\varepsilon_k)_{k\in\mathbb{N}}$ such that

$$\delta_k \to 0, \qquad \varepsilon_k \leq \underline{\varepsilon}\delta_k.$$

We denote by $(u^k, c^k)_{k\in\mathbb{N}}$ the corresponding solutions of $(\mathbf{P}_\varepsilon)$.

According to Proposition 15 we have:

$$\begin{aligned} \max\left(0, 1 - \frac{1}{\kappa^k} e^{-\gamma^k(c^k)(e\cdot x - c^k(t+\rho^*\delta_k))}\right) \\ \leq u^k(x,y,t) \leq \\ \max\left(\varepsilon, 1 - \kappa^k e^{-\gamma^k(c^k)(e\cdot x - c^k(t-M_*\varepsilon_k)}\right). \end{aligned} \qquad (28)$$

Moreover, thanks to Proposition 6, Lemmas 7 and 8, it is easy to prove that

$$\begin{cases} (u^k) \text{ converges uniformly on any compact set to } u(x,t), \\ (c^k) \text{ converges to a positive constant } c. \end{cases}$$

Next, we recall that $\gamma^k(c^k)$ and κ^k in (28)were determined by solving (25), which amounts to solving (after rescaling $\psi(x) = \Phi^k(x/\delta_k)$)

$$\triangle\Phi^k - (\delta_k q + \delta_k \gamma e)\cdot\nabla_x \Phi^k + ((\delta_k\gamma)^2 - \delta_k\gamma(\delta_k c^k - e.\delta_k q))\Phi^k = 0 \qquad (29)$$

with Φ $\mathbb{Z}^n$-periodic. Adapting the result from [BH], one can show that there exits a unique such $\gamma^k(c^k)$ and that $\gamma^k(c^k) \to c$ as $c^k \to c$. Finally, it is easy to check that the corresponding function Φ^k converges to a constant, and therefore, $\kappa^k \to 1$.

Hence we have:

$$u^k(x,t) \longrightarrow u(x,t) = \max\left(0, 1 - e^{-c(x\cdot e - ct)}\right).$$

4 A weak Nondegeneracy estimate

In the previous section (see Corollary 14), we showed that when $\Omega = \mathbb{R}^n$, u^ε grows linearly away from the free boundary except maybe in a small neighbourhood of size $\rho^*|L|$.

In this section, we generalize this result for general domains $\Omega = \mathbb{R}^d \times \omega$:

Proposition 17 *There exists L_o, $\underline{\varepsilon}$, ρ_o, R_o and C_o such that if $|L| \le L_o$ and $\varepsilon \le \underline{\varepsilon}|L|$, then for any (x_o, y_o, t_o) satisfying*

$$u^\varepsilon(x_o, y_o, t_o) = \varepsilon,$$

we have:

$$\sup_{B_r(x_o,y_o)} u^\varepsilon(x,y,t_o) \ge C_o r,$$

for all r such that

$$\rho_o|L| \le r \le R_o.$$

Throughout the proof, we shall always assume that ρ_o is big enough in such a way that any ball $B_{\rho_o|L|}$ in $\mathbb{R}^d$ contains a cell of the lattice L. Moreover, we denote by γ a positive number that will be fixed later, and will satisfy

$$0 < \gamma < 1/4.$$

The starting point of the proof is the following claim: There exists $M_o \in \mathbb{R}$ such that

$$\Theta^n \left(\{u^\varepsilon(x,y,0) \geq \frac{\varepsilon}{2}\} \cap B_{2r}(x_o + M_o e, y_o) \right) = \gamma. \tag{30}$$

(We recall that Θ^n is defined by (11).)

We find such a point by sliding a ball B_{2r} in the e-direction, noticing that the quantity

$$\Theta^n \left(\{u^\varepsilon(x,y,0) \geq \frac{\varepsilon}{2}\} \cap B_{2r}(x_o + Me, y_o) \right)$$

is zero for large negative M and 1 for large positive M.

We note $\tilde{x}_o = x_o + M_o e$. Proposition 17 will then be a consequence of the following proposition, the proof of which is the object of the remainder of this section:

Proposition 18 *There exist $|L_o|$, $\underline{\varepsilon}$, ρ_o, R_o and C such that if $|L| \leq L_o$, $\varepsilon \leq \underline{\varepsilon}|L|$ and $\rho_o|L| \leq r \leq R_o$, then*

$$\begin{array}{ll} (i) & u^\varepsilon(x,y,t_o + r^2) < \varepsilon \quad \text{in } B_r(\tilde{x}_o, y_o) \\ (ii) & \sup_{Q_{4r}(\tilde{x}_o, y_o, t_o)} u^\varepsilon(x,y,t) \geq Cr \end{array} \tag{31}$$

Proof of Proposition 17: With k such that $\frac{k\cdot e}{c^\varepsilon} \geq (\rho_o|L|)^2$, $k \leq C|L|$ and $k - k\cdot e \leq |L|$, we have (applying (i) with $r = \rho_o|L|$):

$$u^\varepsilon(x,y,t_o + \frac{k\cdot e}{c^\varepsilon}) < \varepsilon \quad \text{in } B_{\rho_o|L|}(\tilde{x}_o, y_o)$$

and therefore

$$u^\varepsilon(x,y,t_o) < \varepsilon \quad \text{in } B_{\rho_o|L|}(\tilde{x}_o - k, y_o).$$

The Birkoff property (Lemma 10) and the choice of ρ_o yield:

$$u^\varepsilon(x,y_o,t_o) < \varepsilon \quad \text{in } \{x \in \mathbb{R}^d \mid (x - \tilde{x}_o)\cdot e \leq -C|L|\}.$$

Since $u^\varepsilon(x_o,y_o,t_o) = \varepsilon$, we deduce

$$(x_o - \tilde{x}_o)\cdot e \geq -C|L|.$$

Next, using (ii) and the Birkoff property, we have:

$$\sup_{Q_{8r}(x_1,y_o,t_o)} u^\varepsilon \geq Cr, \qquad \text{for all } x_1 \in \mathbb{R}^d \text{ such that } (x_1 - \tilde{x}_o)\cdot e \geq 0.$$

Hence, for some large (universal) P, we also have

$$\sup_{Q_{Pr}(x,y_o,t_o)} u^\varepsilon \geq Cr, \qquad \text{for all } x \in \mathbb{R}^d \text{ such that } (x - \tilde{x}_o)\cdot e \geq -C|L|$$

(since $r \geq \rho_o|L|$). The proposition follows. □

4.1 Proof of Proposition 18.

We have to consider the cases when $B_{2r}(x_o, y_o)$ lies entirely in Ω or not. For the sake of simplicity we only treat the first case.

The first point follows from the following cleaning lemma:

Lemma 19 *There exist $\gamma > 0$ such that, if*

$$\Theta^n\left(\{u^\varepsilon(x,y,0) \geq \frac{\varepsilon}{2}\} \cap B_{2r}(x_o, y_o)\right) \leq \gamma$$

then

$$\Theta^n\left(\{u^\varepsilon(x,y,r^2) \geq \frac{3\varepsilon}{4}\} \cap B_r(x_o, y_o)\right) = 0.$$

Proof: First, we rescale according to (40), which amounts to assuming that u^ε is solution to

$$\partial_t u^\varepsilon + rq_r \cdot \nabla u = \triangle u^\varepsilon - f_r(x,y)\beta_\varepsilon(u^\varepsilon) \tag{32}$$

with $\varepsilon \leq \underline{\varepsilon}/\rho_o$.

The proof relies on a De Giorgi type argument. Let us introduce

$$\begin{aligned} \varepsilon_{m+1} &= \varepsilon_m + \frac{\varepsilon_0}{2^{m+1}} \quad & \varepsilon_0 = \varepsilon/2 \\ r_{m+1} &= r_m - \frac{r_0}{2^{m+1}} \quad & r_0 = 1 \\ t_{m+1} &= t_m + \frac{1}{2^m} \quad & t_0 = 0. \end{aligned}$$

and define

$$V_m = \mathcal{L}^n(\Omega_{\varepsilon_m}(t_m) \cap B_{r_m}(x_o, y_o))$$

(we recall that $\Omega_\delta(t) = \{u^\varepsilon(x,y,t) \geq \delta\}$). We are going to prove that

$$V_{m+1}^{\frac{n-1}{n}} \leq C_m V_m, \tag{33}$$

with $C_m = C2^{2m}$, which gives the result if γ is small enough.

Integrating (32) over $\Omega_{\varepsilon_m} \cap (B_r \times [t_m, t_{m+1}]) = \{u \geq \varepsilon_m\} \cap (B_r \times [t_m, t_{m+1}])$, for $r \in [r_{m+1}, r_m]$ the left hand side gives:

$$\begin{aligned} -\int_{t_m}^{t_{m+1}} \int_{\Omega_{\varepsilon_m}(t)\cap B_r} \partial_t u \, dx\, dy\, dt \quad &\leq \int_{\Omega_{\varepsilon_m}(t_m)\cap B_r} u(x,y,t_m)\, dx\, dy \\ &\quad - \int_{\Omega_{\varepsilon_m}(t_{m+1})\cap B_r} u(x,y,t_{m+1})\, dx\, dy \\ &\leq \mathcal{L}^n(\Omega_{\varepsilon_m}(t_m) \cap B_{r_m}) \\ &\leq V_m, \end{aligned} \tag{34}$$

and

$$\int_{t_m}^{t_{m+1}} \int_{\Omega_{\varepsilon_m}(t)\cap B_r} rq\cdot\nabla_x u\,dx\,dy\,dt \quad \le |q|_\infty r N_o \int_{t_m}^{t_{m+1}} \mathcal{L}^n(\Omega_{\varepsilon_m}(t)\cap B_{r_m})\,dt \le \frac{CR_o}{2^m} V_m. \tag{35}$$

For the right hand side, we have:

$$\begin{aligned}\int_{t_m}^{t_{m+1}} \int_{\Omega_{\varepsilon_m}(t)\cap B_r} \triangle u\,dx\,dy\,dt \quad &\le \int_{t_m}^{t_{m+1}} \int_{\partial\Omega_{\varepsilon_m}(t)\cap B_r} \nabla u\cdot\nu\,d\sigma(x,y)\,dt \\ &\quad + \int_{t_m}^{t_{m+1}} \int_{\Omega_{\varepsilon_m}(t)\cap\partial B_r} |\nabla u| d\sigma(x,y)\,dt \\ &\le N_o \int_{t_m}^{t_{m+1}} \mathcal{H}^{n-1}(\Omega_{\varepsilon_m}(t)\cap\partial B_r)dt \\ &\le \frac{N_o}{2^m}\mathcal{H}^{n-1}(\Omega_{\varepsilon_m}(t_m)\cap\partial B_r),\end{aligned} \tag{36}$$

(since $\nabla u\cdot\nu \le 0$ along $\partial\Omega_{\varepsilon_m}(t)\cap B_r$). Finally, the coarea formula and (12) yield:

$$\begin{aligned}&\int_{t_m}^{t_{m+1}} \int_{\Omega_{\varepsilon_m}(t)\cap B_r} f(x,y)\beta_\varepsilon(u^\varepsilon)\,dx\,dy\,dt \\ &\qquad \ge \frac{\lambda}{N_o}\int_{t_m}^{t_{m+1}} \int_{\varepsilon_m}^{\varepsilon_{m+1}} \beta_\varepsilon(z)\mathcal{H}^{n-1}(\{u^\varepsilon(t)=z\}\cap B_r)\,dz\,dt \\ &\qquad \ge \frac{\lambda K}{N_o\varepsilon}\int_{t_m}^{t_{m+1}} \int_{\varepsilon_m}^{\varepsilon_{m+1}} \mathcal{H}^{n-1}(\{u^\varepsilon(t)=z\}\cap B_r)\,dz\,dt.\end{aligned}$$

Since $\{u^\varepsilon(t)\ge s\}\subset\{u^\varepsilon(t_m)\ge\varepsilon/2\}$ for all $s\in[\varepsilon_m,\varepsilon_{m+1}]$, we have

$$\mathcal{L}^n(\{u^\varepsilon(t)\ge s\}\cap B_r)\le\mathcal{L}^n(B_r)/2,$$

for all $r\in[1,2]$ (we have to choose γ such that $\gamma\le\mathcal{L}^n(B_1)/(2\mathcal{L}^n(B_2))$). Lemma 31 (ii) thus implies:

$$\begin{aligned}&\int_{t_m}^{t_{m+1}} \int_{\Omega_{\varepsilon_m}(t)\cap B_r} f(x,y)\beta_\varepsilon(u^\varepsilon)\,dx\,dy\,dt \\ &\qquad \ge \frac{C\lambda}{N_o}\int_{t_m}^{t_{m+1}} \int_{\varepsilon_m}^{\varepsilon_{m+1}} \beta_\varepsilon(z)\mathcal{L}^n(\Omega_{\varepsilon_{m+1}}(t)\cap B_r))^{(n-1)/n}\,dz\,dt \\ &\qquad \ge \frac{C\lambda K}{2^{2m}N_o}\mathcal{L}^n(\Omega_{\varepsilon_{m+1}}(t_{m+1})\cap B_{r_{m+1}}))^{(n-1)/n} \\ &\qquad \ge \frac{C\lambda K}{2^{2m}N_o}V_{m+1}^{(n-1)/n}.\end{aligned} \tag{37}$$

Putting (34), (35), (36) and (37) together, we deduce:

$$V_{m+1}^{(n-1)/n}\le C2^{2m}\left(V_m+\frac{N_o}{2^m}\mathcal{H}^{n-1}(\Omega_{\varepsilon_m}(t_m)\cap\partial B_r)\right).$$

Inequality (33) follows by integrating with respect to $r \in [r_{m+1}, r_m]$ and using the coarea formula. □

In order to prove the second point in Proposition 18, we start by noticing that

$$\Theta^n\left(\{u^\varepsilon(x,y,t) \geq \frac{\varepsilon}{2}\} \cap B_{2r}(x_o,y_o)\right) \begin{array}{l} \geq \gamma, \\ \leq \frac{1}{2} \quad \text{for all } t \in [-(2r)^2, 0], \end{array} \tag{38}$$

and

$$\Theta^n\left(\{u^\varepsilon(x,y,t) \leq \frac{\varepsilon}{4}\} \cap B_{2r}(x_o,y_o)\right) \begin{array}{l} \leq (1-\gamma), \\ \geq \frac{1}{4}, \quad \text{for all } t \in [-(2r)^2, 0]. \end{array} \tag{39}$$

These inequalities are derived in the same way we obtained (14) and (15). Note that in order to get the second inequality in (38), we need to find a $k_o \in L$ such that $k_o \cdot e \geq (2r)^2 c_{max}$ and $k_o \leq 2r$. Such a k_o exists if ρ_* is large enough and R_o is small enough.

Proposition 18 is now a consequence of the Green formula. First, we rescale according to the following **parabolic rescaling**:

$$v(x,y,t) = \frac{1}{r} u^\varepsilon(x_o + rx, y_o + ry, t_o + r^2 t), \tag{40}$$

which amounts to assuming that $r = 1$, $\varepsilon \leq \underline{\varepsilon}/\rho_o$ and $(x_o, y_o) = (0,0)$. Then we have:

$$\begin{aligned} u^\varepsilon(0,0,0) \quad &= \int_{B_4} u^\varepsilon(x,y,-4^2) G_{(0,0,0)}(x,y,-(4)^2)\, dx\, dy \\ &+ \int_{-4^2}^0 \int_{\partial B_4} u^\varepsilon(x,y,t) \partial_\nu G_{(0,0,0)}(x,y,t)\, d\sigma(x,y) dt \\ &- \int_{Q_4} f(x,y) \beta_\varepsilon(u^\varepsilon) G_{(0,0,0)}(x,y,t)\, dx\, dy\, dt. \end{aligned}$$

Since $u^\varepsilon(0,0) \geq 0$, $G_{(0,0,0)} \geq C$ in $Q_{(2)}(0,0,0)$ and $f(x,y) \geq \lambda$, we deduce:

$$\int_{Q_2} \beta_\varepsilon(u^\varepsilon)\, dx\, dy\, dt \leq C \sup_{Q_4(0,0,0)} u^\varepsilon(x,y,t). \tag{41}$$

Therefore, we are left with the task of proving that the left hand side is bounded by below: this follows from (38), the coarea formula and the isoperimetric inequality:

First, using the coarea formula (see [F]) we can write

$$\int_{Q_2} \beta_\varepsilon(u^\varepsilon)\, dx\, dy\, dt \geq \frac{1}{N_o} \int_{-2^2}^0 \int_{\mathbb{R}} \beta_\varepsilon(z) \mathcal{H}^{n-1}(\{u^\varepsilon(t) = z\} \cap B_2) dz dt. \tag{42}$$

Then, we notice that for all $z \in [\frac{\varepsilon}{4}, \frac{\varepsilon}{2}]$ and $t \in [-2^2, 0]$, we have

$$\mathcal{L}^n(\{u^\varepsilon(t) \geq z\} \cap B_2) \geq \gamma \mathcal{L}^n(B_2)$$
$$\mathcal{L}^n(\{u^\varepsilon(t) \leq z\} \cap B_2) \geq \frac{1}{4}\mathcal{L}^n(B_2).$$

Therefore, using Lemma 31 (i) (consequence of the isoperimetric inequality) we have (for all $z \in [\frac{\varepsilon}{4}, \frac{\varepsilon}{2}]$):

$$\begin{aligned} \mathcal{H}^{n-1}(\{u^\varepsilon(t) = z\} \cap B_2) \quad & \geq \kappa \left(\mathcal{L}^n(\{u^\varepsilon(t) \geq z\} \cap B_2)\right)^{(n-1)/(n)} \\ & \geq \kappa(\gamma \mathcal{L}^n(B_2))^{(n-1)/(n)}. \end{aligned}$$

Together with (42), it gives

$$\int_{Q_2} \beta_\varepsilon(u^\varepsilon)\, dx\, dy\, dt \geq \frac{K}{\varepsilon} \frac{\varepsilon}{4} \kappa \left(\gamma \mathcal{L}^n(B_2)\right)^{(n-1)/n} = C_o,$$

and (41) gives Proposition 18 (ii). □

5 Singular limit

This section is devoted to the proof of Theorem 3. First of all, we notice that the convergence of u^ε as ε goes to zero is a consequence of Proposition 6 and Arzela-Ascoli's theorem, and that the convergence of c^ε is a consequence of Lemma 7. Further estimates can be derived; they are summarized in the following proposition:

Proposition 20 (Convergence) *For any sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ such that $\varepsilon_i \to 0$, there exists a subsequence (still denoted by ε_i), a function $u \in \mathcal{C}^{1,1/2}(\Omega \times \mathbb{R})$ and a real $c \geq 0$ such that:*

i) $u^{\varepsilon_j} \longrightarrow u$ uniformly on compact subsets of $\Omega \times \mathbb{R}$, $c^{\varepsilon_i} \to c$,

ii) $\nabla u^{\varepsilon_j} \longrightarrow \nabla u$ in $L^2_{loc}(\Omega \times \mathbb{R})$,

iii) $\partial_t u^{\varepsilon_j} \longrightarrow \partial_t u$ weakly in $L^2_{loc}(\Omega \times \mathbb{R})$,

iv) $\beta_{\varepsilon_j}(u^{\varepsilon_j})$ is bounded in $L^1_{loc}(\Omega \times \mathbb{R})$ and converges to a positive measure μ with $supp\mu \in \partial\{u > 0\}$.

v) $\partial_t u - \triangle u + q \cdot \nabla u = 0$ in $\{(x,t)\,,\, u(x,t) > 0\}$.

The detail of the proof of Proposition 20 can be found in [CLW1]; the first part of Theorem 3 follows easily. The aim of this section is therefore to prove that the free boundary condition is satisfied at any 'nice' point. From now on, we shall drop the distinction between x and y, and denote by a single x any point in $\Omega = \mathbb{R}^d \times \omega$. We also use the notations

$$B_r(x_o) = \{x \in \mathbb{R}^n\,;\, |x - x_o| < r\}$$
$$Q_r(x_o, t_o) = B_r(x_o) \times [t_o - r^2, t_o]$$

Throughout this section, (x_o, t_o) denotes a point on the free boundary $\partial\{u > 0\} \setminus \partial\Omega$, such that there exists a tangent ball from inside:

$$\exists\, r > 0 \text{ and } y_o \in \Omega \quad \text{s.t. } B_r(y_o) \subset \{u > 0\}, \quad x_o \in \partial B_r(y_o). \tag{43}$$

We also denote

$$\nu = \frac{y_o - x_o}{|y_o - x_o|}.$$

The proof will be divided in three steps: First, we establish a weak nondegeneracy inequality in $B_r(y_o)$. Next, using a monotonicity formula due to G. Weiss [W], we deduce that u has a linear behaviour in $Q_1(x_o, t_o)$ (Proposition 26):

$$u(x,t) = \alpha\langle x - x_o, \nu\rangle^+ + \gamma\langle x - x_o, \nu\rangle^- + o(|x - x_o| + |t - t_o|^{1/2}).$$

Finally, we shall see that $\gamma = 0$ and $\alpha = \sqrt{2f(x_o)M}$.

5.1 Nondegeneracy in the neighbourhood of the free boundary.

First, we start with the following consequence of our nondegeneracy lemma 12:

Corollary 21 *If $B_r(y_o) \subset \Omega_{\varepsilon,t_o}(u^\varepsilon)$, then there exists a constant $C(r) > 0$ (independent of ε) such that*

$$u^\varepsilon(x, t_o) \geq C(r)\,, \text{ for all } x \in B_{\frac{r}{2}}(y_o)$$

Proof: First of all, we notice that $u^\varepsilon(x,t) > 0$ in $B_r(y_o)$ for all time $t \leq t_o$, and therefore u^ε is solution to $u^\varepsilon_t + q \cdot \nabla u^\varepsilon = \triangle u^\varepsilon$ in $B_r(y_o) \times (-\infty \times t_o]$. Moreover, thanks to Proposition 15, it is easy to see that for large universal T, we have:

$$u^\varepsilon(x, t_o - T) \geq A, \quad \text{in } B_r(x_0).$$

Let us now define

$$h(x,t) = Ae^{-\gamma(t - t_o + T)}\Big(e^{-\frac{(x-y_o)^2}{ar^2}} - e^{-\frac{1}{a^2}}\Big)\Big/\Big(1 - e^{-\frac{1}{a^2}}\Big)$$

Then, one can check that with $a = 1/(2\sqrt{2})$ and $\gamma \geq \frac{16}{r^2(e^{-2} - e^{-8})}$, $h(x,t)$ satisfies:

$$h_t + q \cdot \nabla h \leq \triangle h.$$

Moreover, h vanishes along $\partial B_r(y_o)$, and satisfies $h(x, t_o - T) \leq A \leq u^\varepsilon(x, t_o - T)$ for all $x \in B_r(y_o)$. Therefore

$$u^\varepsilon(x, t_o) \geq h(x, t_o), \quad \text{for all } x \in B_r(y_o),$$

which gives the result with $C(r) \sim ce^{-\frac{c}{r^2}}$. $\square$

Note also that for $x_o \in \partial B_r(y_o)$, the proof gives the existence of $\alpha > 0$ such that

$$u(x,t) \geq \alpha \left\langle x - x_o, \frac{y_o - x_o}{|y_o - x_o|} \right\rangle^+$$

in any non tangential cones.

5.2 Blow-up limit and monotonicity formula

Let (x_o, t_o) satisfy (43). We introduce

$$u_r^\varepsilon(x,t) = \frac{1}{r} u^\varepsilon(x_o + rx, t_o + r^2 t),$$
$$u_r(x,t) = \frac{1}{r} u(x_o + rx, t_o + r^2 t),$$

which are respectively solution to

$$\partial_t u_r^\varepsilon + rq(x_o + rx) \cdot \nabla u_r^\varepsilon = \triangle u_r^\varepsilon - f(x_o + rx)\beta_{\varepsilon/r}(u_r^\varepsilon), \quad \text{in } \Omega, \tag{44}$$

and

$$\partial_t u_r + rq(x_o + rx) \cdot \nabla u_r = \triangle u_r, \quad \text{in } \{u_r > 0\}.$$

Since the rescaling preserves the $\mathcal{C}^{1,1/2}$ norm, and $u_r(0,0) = 0$, it is easy to check that u_r is uniformly bounded in $\mathcal{C}^{1,1/2}$. It follows from Ascoli-Arzela's theorem that there exists a sequence $r_j \to 0$ such that

$$u_{r_j}(x,t) \longrightarrow U(x,t), \quad \text{uniformly on compact subsets.}$$

More precisely, one can prove (the proof being similar to the one of Proposition 20, details can be found in [CLW1]):

Lemma 22 *For every sequence $(r_i)_{i\in\mathbb{N}}$ such that $r_i \to 0$, there exists a subsequence u_{r_j}, and $U \in \mathcal{C}^{1,1/2}$ such that:*

i) $u_{r_j} \longrightarrow U$ uniformly on compact subsets of $\Omega \times \mathbb{R}$,

ii) $\nabla u_{r_j} \longrightarrow \nabla U$ in $L^2_{loc}(\Omega \times \mathbb{R})$,

iii) $\partial_t u_{r_j} \longrightarrow \partial_t U$ weakly in $L^2_{loc}(\Omega \times \mathbb{R})$,

iv) $\partial_t U - \triangle U = 0$ in $\{(x,t) \in \Omega \times \mathbb{R}, U(x,t) > 0\}$.

We want to show that U is linear. First, we note that Corollary 21 and the monotonicity of U with respect to t gives

$$U(x,t) > 0, \quad \text{in } \mathbb{R}^n \cap \{x \cdot \nu > 0\} \times \mathbb{R}^-.$$

Applying Lemma A.1 in [CLW1], we deduce:

Lemma 23 *$U(x,t)$ has a linear behaviour in $\mathbb{R}^n \cap \{x \cdot \nu > 0\}$: there exists $\alpha > 0$ such that*

$$U(x,t) = \alpha \langle x, \nu \rangle^+ + o(|x| + |t|^{1/2}), \quad \in \mathbb{R}^n \cap \{\langle x, \nu \rangle > 0\} \times \mathbb{R}^-.$$

Moreover, we shall see that the blow-up limit $U(t,x)$ is homogeneous along parabolic paths:

Lemma 24 *For all $(x,t) \in \mathbb{R}^n \times \mathbb{R}^-$, and all $\theta \in \mathbb{R}$, we have*

$$U(\theta x, \theta^2 t) = \theta U(x,t).$$

Let us postpone the proof of this result. Together with Lemma 23, it yields:

$$U(x,t) = \alpha \langle x, \nu \rangle^+ \quad \in \mathbb{R}^n \cap \{x \cdot \nu > 0\} \times \mathbb{R}^-,$$

and in particular $U(x,t) = 0$ along $\{x \cdot \nu = 0\}$. Applying Corollary A.1 in [CLW1], it follows:

Lemma 25 *$U(x,t)$ has a linear behaviour in $\mathbb{R}^n \cap \{x \cdot \nu < 0\}$: there exists $\gamma \geq 0$ such that*

$$U(x,t) = \gamma \langle x, \nu \rangle^- + o(|x| + |t|^{1/2}), \quad \in \mathbb{R}^n \cap \{x \cdot \nu < 0\} \times \mathbb{R}^-.$$

Finally, putting together Lemma 23, 25, and 24, we get:

$$U(x,t) = \alpha \langle x, \nu \rangle^+ + \gamma \langle x, \nu \rangle^-, \quad \text{in } \mathbb{R}^n \times \mathbb{R}^-, \tag{45}$$

with $\alpha > 0$ and $\gamma \geq 0$, which is equivalent to

Proposition 26 *Let (x_o, t_o) satisfy (43), then there exists $\alpha > 0$ and $\gamma \geq 0$ such that*

$$u(x,t) = \alpha \langle x - x_o, \nu \rangle^+ + \gamma \langle x - x_o, \nu \rangle^- + o(|x - x_o| + |t - t_o|^{1/2}),$$

in $Q_1(x_o, t_o)$.

The rest of this section is devoted to the proof of Lemma 24, which is a consequence of a parabolic monotonicity formula, similar to the monotonicity formula introduced by G. Weiss in [W]. For the sake of simplicity, we shall write the result in $\mathbb{R}^n$. The case of a domain Ω with Neumann boundary condition could be treated similarly.

Lemma 27 *(Weiss monotonicity formula) Let u be a solution of*

$$\partial_t u + q \cdot \nabla_x u = \triangle_x u - f(x)\beta_\varepsilon(u), \quad \text{in } \mathbb{R}^n \times \mathbb{R}.$$

For $(x_o, t_o) \in \mathbb{R}^n \times \mathbb{R}$, we define the quantity

$$\Psi_\varepsilon(r) = \frac{1}{r^2} \int_{t_o-4r^2}^{t_o-r^2} \int_{\mathbb{R}^n} \Big(|\nabla u|^2 + 2f(x)B_\varepsilon(u) - \frac{u^2}{2(t_o-t)} \Big) G(x,t)dxdt$$

with

$$G(x,t) = G_{(x_o,t_o)}(x,t) = \frac{4\pi(t_o-t)}{(4\pi(T-t))^{\frac{n}{2}-1}} \exp\Big(- \frac{|x-x_o|^2}{4(t_o-t)} \Big)$$

and $B_\varepsilon(u) = \int_0^u \beta_\varepsilon(s)ds = \int_0^{u/\varepsilon} \beta(s)ds$. Then we have:

$$\begin{aligned} \Psi'_\varepsilon(r) = \quad & \frac{2}{r^3} \int_{t_o-4r^2}^{t_o-r^2} \int_{\mathbb{R}^n} \Big(2t\partial_t u + x \cdot \nabla u - u \Big)^2 \frac{G(x,t)}{2(t_o-t)} \, dx\, dt \\ & + \frac{2}{r^3} \int_{t_o-4r^2}^{t_o-r^2} \int_{\mathbb{R}^n} u f(x) \beta_\varepsilon(u) G(x,t) \, dx \, dt \\ & + \frac{2}{r^3} \int_{t_o-4r^2}^{t_o-r^2} \int_{\mathbb{R}^n} B_\varepsilon(u) x \cdot \nabla f(x) G(x,t) \, dx \, dt \\ & + \frac{2}{r^3} \int_{t_o-4r^2}^{t_o-r^2} \int_{\mathbb{R}^n} q \cdot \nabla u (2t\partial_t u + x \cdot \nabla u - u) G(x,t) \, dx \, dt. \end{aligned} \tag{46}$$

Proof: The proof is similar to the proof of Theorem 3.1 in [W], with only minor modifications. It relies on the weak formulation obtained by multiplying (2) by

$$(\partial_t u) 2t G(x,t) + (x \cdot \nabla_x u) G(x,t),$$

and it requires the following estimate:

$$\int_{(t_1, t_o-\eta)\cup(t_o+\eta, t_2)} \int_{\mathbb{R}^n} \exp\left(- \frac{|x-x_o|^2}{4(t_o-t)} \right) \Big((\partial_t u)^2 + u^2 \Big)(t,x) \, dx \, dt < C, \tag{47}$$

(for all t_1, t_2 such that $t_1 < t_o < t_2$ and small η), which is derived by multiplying (2) by $\exp\left(-\frac{|x-x_o|^2}{4(T-t)} \right) \partial_t u$.
□

Proof of Lemma 24: Let $0 < \sigma < \rho$ be two constants. The previous lemma gives (after change of variable, and noting $G(x,t) = G_{(0,0)}(x,t)$):

$$
\begin{aligned}
&\Psi_\varepsilon(\rho s) - \Psi_\varepsilon(\sigma s) = \\
&\int_\sigma^\rho \frac{2}{r^3} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^n} \Big(2t\partial_t u_s^\varepsilon + x\cdot\nabla u_s^\varepsilon - u_s^\varepsilon\Big)^2 \frac{G(x,t)}{2(-t)}\,dx\,dt\,dr \\
&+ \int_\sigma^\rho \frac{2}{r^3} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^n} u_s^\varepsilon f(sx)\beta_{\varepsilon/s}(u_s^\varepsilon) G(x,t)\,dx\,dt\,dr \\
&+ \int_\sigma^\rho \frac{2}{r^3} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^n} B_{\varepsilon/s}(u_s^\varepsilon) sx\cdot(\nabla f)(sx) G(x,t)\,dx\,dt\,dr \\
&+ \int_\sigma^\rho \frac{2}{r^3} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^n} sq(sx)\cdot\nabla u_s^\varepsilon(2t\partial_t u_s^\varepsilon + x\cdot\nabla u_s^\varepsilon - u_s^\varepsilon) G(x,t)\,dx\,dt\,dr.
\end{aligned}
\tag{48}
$$

In view of Lemma 5, estimate (47), and the fact that $B_\varepsilon(u) \le M$ for all u, it is easy to check that the last two terms are bounded by Cs, with C constant which does not depend on ε. Moreover, we have:

$$
\begin{aligned}
&\int_\sigma^\rho \frac{2}{r^3} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^n} u_s^\varepsilon f(sx)\beta_{\varepsilon/s}(u_s^\varepsilon) G(x,t)\,dx\,dt\,dr \\
&= -\int_\sigma^\rho \frac{2}{r^3} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^n} u_s^\varepsilon(\partial_t u_s^\varepsilon - \triangle u_s^\varepsilon + sq(sx)\cdot\nabla u_s^\varepsilon) G(x,t)\,dx\,dt\,dr \\
&= -\int_\sigma^\rho \frac{2}{r^3} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^n} (u_s^\varepsilon\partial_t u_s^\varepsilon + |\nabla u_s^\varepsilon|^2) G(x,t) \\
&\qquad\qquad + (u_s^\varepsilon\nabla u_s^\varepsilon - (u_s^\varepsilon)^2 sq(sx))\cdot\nabla G(x,t))\,dx\,dt\,dr \\
&\longrightarrow -\int_\sigma^\rho \frac{2}{r^3} \int_{-4r^2}^{-r^2} \int_{\{U>0\}} (u_s\partial_t u_s + |\nabla u_s|^2) G(x,t) \\
&\qquad\qquad + (u_s\nabla u_s - (u_s)^2 sq(sx))\cdot\nabla G(x,t)\,dx\,dt\,dr \\
&= -\int_\sigma^\rho \frac{2}{r^3} \int_{-4r^2}^{-r^2} \int_{\{u_s>0\}} u_s(\partial_t u_s - \triangle u_s + sq(sx)\cdot\nabla u_s) G(x,t)\,dx\,dt\,dr \\
&= 0.
\end{aligned}
$$

Therefore, denoting $\Psi_o(r) = \lim_{j\to\infty}\Psi_{\varepsilon_j}(r)$, when $\varepsilon = \varepsilon_j$ goes to zero (48) yields:

$$
\begin{aligned}
&\Psi_o(\rho s) - \Psi_o(\sigma s) = \\
&\quad \int_\sigma^\rho \frac{2}{r^3} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^n} \Big(2t\partial_t u_s + x\cdot\nabla u_s - u_s\Big)^2 \frac{G(x,t)}{2(-t)}\,dx\,dt\,dr \\
&\quad +\mathcal{O}(s),
\end{aligned}
\tag{49}
$$

hence

$$\liminf_{s\to 0}\Big(\Psi_o(\rho s) - \Psi_o(\sigma s)\Big) \ge 0.$$

Assume that the limit is positive. Then, for s small enough, we have

$$\Psi(s) - \Psi\left(\frac{\sigma}{\rho}s\right) > \delta > 0,$$

and denoting $\theta = \frac{\sigma}{\rho} < 1$, we get

$$\sum_{n=0}^{N} \left(\Psi\left((\theta)^n s\right) - \Psi\left((\theta)^{n+1} s\right) \right) = \Psi(s) - \Psi\left((\theta)^{N+1} s\right) \geq N\delta.$$

We get a contradiction as $N \to \infty$, since it is easy to see (after rescaling) that $\Psi_o(r)$ is uniformly bounded with respect to r.

Finally, taking the limit in (48) with $s = r_n$, the lower semicontinuity of the L^2-norm with respect to the weak convergence gives:

$$\int_\sigma^\rho \frac{2}{r^3} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^n} \left(2t\partial_t U + x \cdot \nabla U - U\right)^2 \frac{G(x,t)}{2(-t)}\, dx\, dt\, dr = 0,$$

i.e. $2t\partial_t U(x,t) + x\cdot\nabla U(x,t) - U(x,t) = 0$ for almost all $(x,t) \in \mathbb{R}^n \times (-4\rho^2, -\sigma^2)$. Since the result holds for any $0 < \sigma < \rho$, the lemma follows. □

5.3 The free boundary condition

In order to recover the free boundary condition, we have to determine the value of the coefficients α and γ introduced by Proposition 26. This will be done in two steps:
- First, we show that u is degenerate on one side, i.e. that $\gamma = 0$,
- then, we will see that we have $\alpha = \sqrt{2f(x_o)M}$.

In this section, we use the following lemma, which combines the convergence with respect to ε and r (see [CLW1] for details):

Lemma 28 *Let $(u^{\varepsilon_j})_{j\in\mathbb{N}}$ be a sequence of solution to (2) such that $u^{\varepsilon_j} \to u$ as $j \to \infty$ (in the sense of proposition 20). Assume $(x_o, t_o) \in \partial\{u > 0\}$, and define*

$$\begin{cases} u_r(x,t) &= \frac{1}{r} u(x_o + rx, t_o + r^2 t), \\ u_r^{\varepsilon_j}(x,t) &= \frac{1}{r} u^{\varepsilon_j}(x_o + rx, t_o + r^2 t) \end{cases}$$

Suppose $r_n \to 0$ and $u_{r_n} \to U$ as $n \to \infty$. Then there exists $j(n) \to \infty$ such that for every $j_n \geq j(n)$ there holds that $\frac{\varepsilon_{j_n}}{r_n} \to 0$ and

i) $u_{r_n}^{\varepsilon_{j_n}} \longrightarrow U$ uniformly on compact subsets of $\Omega \times \mathbb{R}$,

ii) $\nabla u_{r_n}^{\varepsilon_{j_n}} \longrightarrow \nabla U$ in $L^2_{loc}(\Omega \times \mathbb{R})$,

iii) $\partial_t u_{r_n}^{\varepsilon_{j_n}} \longrightarrow \partial_t U$ weakly in $L^2_{loc}(\Omega \times \mathbb{R})$.

In the sequel, we shall denote

$$u^n(x,t) = u_{r_n}^{\varepsilon_{j_n}}(x,t),$$

which is solution to

$$\partial_t u^n + r_n q_n \cdot \nabla u^n = \triangle u^n - f_n \beta_{\frac{\varepsilon_{j_n}}{r_n}}(u^n), \tag{50}$$

with $q_n(x) = q(x_o + r_n x)$ and $f_n(x) = f(x_o + r_n x)$.

As a consequence of the previous section, we have:

$$u^n(x,t) \longrightarrow \alpha\langle x - x_o, \nu\rangle^+ + \gamma\langle x - x_o, \nu\rangle^-, \qquad \text{as } n \to \infty.$$

The next lemma shows that if $\gamma > 0$, $u^\varepsilon(x_o, t_o)$ is bounded away from zero by a positive constant independent of ε, which contradicts the fact that (x_o, t_o) belongs to the free boundary. Hence $\gamma = 0$.

Lemma 29 *Let $(x_o, t_o) \in \Omega \times \mathbb{R}$ be such that*

$$u(x, t_o) \geq \alpha\langle x - x_o, \nu\rangle^+ + \gamma\langle x - x_o, \nu\rangle^- + o(|x - x_o| + |t - t_o|^{1/2}),$$

in $B_1(x_o)$, with α and γ positive. Then $u^\varepsilon(x_o, t_o) \geq \eta > 0$ with η independent of ε (and therefore $u(x_o, t_o) \neq 0$).

Proof: After translation and rotation, we may assume $(x_o, t_o) = (0, 0)$, and $\nu = e_n$. Using the monotonicity of u^ε with respect to t, we note that for any small δ, we have

$$u^\varepsilon(x,t) \geq (\alpha x_n^+ + \gamma x_n^- - \delta r)_+ \quad \text{ in } B_r(0) \times \mathbb{R}^-,$$

for all $r \leq r_o(\delta)$ and $\varepsilon \leq \varepsilon_o(\delta)$.

Let φ_r solve

$$\begin{cases} -\triangle \varphi_r + r q(x/r) \cdot \nabla \varphi_r = 0, & \text{in } B_1(0), \\ \varphi_r(x) = (\alpha x_n^+ + \gamma x_n^- - \delta)^+ & \forall x \in \partial B_1(0). \end{cases}$$

We want to show that φ_r can be approached by a subsolution of the ε-problem:

$$\triangle \varphi_r^\varepsilon - q(x/r) \cdot \nabla \varphi_r^\varepsilon \geq f(x/r) \beta_\varepsilon(\varphi_r^\varepsilon).$$

To that purpose, we introduce

$$\Gamma_\varepsilon(t) = \begin{cases} 0 & t \leq 0 \\ \dfrac{1}{2\varepsilon} t^2 & t \in]0, \varepsilon[\\ t - \dfrac{\varepsilon}{2} & t \geq \varepsilon, \end{cases}$$

with

$$B(t) = \int_0^t \beta(s) ds. \tag{51}$$

It is easy to check that

$$\triangle(\Gamma_\varepsilon(\varphi_r)) - rq(x/r)\cdot\nabla(\Gamma_\varepsilon(\varphi_r)) = \frac{1}{\varepsilon}\chi_{\{0\le\varphi_r\le\varepsilon\}}|\nabla\varphi_r|^2.$$

Moreover, with $Q = \sup\beta(s)$, we have

$$\frac{1}{\varepsilon}\chi_{\{0\le\varphi_r\le\varepsilon\}} \ge \frac{1}{2Q}\beta_{\varepsilon/2}(\Gamma_\varepsilon(\varphi_r)),$$

since $\{0\le\varphi_r\le\varepsilon\} = \{0\le\Gamma_\varepsilon(\varphi_r))\le\varepsilon/2\}$.

Therefore, $\varphi_r^\varepsilon = \Gamma_{2\varepsilon}(\varphi_r)$ is a subsolution of the ε-problem if we can show that:

$$|\nabla\varphi_r|^2 \ge 2Q\Lambda \quad \text{in } \{0<\varphi<\varepsilon\}.$$

Classic elliptic estimates tells us that φ_r converges as $r\to 0$ (uniformly and in $H^1(B_1(0))$) to a harmonic function on B_1; the Poisson formula gives:

$$\lim_{r\to 0}\varphi_r(x) = \varphi(x) = C(1-|x|^2)\int_{\partial B_1(0)}\frac{\varphi_r(y)}{|x-y|^n}ds_y.$$

We easily deduce that $|\nabla_x\varphi(x)|\to\infty$ as $x\to y_o$, if y_o is an angular point (a point for which $\nabla_y\varphi_r$ is discontinuous). Therefore, if $\delta\le\delta_1$ and $r\le r_1$, there exists a small neighbourhood $\mathcal{N}$ of $\{x\in\partial B_1; |x_n|\le\delta/\alpha, |x_n|\le\delta/\gamma\}$ such that

$$|\nabla\varphi_r|^2 \ge 2Q\Lambda \quad \text{in } \{0<\varphi<\varepsilon\}, \quad \text{in } \mathcal{N}.$$

Moreover, when $\varepsilon\le\varepsilon_1$ small enough, we have

$$\{0<\varphi_r<\varepsilon\}\subset\mathcal{N}.$$

Introducing $\psi_r^{\varepsilon r}(x) = r\varphi_r^\varepsilon(x/r)$, we get (with $\delta=\delta_1$, $r\le\min(r_o(\delta_1), r_1)$ and $\varepsilon\le r\min(\varepsilon_o(\delta_1),\varepsilon_1)$):

$$\begin{cases} \triangle\psi_r^\varepsilon - q(x)\cdot\nabla\psi_r^\varepsilon \ge f(x)\beta_\varepsilon(\psi_r^\varepsilon), & \text{in } B_r(0) \\ \psi_r^\varepsilon(x) \ge (\alpha x_n^+ + \gamma x_n^- - \delta r)^+, & \text{on } \in\partial B_r(0). \end{cases}$$

We now want to apply the maximum principle and show that $\psi_r^\varepsilon\le u^\varepsilon$ in $B_r(0)\times\mathbb{R}^-$. To that purpose, we introduce $z(x,t) = u^\varepsilon(x,t) - \psi_r^\varepsilon(x)$. Then we have $z(x,t)>0$ along $\partial B_r(0)$ for all $t\le 0$, and since $u^\varepsilon(x,t)\to 1$ as $t\to-\infty$, we have $z(x,T)>0$ in $B_r(0)$ for large negative T. Denote by t_1 the first time such that z vanishes at some point $x_1\in B_1(0)$. Then in $B_r\times[T,t_1]$, $z(x,t)$ solves

$$\partial z - \triangle z + q\cdot\nabla z \ge -\frac{C\Lambda}{\varepsilon^2}z.$$

If $t_1\le 0$, then x_1 belongs to the interior of $B_1(0)$, and therefore $z(x_1,t_1)=0$, $\nabla z(x_1,t_1)=0$, $\triangle z(x_1,t_1)\ge 0$ and $\partial_t z(x_1,t_1)<0$, which gives a contradiction, and yields $t_1\ge 0$.

Finally, we have $u^\varepsilon(0,0) \geq \psi_r^\varepsilon(0)$, and since

$$\lim_{\varepsilon\to 0} \psi_r^\varepsilon(0) = r\varphi_r(0) \geq \eta > 0,$$

the proof of Lemma 29 is complete. □

Proof of Theorem 3: We are now in position to complete the proof of Theorem 3. Collecting our previous results, we know that there exist two sequences $r_n \to 0$ and $\varepsilon_n \to 0$ such that

$$u^n(x,t) = u_{r_n}^{\varepsilon_n}(x,t) \longrightarrow U(x,t) = \alpha\langle x - x_o, \nu\rangle^+, \quad \text{as } j \to +\infty.$$

We are left with the task of proving that $\alpha = \sqrt{2f(x_o)M}$.

Let $\zeta(x,t) \in \mathcal{C}_c^\infty(\Omega\times\mathbb{R})$. Multiplying (50) by $u_{x_k}^n\zeta$, and integrating by part, we get:

$$\iint u_t^n u_{x_k}^n \zeta + \frac{1}{2}(\nabla u^n)_{x_k}^2\zeta + (u^n)_{x_k}\nabla u^n \nabla\zeta \\ + \iint rq_r\cdot\nabla u^n u_{x_k}^n\zeta = \iint (f_{r_n}(x)\zeta)_{x_k} B_{\frac{\varepsilon_n}{r_n}}(u^n)$$

and therefore

$$\iint (u^n)_t (u^n)_{x_k}\zeta - \frac{1}{2}(\nabla u^n)^2\zeta_{x_k} + (u^n)_{x_k}\nabla u^n\nabla\zeta \\ + \iint rq_r\cdot\nabla u^n (u^n)_{x_k}\zeta = \iint (f_{r_n}(x)\zeta)_{x_k} B_{\frac{\varepsilon_n}{r_n}}(u^n),$$

where we recall that

$$B_\varepsilon(u) = \int_0^u \beta_\varepsilon(s)ds = \int_0^{u/\varepsilon} \beta(s)ds.$$

Introducing $\rho^n(x,t) = B_{\frac{\varepsilon_n}{r_n}}(u^n)$, we have $0 \leq \rho^n(x,t) \leq M$, and therefore

$$\rho^n(x,t) \rightharpoonup \rho(x,t) \qquad *\text{-weakly in } L^\infty.$$

Taking the limit as in Lemma 28, we deduce:

$$\iint U_t U_{x_k}\zeta - \frac{1}{2}(\nabla U)^2\zeta_{x_k} + U_{x_k}\nabla U\nabla\zeta = \iint f_o\zeta_{x_k}\rho(x,t) \tag{52}$$

where $f_{r_n} \to f_o = f(x_o)$. Moreover, since $U(x,t) = \alpha x_n^+$, we have

$$\begin{cases} \nabla_x U = \alpha\chi_{\{x_n>0\}}e_n \\ \partial_t U = 0 \\ \rho(x,t)|_{\{x_n>0\}} = M. \end{cases}$$

When ζ is such that $\text{supp}\zeta \subset Q_{r_o}(x_o,t_o) \cap \{x_n < 0\}$, (52) gives:

$$\iint_{Q_r^-} f_o \zeta_{x_k} \rho(x,t) = 0$$

which implies that $\rho(x,t) = M_1(t)$ in $Q_{r_o}(x_o,t_o) \cap \{x_n < 0\}$.

With $k = n$, (52) leads to:

$$\iint_{\{x_n>0\}} \frac{1}{2}\alpha^2 \zeta_{x_n} = \iint f_o \zeta_{x_n} \rho(x,t) = \int_{\{x_n=0\}} f_o(M - M_1(t))\zeta,$$

which also reads:

$$\alpha^2 = 2f_o(M - M_1(t)). \tag{53}$$

In particular, $M_1(t) = M_1$ is constant in $Q_{r_o}(x_o,t_o) \cap \{x_n < 0\}$

Note that $\nabla_x \rho^n = \beta_{\varepsilon_n/r_n}(u^n)\nabla_x u^n$ is bounded in L^1 (Proposition 20 (iv)), and for any $\sigma \geq \tau$ and K compact set we have:

$$\int_\sigma^\tau \int_K |\partial_t \rho^n| dt dx = -\int_\sigma^\tau \int_K \partial_t \rho^n dx dt = -\left[\int_K \rho^n dx\right]_\sigma^\rho \leq 2M\mathcal{L}^n(K).$$

It follows that ρ^n is bounded in $W^{1,1}_{loc}(\Omega \times \mathbb{R})$, and therefore

$$\rho^n(x,t) \to \rho(x) \text{ strongly in } L^1(Q_1), \text{ and almost everywhere.}$$

Now, in view of (53), and since $\alpha > 0$ (nondegeneracy), we have $M_1 < M$. We want to prove that $M_1 = 0$. Assume that $M_1 > 0$, and let $\kappa > 0$ be such that $M_1 \in]\kappa, M - \kappa[$. There exist a, and b positive such that

$$B_\varepsilon(u) \in]\kappa, M-\kappa[\quad \Longleftrightarrow \quad a\varepsilon \leq u \leq b\varepsilon,$$

and

$$\rho^n(x,t) \in]\kappa, M-\kappa[\quad \Longleftrightarrow \quad a\frac{\varepsilon_n}{r_n} \leq u^n(x,t) \leq b\frac{\varepsilon_n}{r_n}.$$

The next lemma, the proof of which is similar to that of Lemma 9, gives a contradiction. It follows that $M_1 = 0$, and (53) yields

$$\alpha = \sqrt{2f(x_o)M}.$$

The theorem is therefore proved. □

Lemma 30 *For any a, $b > 0$, there exists a constant K such that*

$$\mathcal{L}^{n+1}\left(\{a\frac{\varepsilon}{r} \leq u_r^\varepsilon \leq b\frac{\varepsilon}{r}\} \cap Q_s(x_o,t_o)\right) \leq K\frac{\varepsilon}{r}s^{n+1},$$

for all $s \leq s_o$.

A Approximation of supersolutions

In the proof of Lemma 12, we omitted the construction of the supersolution h^ε. We recall that $h(t,x)$ is a supersolution of (1) which reads $h(t,x) = \varphi(x\cdot e - \nu t)$, with

$$\varphi(s) = \frac{\sqrt{A}}{\nu + |q|_\infty}\Big[1 - \exp(-(\nu + |q|_\infty)s)\Big]\,.$$

Therefore, it is enough to construct a supersolution φ^ε of

$$-\varphi'' + (q\cdot e - \nu)\varphi' = -f(x)\beta_\varepsilon(\varphi)\,. \tag{1}$$

Let us define

$$\varphi^\varepsilon(s) = \begin{cases} \varphi(s) & \text{for } s \geq s_o, \\ \dfrac{K\lambda}{4\varepsilon}(s - s_o)^2 + \sqrt{A}(s - s_o) + b\varepsilon & \text{for } s_1 \leq s \leq s_o, \\ a\varepsilon & \text{for } s \leq s_1, \end{cases}$$

Where s_0 is such that $\varphi(s_0) = b\varepsilon$. First of all, if $\varepsilon \leq \varepsilon_o$ small enough, we have $\varphi'(s_0^+) \leq \sqrt{A} = \varphi'(s_0^-)$, and therefore the jump of the derivatives at s_0 has the right sign.

Next, choosing $s_1 = s_o - \varepsilon\frac{2\sqrt{A}}{\lambda K}$, we check that with $A = \lambda K(b-a)$, we have $\varphi(s_1^+) = a\varepsilon$, and $\varphi'(s_1^+) = \varphi'(s_1^-) = 0$.

In order to conclude, it only remains to check that φ^ε is a supersolution of (1) in $[s_1, s_o]$:

$$\begin{aligned} -\varphi'' + (q\cdot e - \nu)\varphi' \quad & \geq -\tfrac{K\lambda}{\varepsilon}\big(\tfrac{1}{2} + \varepsilon(q\cdot e - \nu)\tfrac{\sqrt{A}}{\lambda K}\big) \\ & \geq -K\lambda/\varepsilon \\ & \geq -f(x)\beta_\varepsilon(\varphi^\varepsilon)\,, \end{aligned}$$

if $\varepsilon \leq \varepsilon_0$, with ε_0 small enough (depending only on $|q|_\infty$ and Λ).

B The isoperimetric inequality

Let us first recall that given an Euclidean ball B in $\mathbb{R}^n$ and a subset Ω of $\mathbb{R}^n$, the isoperimetric inequality gives:

$$(\mathcal{L}^n(B\cap\Omega))^{(n-1)/n} \leq \mu_n(\mathcal{H}^{n-1}(\partial\Omega\cap B) + P(\Omega\cap\partial B))$$

with equality when $B \subset \Omega$.

In this section, we establish a couple of results that allow us to control the perimeter $P(\Omega\cap\partial B)$ by the Hausdorff measure of $\partial\Omega\cap B$ in some situations.

We note

$$V_1 = \mathcal{L}^n(B\cap\Omega), \quad V_2 = \mathcal{L}^n(B\setminus\Omega) \quad \text{and } V = \mathcal{L}^n(B),$$

and

$$S_1 = P(\Omega \cap \partial B), \quad S_2 = P(\partial B \setminus \Omega), \quad S = P(\partial B) \text{ and } A = \mathcal{H}^{n-1}(\partial\Omega \cap B).$$

Then, we have the following result:

Lemma 31 *(i) For any $\gamma > 0$, there exists a constant $C(\gamma)$ such that if*

$$\gamma \leq V_1 \leq V - \gamma,$$

then

$$V_1^{(n-1)/n} \leq CA.$$

(ii) if $V_1 < V/2$, there exists a constant C such that

$$V_1^{(n-1)/n} \leq CA$$

Proof:
(i) Noting $p = (n-1)/n$, the isoperimetric inequality gives:

$$V_i^p \leq \mu(A + S_i), \quad \text{for } i = 1, 2, \quad \text{and} \quad V^p = \mu S.$$

It follows

$$V_1^p + V_2^p \leq \mu(A + S) \leq \mu A + V^p,$$

and therefore

$$V_1^p + (V - V_1)^p - V^p \leq \mu A.$$

It remains to see that for $\gamma \leq V_1 \leq V - \gamma$, there exists a constant C such that:

$$V_1^p \leq C(V_1^p + (V - V_1)^p - V^p)$$

(ii) If $V_1 = 0$, the result holds. Otherwise, since $V_1 < V/2$, we have $A > 0$, and we may write $S_1 = MA$. The isoperimetric inequality gives:

$$V_1^p \leq \mu(M + 1)A, \quad V_2^p \leq \mu(S - (M - 1)A).$$

Writing $V^p = \mu S$, we deduce:

$$S^{1/p} \leq ((M + 1)A)^{1/p} + (S - (M - 1)A)^{1/p},$$

and therefore

$$1 \leq \left((M + 1)\frac{A}{S}\right)^{1/p} + (1 - (M - 1)\frac{A}{S})^{1/p}.$$

If M is large (and $M + 1 \sim M - 1 \sim M$), it implies that A/S is bounded by below (since $A > 0$) (for example, we check that, with $p = 3/2$, if $M \geq 10$ we have $\frac{A}{S} \geq 0.08$), and therefore

$$A \geq CS = \frac{C}{\mu}V^p \geq \frac{C}{\mu}V_1^p.$$

Otherwise, M is bounded, and since $S_1 = MA$ we have

$$V_1^p \leq \mu(M + 1)A.$$

□

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