HOMOGENIZATION AND FLAME PROPAGATION IN PERIODIC EXCITABLE MEDIA: THE ASYMPTOTIC SPEED OF PROPAGATION.

L. A. CAFFARELLI, K.-A. LEE, AND A. MELLET

ABSTRACT. We study the effect of homogenization on flame propagation in periodic excitable media, when the width of the flame is much smaller than the characteristic size of the heterogeneities.

1. INTRODUCTION

The following equation arises in the modeling of the combustion of a premixed gas (thermo-diffusive approximation):

$$\partial_t u = \Delta u - g(x)\beta_\delta(u) \qquad x \in \mathbb{R}^n. \tag{1}$$

The reaction term is given by $\beta_\delta(s) = \frac{1}{\delta}\beta(\frac{s}{\delta})$, with $\delta < 1$ and with $\beta(s)$ a Lipschitz function satisfying:

$$\begin{cases} \beta(s) > 0 \text{ if } x \in (0,1),\ \beta(s) = 0 \text{ otherwise,} \\ M = \int_0^1 \beta(s)ds. \end{cases} \tag{2}$$

This model is usually refered to as the ignition temperature model. The function $g(x)$ is positive, and is related to the combustion rate; it is independent of the space variable when the media is perfectly homogeneous. In this paper, we assume that heterogeneities arise in the premixed gas, over a small scale (of order ε) and in a periodic manner. This amounts to writing

$$g(x) = f\left(\frac{x}{\varepsilon}\right), \qquad \text{with } f(x+k) = f(x) \quad \forall k \in \mathbb{Z}^n.$$

In this framework we can define **Pulsating Traveling Fronts** which are particular solutions of (1) satisfying:

$$\begin{array}{ll} u \longrightarrow 0 & \text{as } x\cdot e \to -\infty, \\ u \longrightarrow 1 & \text{as } x\cdot e \to +\infty, \\ u(x+k,t) = u\left(x, t - \dfrac{k\cdot e}{c^\varepsilon}\right), & \forall k \in \varepsilon\mathbb{Z}^n, \end{array} \tag{3}$$

where e is a unit vector (direction of propagation). In [2], H. Berestycki and F. Hamel proved that for any $e \in S^{n-1}$, $\varepsilon > 0$ and $\delta > 0$ there exists a unique real $c^{\varepsilon,\delta}(e)$ and a unique function $u^{\varepsilon,\delta}$ (up to a translation in time) solutions of (1,3). Furthermore, they showed that $c^{\varepsilon,\delta}(e)$ is positive and that $u^{\varepsilon,\delta}(x,t)$ is a decreasing function of t.

The real $c^{\varepsilon,\delta}$ is refered to as the effective speed of propagation, while the unit vector e denotes the direction of propagation. In the present paper we investigate asymptotic behavior of $c^{\varepsilon,\delta}$ when ε and δ go to zero.

It is well known that the limit $\delta \to 0$ in (1) gives rise (at least formally) to the free boundary problem:

$$u^\varepsilon_t = \Delta u^\varepsilon, \quad \text{in } \{u^\varepsilon > 0\}, \tag{4}$$
$$|\nabla_x u^\varepsilon|^2 = 2f(x/\varepsilon)M, \qquad \text{on } \partial\{u^\varepsilon > 0\}. \tag{5}$$

On the other side, uniform hölder estimates for $u^{\varepsilon,\delta}$ (see [4]) allow us to prove that when ε goes to zero with δ fixed, (1) yields:

$$u^\delta_t = \Delta u^\delta - \langle f\rangle \beta_\delta(u^\delta), \tag{6}$$

where $\langle f\rangle$ denotes the average of f.

Thus, the limit $\delta \to 0$ with $\varepsilon \ll \delta$ leads to:

$$u_t = \Delta u, \quad \text{in } \{u > 0\},$$
$$|\nabla_x u|^2 = 2\langle f\rangle M, \qquad \text{on } \partial\{u > 0\}.$$

In other words, the asymptotic regime when the size of the heterogeneities is much smaller than the width of the flame depends on the average value of the reaction rate.

When δ and ε are comparable, or when $\delta \ll \varepsilon$ (i.e. when the width of the flame is comparable or much smaller than the size of the heterogeneities), the limit $\varepsilon \to 0$ corresponds to the homogenization of the free boundary problem (4-5). In this paper, we will characterize the asymptotic regime when $\varepsilon \ll 1$, with $\delta/\varepsilon = \tau$, and we will show that the limits $\varepsilon \to 0$ and $\delta \to 0$ do not commute.

Note that similar regime are investigated for the viscosity solutions of the elliptic free boundary problem in [6] and [7].

The characterization of the effective speed of propagation was initiated by some numerical results (which we present in the appendix) obtained for a one-dimensional front, which showed that the homogenized speed of propagation for the free boundary problem depends on the minimum value of the combustion rate $f(x)$ and not on its average.

The numerical computations also suggested the proof: As ε goes to zero, the free boundary motion is given by a step function, the free boundary stopping for a long time where f reaches its minimum and travelling quickly through the other values. Moreover, while sticking, the rescaled solution $\frac{1}{\varepsilon}u(\varepsilon x, \varepsilon^2 t)$ approaches a solution of the stationary problem, which in one-dimension, is simply a linear function γx^+ with $\gamma = \sqrt{2\inf(f)M}$.

In higher dimension, a similar behavior is expected, with the linear function being replaced by a planelike solution of the stationary problem (a solution that stays at a finite distance of a plane). Instead of the minimum of f, it is the smallest slope of such solutions that will give the homogenized free boundary condition (and the homogenized speed of propagation).

In the next Section, we recall previous results established in [8], and we state the main theorem, the proof of which consists of two steps: In Section 3, we show that the homogenized slope along the free boundary is bounded above by the slope of any planelike solution of the stationary problem. Then, in Section 4, we construct a planelike solution with the same slope as the homogenized Pulsating Travelling Front. It follows that the homogenized slope is equal to the smallest slope among all planelike solutions of the stationary problem. Finally, Section 5 is devoted to the construction of planelike solutions. Technical lemmas and numerical simulations are presented in Appendix.

Finally, let us point out that advection phenomena could be taken into account as follows:

$$\partial_t u + p(x)\cdot\nabla_x u = \Delta u - g(x)\beta_\delta(u).$$

When $p(x) = q(x/\varepsilon)$, with $q(x)$ periodic, divergence free and with null average, the results presented in this paper can be adapted via minor corrections. In particular, the asymptotic speed and slope along the free boundary are the same with and without the advection term, when ε and δ go to zero under the condition $\delta = \tau\varepsilon$. Nevertheless, the presence of the advection term adds a lot of technical (though rather straightforward) details to the proof, and for the sake of clarity, the authors decided to treat the case when $q = 0$.

2. Main results

In this section, we recall previous results and state our main theorem. In what follows, $f(x)$ satisfies:

$$\begin{cases} f(x+k) = f(x), \text{ for all } k \in \mathbb{Z}^n. \\ \text{There exist two constants } \lambda \text{ and } \Lambda \text{ such that} \\ \qquad 0 < \lambda \le f(x) \le \Lambda. \end{cases}$$

For technical purpose, we also assume:

$$\text{There exists } b > 0 \text{ such that } \beta(s) \text{ is increasing for } s \in [0,b]. \tag{7}$$

From now on, e is a fixed vector in S^{n-1}, and $c^{\varepsilon,\delta}, u^{\varepsilon,\delta}$ are the corresponding solutions of (1,3). In [8], we established several results regarding the asymptotic behavior of $u^{\varepsilon,\delta}$. The relevant results for our present purpose can be summarized as follows:

Theorem 2.1.
(i) There exists $c_{\min}, c_{\max} > 0$ universal constants such that

$$c_{\min} \le c^{\varepsilon,\delta} \le c_{\max}, \qquad \forall \delta, \varepsilon > 0.$$

(ii) There exists ρ such that for all ε_o, if $\varepsilon \le \varepsilon_o$ and $\delta \le \rho\varepsilon$, then

$$\begin{aligned} &\max\left(0, 1 - \frac{1}{\kappa_\varepsilon} e^{-\gamma^{\varepsilon,\delta}(x\cdot e - c^{\varepsilon,\delta}(t+M^*\varepsilon))}\right) \\ &\qquad \le u^{\varepsilon,\delta}(x,t) \le \\ &\qquad\qquad \max\left(\delta, 1 - \kappa_\varepsilon e^{-\gamma^{\varepsilon,\delta}(x\cdot e - c^{\varepsilon,\delta}(t - M_*\delta))}\right), \end{aligned} \tag{8}$$

where M^*, M_* are universal constants.

In [8], the slope $\gamma^{\varepsilon,\delta}$ and the constant κ_ε were determined through an eigenvalue problem. In the advection free case, however, we simply have

$$\gamma^{\varepsilon,\delta}(e) = c^{\varepsilon,\delta}(e) \quad \text{ and } \kappa_\varepsilon = 1.$$

Nevertheless, to avoid confusion, we shall use the notation $\gamma^{\varepsilon,\delta}(e)$ when referring to the slope of the solution, and $c^{\varepsilon,\delta}(e)$ when referring to the effective speed of propagation.

We also recall that since $0 \le u^{\varepsilon,\delta} \le 1$, $u^{\varepsilon,\delta}$ is bounded in $\mathscr{C}^{1,\frac{1}{2}}$ (see [4]):

Lemma 2.2. *There exists a universal constant N_0 such that*

$$|u^{\varepsilon,\delta}(x,t) - u^{\varepsilon,\delta}(x',t')| \le N_0\left(|x - x'| + |t - t'|^{1/2}\right),$$

for all $(x,t), (x',t') \in (\mathbb{R}^n \times \mathbb{R})^2$.

As a consequence, there exist $\tilde{\gamma}^\tau = \tilde{c}^\tau$ positive constant such that if $\delta = \varepsilon\tau$ and $\varepsilon \to 0$ then the solution $u^{\varepsilon,\delta}$ converges, up to a subsequences to the function

$$u^\tau(x,t) = \left(1 - e^{-\tilde{\gamma}^\tau(x\cdot e - \tilde{c}^\tau t)}\right)_+. \tag{9}$$

This paper addresses the question of the determination of the slope $\tilde{\gamma}^\tau$ and the effective speed $\tilde{c}^\tau$. It turns out that the answer relies on the properties of the solutions of the elliptic equation

$$\Delta u = f(x/\varepsilon)\beta_\delta(u), \tag{10}$$

and of the corresponding free boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \{u>0\} \\ |\nabla u|^2 = 2f(x/\varepsilon)M & \text{on } \partial\{u>0\}. \end{cases} \tag{11}$$

Note that if u is solution to (10), then the rescaled function $h(x) = \frac{1}{\varepsilon}u(\varepsilon x)$ solves

$$\Delta h = f(x)\beta_\tau(h), \tag{12}$$

with $\tau = \delta/\varepsilon$. Note that in the presence of an advection term, we would have an extra term $\varepsilon q(x)\cdot\nabla h$. This term, though of a smaller order, would introduce a lot of technical change in the proof, since ε would have to be taken into account when dealing with (12).

As mentioned in the Introduction, we are concerned with particular solutions of (12):

Definition 2.3 (Planelike Solutions)**.**
A function $h(x)$ is a planelike solution of (12) in the direction e if $h(x)$ solves (12) and if there exists $\gamma > 0$, A and B such that

$$\begin{aligned} & h(x) \longrightarrow 0 \qquad \text{as } x\cdot e \to -\infty, \\ & \max(0, \gamma x\cdot e - A) \le h(x) \le \max(\tau, \gamma x\cdot e + B). \end{aligned} \tag{13}$$

A function $h(x)$ is a planelike solution for the free boundary problem (11) if there exist sequences $(\tau_n)_{n\in\mathbb{N}}$, and $(h_n)_{n\in\mathbb{N}}$ such that $h_n(x)$ is a planelike solution of (12) with $\tau = \tau_n$ and

$$\tau_n \xrightarrow[n\to\infty]{} 0, \qquad h_n \xrightarrow[n\to\infty]{} h \quad \text{uniformly on compact set.}$$

Remark 2.4. There are other possible definitions for planelike solutions of the free boundary problem, such as viscosity solutions (see [7]), and it is not clear whether all definitions would lead to the same class of functions. Anyway, as we will see later on, the present definition is the most convenient one for our purpose.

Of course, there is no uniqueness for (12). Actually, there is no maximum principle. Also, the convergence of solutions of (12) to (11) is a tricky question (Berestycki et Al., in [1], established the convergence for the smallest supersolution).

For our purpose, we need solutions $v(x)$ of (12) that satisfy the following maximum principle:

$$\begin{aligned} & \textit{If } G \textit{ is a smooth subset of } \mathbb{R}^n \textit{ and } h \textit{ satisfies} \\ & \qquad \Delta h \ge f(x)\beta_\tau(h) \qquad \text{in } G \\ & \textit{with} \\ & \qquad h \le v \qquad \text{on } \partial G, \\ & \textit{then} \\ & \qquad h \le v \qquad \text{in } G \end{aligned} \tag{14}$$

A solution $v(x)$ of (12) that satisfies condition (14) will be said to be a **largest subsolution**.

Then, we have the following theorem:

Theorem 2.5. *There exists a planelike solution $h^\tau(x)$ of (12) that satisfies Property (14). Moreover, we have*

(i) *There exists* $\gamma^\tau_{\min}(e) \in [\sqrt{2\lambda M} - \mathcal{O}(\tau), \sqrt{2\Lambda M} + \mathcal{O}(\tau)]$ *and* $R_\star$ *universal constant such that*

$$\max(0, \gamma^\tau_{\min}(e)\, x\cdot e - R_\star) \le h^\tau(x) \le \max(\tau, \gamma^\tau_{\min}(e)\, x\cdot e). \tag{15}$$

(ii) *Any other planelike solution (and subsolution) has a slope greater than* $\gamma^\tau_{\min}(e)$.

(iii) *The sequence* $\gamma^\tau_{\min}(e)$ *converges as* τ *goes to zero. We write*

$$\lim_{\tau\to 0} \gamma^\tau_{\min}(e) = \gamma_{\min}(e).$$

This theorem is proved in Section 5. The real number $\gamma_{\min}(e)$ can be interpreted as the **smallest slope** of all planelike solutions of the homogenized free boundary problem in the direction e.

We now state our main result:

Theorem 2.6. *For any* τ *and* η *positive constants, there exists* ε_o *such that if*

$$\begin{aligned} \varepsilon &\le \varepsilon_o \\ \delta &= \varepsilon\tau \end{aligned}$$

then

$$\gamma^\tau_{\min}(e) - \eta < \gamma^{\varepsilon,\delta}(e) < \gamma^\tau_{\min}(e) + \eta. \tag{16}$$

It is worth mentioning that the upper bound in (16) is actually uniform in τ (but the lower one is not).

To be complete, it remains to evaluate $\gamma^\tau_{\min}$ for a given reaction rate $f(x)$, which is a delicate task. Since $\gamma^\tau_{\min} = \gamma_{\min} + \mathcal{O}(\tau)$, we only wish to evaluate $\gamma_{\min}$. For a one dimensional front, we prove in Section A.1 that

$$\gamma_{\min} = \sqrt{2M\inf(f)}.$$

In $\mathbb{R}^n$, we always have

$$\gamma_{\min}(e) \ge \sqrt{2M\inf(f)}, \quad \text{for all } e \in S^{n-1}$$

but equality does not always hold. In $\mathbb{R}^2$ for instance, and when f is constant in one direction ($f(x_1,x_2) = f(x_1)$), we prove that (see Section A.2 for details):

$$\gamma_{\min}(\pm e_1) = \sqrt{2M\inf(f)}, \quad \text{and} \quad \gamma_{\min}(e) = \sqrt{2M\langle f\rangle} \quad \text{for } e \ne \pm e_1.$$

For more general $f(x)$, the determination of $\gamma_{\min}$ is an open question.

The next two sections of the paper are devoted respectively to the upper bound and the lower bound in (16). Detailed proof of Theorem 2.5 is given in Section 5.

3. Upper bound

In this section, we prove that $\gamma^{\varepsilon,\delta}$ is smaller than the slope of any planelike solution of the stationary problem. Applying this result to the largest subsolution as defined in Theorem 2.5, we thus obtain the upper bound in (16).

Let h^τ be a planelike solution of (12), with slope γ_s. We assume that

$$\gamma^{\varepsilon,\delta} > \gamma_s + \eta.$$

Assuming that $\delta = \tau\varepsilon$, we introduce the (hyperbolic) rescaled function

$$v^{\varepsilon,\tau} = \frac{1}{\varepsilon} u^{\varepsilon,\delta}(\varepsilon x, \varepsilon t), \tag{17}$$

we will show that when ε is small enough, we can 'cut' $v^{\varepsilon,\tau}$ by h^τ, in the following sense:

$$\text{There exists } A<B \text{ and } t_o \text{ s.t.} \quad \begin{array}{ll} v^{\varepsilon,\tau}(x,t_o)\le h^\tau(x), & \text{for } x\cdot e=A,\\ v^{\varepsilon,\tau}(x,t_o)\ge h^\tau(x), & \text{for } x\cdot e=B.\end{array} \tag{18}$$

The monotonicity of $v^{\varepsilon,\tau}$ with respect to t together with the maximum principle will give a contradiction.

Proof. The function $v^{\varepsilon,\tau}$ propagates with speed $c^{\varepsilon,\delta}$, and satisfies

$$\varepsilon\partial_t v=\Delta v-f(x)\beta_\tau(v) \qquad x\in\mathbb{R}^n. \tag{19}$$

Moreover, (8) becomes

$$\begin{aligned}\max\left(0,\frac{1}{\varepsilon}\left(1-e^{-\varepsilon\gamma^\varepsilon(x\cdot e-c^\varepsilon t-M^*)}\right)\right)&\\ \le v^{\varepsilon,\tau}(x,t)\le&\\ \max\left(\tau,\frac{1}{\varepsilon}\left(1-e^{-\varepsilon\gamma^\varepsilon(x\cdot e-c^\varepsilon t+M_*\tau)}\right)\right).\end{aligned} \tag{20}$$

When ε goes to zero, (20) leads to

$$\tilde\gamma(x\cdot e-\tilde c t-M^*)\le v^\tau(x,t)\le\max\left(\tau,\tilde\gamma(x\cdot e-\tilde c t+M_*\tau)\right).$$

Finally, we point out that since $\partial_t v^{\varepsilon,\tau}<0$, $v^{\varepsilon,\tau}$ is a supersolution of (12).

Let A be such that

$$\gamma_s x\cdot e-R_\star\ge 10 \quad \text{when } x\cdot e=A,$$

(with $R_\star$ as in Theorem 2.5), we can choose t_o such that

$$v^{\varepsilon,\tau}(x,t_o)\le h^\tau(x), \quad \text{for } x\cdot e=A. \tag{21}$$

This is achieved, for example if t_o satisfies

$$\frac{1}{\varepsilon}\left(1-e^{-\varepsilon\gamma^\varepsilon(R_\star-c^\varepsilon t_o+M_*\tau)}\right)\le\tau,$$

or

$$1-e^{-\varepsilon\gamma^\varepsilon(R_\star-c^\varepsilon t_o+M_*\tau)}\le\delta.$$

Note that t_o is independent on ε, and only depends on ρ.

Next, for ε small enough, and since $\gamma^{\varepsilon,\tau}\ge\gamma_s+\eta$ there exist $B\ge A$ such that

$$\frac{1}{\varepsilon}\left(1-e^{-\varepsilon\gamma^\varepsilon(x\cdot e-c^\varepsilon t_o-M^*)}\right)\ge\gamma_s x\cdot e \quad \text{for } x\cdot e=B,$$

which yields

$$v^{\varepsilon,\tau}(x,t_o)\ge h^\tau(x), \quad \text{for } x\cdot e=B.$$

In order to conclude, we need to ensure that for some large P and some time $t_1\le t_o$ we have:

$$v^{\varepsilon,\tau}(x,t_1)\ge h^\tau(x) \qquad \text{in } \{x;\, x\cdot e\le -P\}. \tag{22}$$

Admitting this fact for the time being, and thanks to the monotonicity of $v^{\varepsilon,\tau}(x,t)$ with respect to t, we deduce that there exists $t_2\le t_o$ such that

$$v^{\varepsilon,\tau}(x,t_2)\ge h^\tau(x) \quad \text{for } x\cdot e\le B.$$

Letting t increase until $v^{\varepsilon,\tau}$ touches h^τ by above, we find $t^\star<t_o$ such that

$$\begin{cases} v^{\varepsilon,\tau}(x,t^\star)\ge h^\tau(x) & \text{for all } x\cdot e\le B\\ v^{\varepsilon,\tau}(x^\star,t^\star)=h^\tau(x^\star) & \text{for some } x^\star \text{ s.t. } x^\star\cdot e<B\end{cases} \tag{23}$$

(note that the existence of $x^\star$ follows from a compactness argument and the invariance of the problem by integer translation). In particular the function $v^{\varepsilon,\tau}(x,t^\star) - h(x)$ has a minimum for $x = x^\star$. But since $\partial_t v^{\varepsilon,\tau} < 0$, we have

$$\Delta(v^{\varepsilon,\tau} - h) < f(x)(\beta_\delta(v^{\varepsilon,\tau}) - \beta_\delta(h)),$$

and the contradiction follows

To complete the proof, we have to prove that (22) holds. Let P be such that

$$h^\tau(x) \le b\,\delta \qquad \text{in} \quad \{x \in \mathbb{R}^n\,;\, x\cdot e \le -P\}$$

(with b as in (7)), and $t^\star$ such that

$$v^{\varepsilon,\tau}(x,t^\star) \ge h^\tau(x) \qquad \text{in} \quad \{x \in \mathbb{R}^n\,;\, x\cdot e = -P\}.$$

To see that $v^{\varepsilon,\tau}(t^\star) \ge h^\tau$, we introduce

$$\eta^\star = \inf\{\eta\,;\, v^{\varepsilon,\tau}(x,t^\star) \ge h^\tau(x) - \eta \text{ in } \{x\,;\, x\cdot e \le -P\}\},$$

and we claim that $\eta^\star = 0$. As a matter of fact, if $\eta^\star > 0$, we set $h^\star = h^\tau - \eta^\star$, and since $b\delta \ge h^\tau \ge h^\star$, we deduce:

$$\begin{array}{ll} v^{\varepsilon,\tau}(x) \ge h^\star(x) & \text{for all } x\cdot e \le -P \\ v^{\varepsilon,\tau}(x^\star) = h^\star(x^\star) & \text{for some } x^\star \text{ with } x\cdot e \le -P \\ \Delta(v^{\varepsilon,\tau} - h^\star) < f(x)(\beta_\delta(v^{\varepsilon,\tau}) - \beta_\delta(h^\star)). & \end{array}$$

The contradiction follows easily. □

4. Lower bound

The main result of this section is the construction of a planelike solution $U(x)$ of (12), with slope

$$\tilde\gamma^\tau = \liminf_{\substack{\varepsilon\to 0,\\ \delta/\varepsilon=\tau}} \gamma^{\varepsilon,\delta}$$

(see lemma 4.1). The minimality of $\gamma^\tau_{\min}$ (Theorem 2.5 (iii)) immediately yields

$$\tilde\gamma^\tau \ge \gamma^\tau_{\min}(e).$$

Moreover, using the same argument as in the previous section, but with $U(x)$ as a barrier, we can show that

$$\gamma^{\varepsilon,\delta} \le \tilde\gamma^\tau + \eta,$$

for small ε, and in particular

$$\limsup_{\substack{\varepsilon\to 0,\\ \delta/\varepsilon=\tau}} \gamma^{\varepsilon,\delta} \le \tilde\gamma^\tau.$$

Therefore, the whole sequence converges, and the lower bound in Theorem 2.6 follows.

To prove the existence of U, we first introduce the function:

$$w^{\varepsilon,\tau} = \frac{1}{\varepsilon} u^{\varepsilon,\delta}(\varepsilon x, \varepsilon^2 t), \tag{24}$$

which is solution to

$$\partial_t w = \Delta w - f(x)\beta_\tau(w) \qquad x \in \mathbb{R}^n. \tag{25}$$

This parabolic scaling preserves the Hölder estimates ($w^{\varepsilon,\delta}$ is uniformly bounded in $C^{1,\frac{1}{2}}$), but not the propagation property (the effective speed of propagation becomes $\varepsilon c^{\varepsilon,\delta}$). In particular, w^τ satisfies:

$$\max\left(0,\frac{1}{\varepsilon}\left(1-e^{-\varepsilon\gamma^\varepsilon(x\cdot e-\varepsilon c^\varepsilon t-M^*))}\right)\right) \le w^{\varepsilon,\tau}(x,t)\le \max\left(\tau,\frac{1}{\varepsilon}\left(1-e^{-\varepsilon\gamma^\varepsilon(x\cdot e-\varepsilon c^\varepsilon t+M_*\tau)}\right)\right). \tag{26}$$

Thus, when ε goes to 0, with τ fixed, $w^{\varepsilon,\tau}$ converges (uniformly on compact sets) to w^τ, solution to (25), and satisfying

$$\tilde{\gamma}^\tau(x\cdot e-M^*)\le w^\tau(x,t)\le \max(\tau,\tilde{\gamma}^\tau(x\cdot e+M_*\tau)\,. \tag{27}$$

Then, we have the following lemma:

Lemma 4.1.
When $t\to+\infty$, the function $w^\tau(\cdot,t)$ converges (uniformly on compact sets) to a continuous function $U(x)$ solution to (12), that also satisfies

$$\tilde{\gamma}^\tau(x\cdot e-M^*)\le U(x)\le \max(\tau,\tilde{\gamma}^\tau(x\cdot e+M_*\tau))\,.$$

The function $U(x)$ is therefore a planelike solution of (12) with slope $\tilde{\gamma}^\tau$.

Proof. The uniform convergence and the uniqueness of the limit are immediate consequences of the Hölder estimate and the monotonicity of w^τ with respect to t.

It is readily seen that $U(x)$ satisfies (27), and thus is planelike. To complete the proof, we therefore have to prove that $U(x)$ is solution to the elliptic problem (12).

Integrating (25) for t in $[T,T+\tau]$, we have:

$$w^\tau(T+\tau)-w^\tau(T)=\Delta\int_T^{T+\tau}w^\tau(t)dt-f(x)\int_T^{T+\tau}\beta_\tau(w^\tau(t))dt \tag{28}$$

Writing $\int_T^{T+\tau}w^\tau(x,t)dt=\int_0^\tau w^\tau(x,t+T)dt$, and thanks to the uniform convergence with respect to t, deduce:

$$\int_T^{T+\tau}w^\tau(x,t)dt\xrightarrow[T\to\infty]{}\tau U(x),$$

and

$$\int_T^{T+\tau}\beta_\tau(w^\tau)dt\xrightarrow[T\to\infty]{}\tau\beta_\tau(U).$$

Since $w^\tau(x,T+\tau)-w^\tau(x,T)\xrightarrow[T\to\infty]{}0$, (28) becomes

$$\tau\Delta U(x)=\tau f(x)\beta_\tau(U(x)),$$

and the lemma follows. □

5. Planelike solutions of the elliptic problem

In this section, we give the proof of Theorem 2.5. The function $h(x)$ is obtained as the limit of the largest subsolution of the following boundary value problem:

$$\begin{cases}\Delta u=f(x)\beta_\tau(u)\\ u|_{x\cdot e=0}=M\\ \lim_{x\cdot e\to-\infty}u=0\end{cases} \tag{29}$$

In Section 5.1, we prove the existence of h^M. Section 5.2 and 5.3, we establish the main properties of h^M, and finally, in Section 5.4, we pass to the limit $M \to \infty$ and complete the proof of Theorem 2.5.

5.1. **Barriers and existence of h^M.** To begin with, we need barriers for (29): For any slope l and any translation t, the function

$$\Phi_l^t = lx \cdot e + t$$

satisfies

$$\begin{cases} \Delta \Phi_l = 0 \\ |\nabla \Phi_l| = l. \end{cases}$$

Thus, with $l = \sqrt{2\Lambda M} + \eta$ (respectively $l = \sqrt{2\lambda M} - \eta$), we can construct a family of planelike subsolutions $\Phi_1^{t,\tau}$ (respectively a family of supersolutions $\Phi_2^{t,\tau}$) of (12) (see Section B for details).

Note that $\Phi_i^{t,\tau}$ depends continuously on t, which allows us to make use of the sliding method.

Finally, choosing t_1 and t_2 such that

$$\begin{array}{l} M - C \leq \Phi_1^{t_1} \leq M \\ M \leq \Phi_2^{t_2} \leq M + C \end{array} \qquad \text{when } x \cdot e = 0,$$

we can define the largest subsolution of (29) as follows:

$$h^M(x) = \sup\{u(x)\,;\, u \text{ subsolution of (29) s.t. } \Phi_1^{t_1,\tau} \leq u \leq \Phi_2^{t_2,\tau}\}.$$

5.2. **Birkoff's property.** Even though h^M is not monotonic with respect to $x \cdot e$, we have:

Lemma 5.1.

$$\begin{array}{ll} h^M(x+m) \geq h^M(x) & \textit{for all } m \in \mathbb{Z}^n \textit{ such that } m \cdot e \leq 0, \\ h^M(x+m) \leq h^M(x) & \textit{for all } m \in \mathbb{Z}^n \textit{ such that } m \cdot e \geq 0. \end{array}$$

Introducing $\Omega_\eta = \{h(x) \geq \eta\}$, this lemma yields the following property, which is reminiscent of Theorem 8.1 in [5], and which we call Birkoff's property:

$$\begin{array}{lll} T_m(\Omega_\eta) & \subset \Omega_\eta & \text{for all } m \in L \text{ such that } m \cdot e \leq 0, \\ T_m(\Omega_\eta) & \subset \Omega_\eta & \text{for all } m \in L \text{ such that } m \cdot e \geq 0. \end{array}$$

Proof of Lemma 5.1. Let $k \in \mathbb{Z}$ be such that $k \cdot e \geq 0$, then $h_k^M(x) = h^M(x-k)$ satisfies $h_k(x) \leq M$ for $x \cdot e = 0$, and $h_k^M(x) \leq \Phi_2^{t_2}(x-k) \leq \Phi_2^{t_2}$. Therefore $z(x) = \sup(h^M(x-k), h^M(x))$ is a subsolution of (29) and satisfies

$$\Phi_1^{t_1} \leq z \leq \Phi_2^{t_2}.$$

By definition of h^M, it follows that $z(x) \leq h^M(x)$, and therefore

$$h^M(x-k) \leq h^M(x).$$

The second inequality is obtained by replacing $h(x)$ by $h(x+k)$ in the previous argument (since everything is invariant by translation). □

5.3. **Non-degeneracy.** In this section, we prove that part (i) of Theorem 2.5 holds for $h^M(x)$, i.e. that $h^M(x)$ remains at a finite (universal) distance from a plane.

The proof relies on technique first introduced by L. Caffarelli and R. De La Llave in [5], and later used in [8] to establish Theorem 2.1.

Let x_o be the most left point on the free-boundary. Without loss of generality, we can always assume that $x_o = 0$, i.e.

$$h^M(0) = \tau, \qquad h^M(x) < \tau \text{ for all } x \text{ such that } x \cdot e < 0.$$

Then, we have the following lemma:

Lemma 5.2. *There exists a universal constant C such that*

$$\sup_{x \cdot e \leq r} h^M(x) \geq Cr$$

Before proving this lemma, let us see how, using the Birkoff property it will give Theorem 2.5 (i) for h^M. Let R_o be such that

$$CR_o - N_o 2\sqrt{n} \geq \tau,$$

with C as in Lemma 5.2. Then, there exists y such that

$$h^M(y) \geq CR_o, \qquad y \cdot e \leq R_o,$$

and thanks to the choice of R_o, we deduce that

$$h^M \geq \tau \quad \text{ in } B_{2\sqrt{n}}(y).$$

The Birkoff property (Lemma 5.1) yields

$$\cup_{k \in \mathbb{Z}^n, k \cdot e \geq 0} T_k(B_{2\sqrt{n}}(y)) \subset \{h^M > \tau\} \subset \{x; x \cdot e \geq 0\}$$

and since $B_{2\sqrt{n}}$ contains a cube of size 1, it is readily seen that

$$\cup_{k \in \mathbb{Z}^n, k \cdot e \geq 0} T_k(B_{2\sqrt{n}}(y))$$

covers a half plane. It follows that

$$\{x; x \cdot e \geq R_\star\} \subset \{h^M(x) > \tau\} \subset \{x; x \cdot e \geq 0\} \tag{30}$$

for some universal $R_\star$.

Proof of Lemma 5.2. We take x_o such that $B_{r_o}(x_o)$ is tangent to $\partial\{h^M > \tau\}$ at 0 ($r_o = |x_o|$). Then, the function

$$u(x) = \frac{r_o^{n+1}}{n}\left(\frac{1}{r_o^n} - \frac{1}{|x - x_o|^n}\right)$$

is such that

$$u \geq 0 \text{ and } \Delta u = 0, \quad \text{ in } \mathbb{R}^n \setminus B_{r_o}(x_o),$$

and

$$|\nabla u| = 1 \text{ on } \partial B_{r_o}(x_o).$$

Moreover, we have

$$u(x) \geq Cr_o, \qquad \text{if } x \cdot e = R_o = 2r_o.$$

Using Appendix B, we see that the function $v^\tau(x) = \Psi_\tau(\sqrt{2\lambda M}u(x))$ is a supersolution of the τ-problem. We now prove Lemma 5.2 by contradiction: Assume that

$$h^M(x) < v^\tau(x) \text{ on } \{x; x \cdot e = R_o\}$$

Then the translation $v^\tau_T(x) = v(x + Te)$ is still a supersolution, and, for large T satisfies

$$h^M(x) \leq v^\tau_T(x) \text{ in } \{x; x \cdot e \leq R_o\}$$

Sliding v_T^τ by taking smaller and smaller T, we stop when v_T^τ touches h^M by above:

$$T^* = \inf\{T\,;\, h^M(x) \le v_T^\tau(x) \text{ in } \{x;\, x\cdot e \le R_o\}\}.$$

Since $h^M(0) \ge v^\tau(0)$, we have $T^* > 0$, and since $h^M(x) \le v^\tau(x)$ on $x\cdot e = R_o$, we get a contradiction as in the proof of Lemma 2 in [8]. The result follows. □

5.4. **Existence and properties of h.** For all M, there exists $k \in \mathbb{Z}^n$ such that the translation $h_o^M(x) = h^M(x-k)$ satisfies

$$\{h_o^M = \tau\} \subset \{x;\, 0 \le x\cdot e \le R_\star\}. \tag{31}$$

With $\gamma^M = M/(k\cdot e)$, and thanks to the previous section, we deduce

$$\max(\tau, \gamma^M x\cdot e - R_\star) \le h_o^M(x) \le \max(\tau, \gamma^M x\cdot e).$$

Finally, it is easy to see that $\sqrt{2\lambda M} - \mathscr{O}(\tau) \le \gamma^M \le \sqrt{2\Lambda M} + \mathscr{O}(\tau)$.

Passing to the limit $M \to \infty$ is now an easy task, thanks to the gradient estimates (see [3]).

Theorem 2.5 (i) is an immediate consequence of (30). It remains to establish (ii). Let g be a planelike subsolution with slope γ_g, we want to prove that $\gamma_g \ge \gamma^\tau$ Taking A large, we consider the restriction of g to $\{x;\, x\cdot e \le A\}$. Then, there exists a translation of the supersolution $\Phi_2^{t_2,\tau}$ such that

$$g \le \Phi_2^{t_2,\tau} \text{ when } x\cdot e = A,$$

we want to check that this implies

$$g \le \Phi_2^{t_2,\tau} \text{ when } x\cdot e \le A. \tag{32}$$

Clearly, there exists $t \ge t_2$ such that

$$g \le \Phi_2^{t,\tau} \text{ when } x\cdot e \le A$$

(since $\lim_{x\cdot e\to -\infty} \Phi_2^{t,\tau}(x) = \tau/4$). Since $\Phi_2^{t,\tau}$ depends continuously on t, we can slide the supersolution by letting t goes to t_2 and stop whenever there exists $x^\star$ such that $g(x^\star) = \Phi_2^{t,\tau}(x^\star)$. Then the function $z(x) = \Phi_2^{t,\tau}(x) - g(x)$ satisfies

$$\begin{aligned} &z(x) \ge 0 && \text{when } x\cdot e \le A\\ &z(x) > 0 && \text{when } x\cdot e = A\\ &z(x^\star) = 0 && \end{aligned}$$

and

$$\Delta z < f(x)(\beta_\tau(\Phi_2^{t,\tau}) - \beta_\tau(g)).$$

We get a contradiction ($0 < 0$) when $x = x^\star$, hence (32) has to be satisfied.

Next, the function

$$g_1 = \max(g(x), \Phi_1^{t_1,\tau}(x))$$

is a subsolution, and satisfies

$$\Phi_1^{t_1,\tau} \le g_1 \le \Phi_2^{t_2,\tau} \text{ when } x\cdot e \le A.$$

By definition of h^M, it follows that $h^M \ge g$ for $x\cdot e \le A$.

Hence, when A goes to infinity, there exists a sequence of M (also going to infinity) such that

$$g(x) \le h^M(x), \text{ when } x\cdot e \le A, \quad \text{and} \quad g(x) \ge h^M(x) - C, \text{ when } x\cdot e = A$$

The result follows.

5.5. **Convergence of the sequence $\gamma^\tau_{\min}$.** In this section, we investigate the limit of $\gamma^\tau_{\min}$ as τ goes to zero. Since $\gamma^\tau_{\min} \in [\sqrt{2\lambda M} - \mathscr{O}(\tau), \sqrt{2\Lambda M} + \mathscr{O}(\tau)]$, we can define

$$\liminf_{\tau\to 0} \gamma^\tau_{\min} = \gamma^o_{\min} \in [\sqrt{2\lambda M}, \sqrt{2\Lambda M}].$$

In order to prove that the whole sequence converges, we prove the following:

Lemma 5.3. *Assume that f is continuous, and fix $\tau_o > 0$. Then, for any $\tau_1 > \tau_o$, we have:*

$$\gamma^{\tau_1}_{\min} \le \gamma^{\tau_o}_{\min} + \mathscr{O}(\tau_1).$$

In particular, it follows that for all $\tau_1 > 0$, we have

$$\gamma^{\tau_1}_{\min} \le \liminf_{\tau\to 0} \gamma^\tau_{\min} + \mathscr{O}(\tau_1),$$

and therefore

$$\limsup_{\tau\to 0} \gamma^\tau_{\min} \le \liminf_{\tau\to 0} \gamma^\tau_{\min} = \gamma^o_{\min},$$

which proves that the whole sequence converges, and that

$$\lim_{\tau\to 0} \gamma^\tau_{\min} = \gamma^o_{\min}.$$

Proof of Lemma 5.3. Let h^τ be a planelike solution of

$$\Delta h = f(x)\beta_\tau(h),$$

with slope $\gamma^\tau_{\min}$.

Let a be a small positive real. There exists C, depending only on the modulus of continuity of $f(x)$ such that, if t is a vector satisfying $|t| \le Ca$, the function $h_{a,t}(x) = (1+a)h(x+t)$ is a planelike subsolution solution of (12) with slope $(1+a)\gamma^\tau_{\min}$. As a consequence, the function v_η, defined by

$$v_\eta(x) = \sup_{y\in B_\eta(x)} h_{a,t}(y)$$

is a subsolution of (12).

Then, we have the following lemma:

Lemma 5.4. *Let x_o be a point on the τ_o level set of v_η. Then the ball $B_\eta(x_o)$ is tangent from outside to $\{h^{\tau_o}_a = \tau_o\}$. Moreover, if we note $y_o \in \partial B_\eta(x_o) \cap \{h^{\tau_o}_a = \tau_o\}$, we have:*

$$|\nabla_x h^{\tau_o}_a(y_o)| > \sqrt{2f(y_o)M}.$$

It follows that

$$|\nabla_x v_\eta| \ge \sqrt{2f(y_o)M} \qquad \text{along } \{v_\eta = \tau\},$$

and therefore $(v_\eta - \tau)_+$ is a subsolution of the free boundary problem. Moreover,

$$\Delta h^{\tau_o}_a \ge 0$$

and if we denote by ν the outward unit normal vector to each level set $\{h_a = t\}$, and by τ_i the tangential direction to $\{h_a = t\}$, we have

$$|\nabla h^{\tau_o}_a| = (h^{\tau_o}_a)_\nu, \quad \text{and} \quad (h^{\tau_o}_a)_{\tau_i\tau_i} = \kappa_i (h^{\tau_o}_a)_\nu,$$

where κ_i is the directional curvature of $\{h^{\tau_o}_a = t\}$ in the τ_i direction. Therefore

$$0 \le \Delta h^{\tau_o}_a = (h^{\tau_o}_a)_{\nu\nu} + \kappa_i (h^{\tau_o}_a)_\nu.$$

On the other hand, since $\{h^{\tau_o}_a = t\}$ has a touching ball from outside whose radius is greater than η, we have

$$\kappa_i \le \frac{1}{\eta} \quad \text{for all } i.$$

We deduce

$$0 \leq (h_a^{\tau_o})_{\nu\nu} + \frac{n-1}{\eta}(h_a^{\tau_o})_\nu, \qquad \text{and } (h_a^{\tau_o})_\nu(y_o) > \sqrt{2f(y_o)M}.$$

From an ODE argument, it follows that

$$(h_a^{\tau_o})_\nu > \sqrt{2f(y_o)M} \quad \text{in } \{\tau_o \leq h_a^{\tau_o} \leq \tau_o + C\eta\}.$$

Hence $\Psi_{\tau_1}((v_\eta - \tau_o)_+)$ is a subsolution for

$$\Delta u = f(x)\beta_{\tau_1}(u),$$

with slope $(1+a)\gamma_{\min}^{\tau_o}$ (see Appendix B for the definition of Ψ_τ). The minimality of $\gamma_{\min}^{\tau_1}$ gives Lemma 5.3. □

Proof of Lemma 5.4. Define

$$u(r) = \sup_{x \in \partial B_r(x_o)} h_a^{\tau_o}(x).$$

Then, the function $u(r)$ satisfies

$$u(r) \leq \tau_o \text{ in } B_\eta(0), \quad u(\eta) = \tau_o.$$

Moreover, $u(r)$ satisfies:

$$\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right) \geq f(x_o)\beta_{\tau_o}(u) \tag{33}$$

on $]\eta/2, \eta[$. Assuming that $(u'(\eta))^2 \leq (1-a)2f(x_o)M$, we will get a contradiction by proving that $u(\eta/2) > \tau_o$.

First, let us re-scale equation (33): the function $v(r) = \frac{1}{\tau_o}u(\tau_o r))$ satisfies:

$$\frac{(n-1)}{r}v'(r) + v''(r) \geq f(x_o)\beta(v) \tag{34}$$

on $]\eta/(2\tau_o), \eta/\tau_o[$, with

$$v(\eta/\tau_o) = 1, \qquad v'(\eta/\tau_o) < (1-a)2f(x_o)M$$

Next, we prove that if τ_o is small enough, there exists $r_o \in [\eta/(2\tau_o), \eta/\tau_o]$ such that $v'(r_o) = 0$. If not, we have $v'(r) > 0$ in $r_o \in [\eta/(2\tau_o), \eta/\tau_o]$. Hence, multiplying by $v'(r)$, and integrating on $[r, \eta/\tau_o]$, for some $r \in [\eta/(2\tau_o), \eta/\tau_o]$, we get:

$$\int_r^{\eta/\tau_o} \frac{(n-1)}{r}(v'(r))^2 dr + \frac{1}{2}(v'(\eta/\tau_o))^2 - \frac{1}{2}(v'(r))^2 \geq f(x_o)M - f(x_o)B(v(r)),$$

and therefore

$$(v'(r))^2 \leq 2\int_r^{\eta/\tau_o} \frac{(n-1)}{r}(v'(r))^2 dr + 2f(x_o)B(v(r)) - 2af(x_o)M.$$

It is easy to check that $v'(r)$ is bounded, and therefore:

$$(v'(r))^2 \leq C\frac{|\eta/\tau_o - r|}{\eta/\tau_o} + 2f(x_o)B(v(r)) - 2af(x_o)M.$$

In particular, if $|\eta/\tau_o - r| \leq \frac{a\lambda M}{C}\eta/\tau_o$, we have:

$$(v'(r))^2 \leq 2f(x_o)B(v(r)) - af(x_o)M.$$

Hence $B(v) \geq af(x_o)M$, and therefore $v(r) \geq c$ if $|\eta/\tau_o - r| \leq \frac{a\lambda M}{C}\eta/\tau_o$.

Since $\beta(c) > 0$, it is readily seen that (34) implies that $v'(r_o) = 0$ for some $r_o > \eta/(2\tau_o)$ if τ_o is small enough. Finally, a similar argument show that $v(r_1) > 1$ for some $r_1 \in [\eta/(2\tau_o), r_o]$, which is the contradiction we were aiming at. □

5.6. **Remark.** With the same method as the one presented in this section, we could prove the existence of a smallest planelike supersolution of (12), $g^\tau(x)$, satisfying:

(i) There exists $\gamma^\tau_{\max}(e) \in [\sqrt{2\lambda M}, \sqrt{2\Lambda M}]$ and $R_\star$ universal constant such that

$$\max(0, \gamma^\tau_{\max}(e)\, x \cdot e - R_\star) \leq g^\tau(x) \leq \max(\tau, \gamma^\tau_{\max}(e)\, x \cdot e).$$

(ii) Any other planelike solution (or supersolution) has a slope smaller than $\gamma^\tau_{\max}(e)$.

Appendix A. Computation of $\gamma_{\min}$ in some particular cases.

A.1. **One-dimensional fronts.** In this section, we seek to compute $\gamma_{\min}$ when $n = 1$.

Let us define

$$\underline{\alpha}(\mu) = \inf_{x\in[0,1]} \left(\sup_{y\in]x-\mu,x+\mu[} f(y) \right).$$

Then, we prove the following result:

Theorem A.1. *For any μ and η positive constants, there exists τ_o such that*

$$\sqrt{2M\inf(f)} - \eta < \gamma^\tau_{\min} < \sqrt{2M\underline{\alpha}(\mu)} + \eta \qquad \forall\, \tau \leq \tau_o$$

When $f(x)$ is a continuous function,

$$\underline{\alpha}(\mu) \to \inf(f) \qquad \text{when } \mu \to 0,$$

and therefore

$$\gamma_{\min}(e) = \sqrt{2M\inf(f)}.$$

Proof. According to Theorem 2.5, the lower bound is always satisfied. To establish Theorem A.1, we have to construct a subsolution with slope $\sqrt{2M\underline{\alpha}(\mu)} + \eta$. The minimality of $\gamma_{\min}$ will imply the result.

First of all, it is obvious that, if $f(x_o) = \inf f$, the function

$$h(x) = (\sqrt{2M\inf(f)} + \eta)(x - x_o)$$

is a subsolution of the free boundary problem. We wish to bend h in the region where $0 < h < \tau$, to get a solution of the β-problem. To that purpose, we must make sure that $|\nabla h(x)|^2 \leq 2Mf(x)$ in a neighborhood of x_o, hence the introduction of $\alpha(\mu)$:

Fixing μ and η, we assume that the infimum $\alpha(\mu)$ is achieved for $x = 0$:

$$\underline{\alpha}(\mu) = \sup_{y\in]-\mu,\mu[} f(y),$$

and we define

$$\gamma = \sqrt{2M\underline{\alpha}(\mu)} + \eta, \quad \text{and } h(x) = \gamma x_+.$$

Then, $h(x)$ is a subsolution of the free boundary problem, and to construct a solution of the β-problem, we introduce φ solution to

$$\varphi'' = \underline{\alpha}(\mu)\beta_\tau(\varphi) \quad \text{in }]-\infty, x_o[\tag{35}$$

$$\varphi(x_o) = \tau \tag{36}$$

$$\varphi'(x_o) = \gamma, \tag{37}$$

with $x_o = \tau/\gamma$. Then, we claim that for $\tau \le \tau_o$, with τ_o small, there exists $x_1 \in [-\mu,\mu]$ such that

$$\varphi(x_1) = 0, \quad \varphi(x) > 0 \quad \forall x \in]x_1, x_o[. \tag{38}$$

As a matter of fact, multiplying (35) by φ' and integrating, we get

$$\frac{1}{2}\Big((\varphi'(x))^2 - (\varphi'(x_o))^2\Big) = \underline{\alpha}(\mu)\Big(B_\tau(\varphi(x)) - B_\tau(\varphi(x_o)\Big),$$

and using (36) and (37), we deduce:

$$(\varphi'(x))^2 = 2\underline{\alpha}(\mu)B_\tau(\varphi(x)) + \gamma^2 - 2M\underline{\alpha}(\mu).$$

Thanks to the choice of γ it follows that

$$(\varphi'(x))^2 \ge 2\underline{\alpha}(\mu)B_\tau(\varphi(x)) + \eta^2 \ge \eta^2,$$

and the claim follows.

Finally, we define h^τ as follows:

$$h^\tau(x) = \begin{cases} 0 & x \le x_1 \\ \varphi(x) & x_1 \le x \le x_o \\ h(x) & x \ge x_o \end{cases} \tag{39}$$

It is readily seen that h^τ is a subsolution of (12), and Theorem 2.5 (ii) gives the result. □

A.2. **Two-dimensional Front.** In this section, we assume that $n = 2$ and $f(x_1,x_2) = f(x_1)$. Defining

$$\underline{\alpha}(\mu) = \inf_{x_1}\left(\sup_{y\in]x_1-\mu,x_1+\mu[} f(y)\right),$$

we prove the following result:

Theorem A.2. *For any μ and η positive constants we have*

$$\sqrt{2M\inf(f)} - \eta < \gamma_{\min}(\pm e_1) < \sqrt{2M\underline{\alpha}(\mu)} + \eta \tag{i}$$

and

$$\gamma_{\min}(e) = \sqrt{2M\langle f\rangle} \quad \textit{for all } e \ne \pm e_1. \tag{ii}$$

Proof. The proof of (i) is similar to the proof of Theorem A.1, so we only have to prove (ii).

Let $h^\tau(x_1,x_2)$ be the planelike solution of (12) with slope γ^τ_{min}. Since (12) is invariant by translation with respect to x_2, the Birkoff property implies that h^τ is periodic in the direction normal to e. Introducing

$$v(y_1,y_2) = h^\tau(y_1, y_2 - \frac{e_1}{e_2}y_1),$$

we deduce that v periodic with period 1 with respect to y_1, and v is solution to:

$$v_{11} + 2av_{12} + (1+a^2)v_{22} = f(y_1)\beta_\tau(v), \tag{40}$$

with $a = \frac{e_1}{e_2}$ (and $(1+a^2) = 1/e_2^2$). Moreover, since

$$\max\{0, \gamma x\cdot e\} \le h^\tau \le \max\{\tau, \gamma x\cdot e + A^\star\},$$

by a rescaling argument, we show that

$$\nabla_x h^\tau \to \gamma e \text{ when } x\cdot e \to +\infty$$

and therefore

$$\nabla_y v \to \gamma \begin{pmatrix} 0 \\ e_2 \end{pmatrix} \text{ when } y_2 \to +\infty. \tag{41}$$

Multiplying (40) by v_2 and integrating over a large rectangle $[0,1]\times[-R,R]$, we have

$$\int_0^1\int_{-R}^R (v_{11}+2av_{12}+\frac{1}{e_2^2}v_{22})v_2dy_2dy_1 = \int_0^1\int_{-R}^R f(y_1)\beta_\tau(v)v_2dy_2dy_1,$$

and after integration by parts we deduce:

$$\int_0^1\int_{-R}^R -v_1v_{21}+\frac{1}{2}\frac{1}{e_2^2}(|v_2|^2)_2dy_2dy_1 = \int_0^1\int_{-R}^R f(y_1)B_\tau(v)_2dy_2dy_1,$$

which yields

$$\frac{1}{2}\int_0^1\int_{-R}^R\Big[-|v_1|^2+\frac{1}{e_2^2}|v_2|^2\Big]_{y_2=-R}^{y_2=+R}dy_1 = \int_0^1 f(y_1)\Big[B_\tau(v)\Big]_{y_2=-R}^{y_2=+R}dy_1,$$

with $B_\tau(u)=\int_0^u\beta_\tau(s)ds$.

Since

$$\begin{aligned} &B_\tau(v)(y_1,R)=M \text{ for large } R,\\ &B_\tau(v)(y_1,-R)\xrightarrow{R\to\infty}0\\ &\nabla_y v(y_1,-R)\xrightarrow{R\to\infty}0,\end{aligned}$$

it follows that

$$\lim_{R\to\infty}\int_0^1\Big[-|v_1|^2+\frac{1}{e_2^2}|v_2|^2\Big]_{y_2=+R}dy_1 = 2\int_0^1 f(y_1)dy_1M,$$

and (41) yields

$$\gamma = 2\langle f\rangle M.$$

□

Remark A.3. Thanks to the invariance with respect to x_2, we can translate the solution h^τ in the direction x_2 in a continuous manner. Thus, using the sliding method, we could prove:
(i) h^τ is monotone increasing with respect to x_2.
(ii) h^τ is the unique planelike solution of (12) (and in particular, $\gamma_{\min}=\gamma_{\max}=2\langle f\rangle M$).

Appendix B. Proof of some technical lemmas: Construction of subsolutions and supersolutions of (12)

In section 5.1, we omitted the construction of the supersolution and subsolution Φ_i^δ. We present those construction here.

We define $\Gamma_\delta^\eta(s)$ as follows:

$$\Gamma_\delta(s)=a\delta \qquad \text{for } 0\le s\le a\delta$$

and Γ_δ solution to

$$\begin{cases} \Gamma_\delta''(s)=\chi_\delta(\Gamma_\delta(s)) & \text{for } s\ge a\delta,\\ \Gamma_\delta(a\delta)=a\delta \\ \Gamma_\delta'(a\delta)=0, \end{cases}$$

with

$$\chi_\delta(u)=\begin{cases} \frac{1+\eta}{2M}\beta_\delta(u) & \text{if } a\delta\le u\le\delta\\ 0 & \text{otherwise.}\end{cases}$$

The constant $\eta > 0$ will be chosen later, and a is such that

$$\int_a^1 \beta(u)du = \frac{M}{1+\eta}.$$

Similarly, we define $\Psi_\delta^\eta(s)$ as follows:

$$\begin{cases} \Psi_\delta''(s) = \frac{1-\eta}{2M}\beta_\delta(\Psi_\delta(s)), \\ \Psi_\delta(0) = 0 \\ \Psi_\delta'(0) = b, \end{cases}$$

where b is such that if u_o satisfies $\Psi(u_o) = \delta$, then $\Psi'(u_o) = 1$.

Thanks to the choice of a and b, it is easy to check that

$$\Gamma_\delta(t) = t - \mathscr{O}(\delta) \text{ when } t \geq \delta$$

and

$$\Psi_\delta(t) = t - \mathscr{O}(\delta) \text{ when } t \geq 2\delta.$$

In particular, we have

$$\Gamma^\delta(t) \xrightarrow{\delta\to 0} t, \qquad \Psi^\delta(t) \xrightarrow{\delta\to 0} t.$$

Then we have the following lemma:

Lemma B.1. *Assume that $f(x)$ is continuous. Then the following holds:*
(i) Let u be a classical supersolution of the free boundary problem such that there exists τ and η such that

$$|\nabla u|^2 \leq \frac{2f(x)M}{1+\eta} \quad \text{in } \{0 < u < \tau\},$$

then $\Gamma_\delta^\eta(u)$ is a supersolution of (12) for $\delta < \tau$.
(ii) Let v be a classical subsolution of the free boundary problem such that there exists τ and η such that

$$|\nabla u|^2 \geq \frac{2f(x)M}{1-\eta} \quad \text{in } \{0 < u < \tau\},$$

then $\Psi_\delta^\eta(u)$ is a subsolution for (12) for $\delta < \tau$.

Proof. We only prove (i), leaving (ii) to the reader. We have:

$$\begin{aligned} \Delta\Gamma_\delta(u) &= \Gamma_\delta'(u)\Delta u + \Gamma_\delta''(u)|\nabla u|^2 \\ &= \frac{1+\eta}{2M}\beta_\delta(\Gamma_\delta(u))|\nabla u|^2\chi_{a\delta\leq\Gamma_\delta(u)\leq\delta} \\ &\leq f(x)\beta_\delta(\Gamma_\delta(u))\chi_{a\delta\leq\Gamma_\delta(u)\leq\delta} \\ &\leq f(x)\beta_\delta(\Gamma_\delta(u)), \end{aligned}$$

which gives the result. □

APPENDIX C. NUMERICAL COMPUTATIONS

In this appendix, we present some numerical computations for the 1-d Pulsating Travelling Fronts solutions of the free boundary problem. Those computations were done prior to the rest of the paper, and actually led us to the proof presented here.

First, we have the following lemma:

Lemma C.1. *For all t, $x \mapsto u(x,t)$ is monotone increasing.*

Proof. If not, there exists t_o such that $u(\cdot,t_o)$ has a local minimum for some x_o.

If $u(x_o,t_o)>0$, then this contradicts the fact that $u(\cdot,t_o)$ is superharmonic in $\{u>0\}$ (since $u_t\leq 0$).

If $u(x_o,t_o)=0$, then there exists x_1, x_2 such that $u(x_i,t_o)>0$ and $x_o\in]x_1,x_2[$. Since u is decreasing with respect to t, we can find $t_1<t_o$ such that $u(x,t_1)>0$ for all $x\in[x_1,x_2]$, and u has a local minimum in $]x_1,x_2[$, which yields a contradiction as in the first case. □

Thanks to this lemma, we can define the level function $v(z,t)$ by:

$$u(v(z,t),t)=z, \qquad z\in[0,1].$$

Differentiating this relation with respect to t and z, we obtain

$$u_t=-u_xv_t, \quad u_x=1/v_z, \quad u_{xx}=-u_xv_{zz}/v_z^2,$$

and (5) becomes:

$$\begin{cases} v_t=\frac{1}{v_z^2}v_{zz} & \text{for } z\in(0,1)\\ v_z^2(0,t)=\frac{1}{2f(v(0,t))M} \\ v(1,t)=+\infty, \end{cases}$$

which can be solve numerically (note that this method does not work for n-d fronts, since fronts are not monotonous with respect to $x\cdot e$). The evolution of the free boundary (or 0-level set) is then given by the curve $x=v(0,t)$.

All figures were obtained with a reaction rate $f(x)$ such that:

$$2f(x)=0.1+(1+\sin(x))^2,$$

and with $M=1$ (and thus $\sqrt{2\inf f}=\sqrt{0.1}$).

Figure 1 shows the evolution of the free boundary for different values of ε, with respect to the rescaled variables $x'=x/\varepsilon$ and $t'=t/\varepsilon$. We observe that the free boundary motion looks more and more like a step function, sticking only on the minimum value of f.

Figure 2 shows the effective speed of propagation as a function of $\log(\varepsilon)$. We immediately notice that the effective speed of propagation decreases as ε goes to zero, and seems to converges to the asymptotic value $c=\sqrt{0.1}\sim 0.32$.

Figure 3 shows that the profile is asymptotically given by an exponential curve as $\varepsilon\to 0$, while Figure 4 shows the boundary layer near the free boundary.

Remark C.2. A more general computation on the level fuction $v(x,t)$ can be found in [10]. For similar results with the porous medium equations and Stephan problem we refere to [9], [11].

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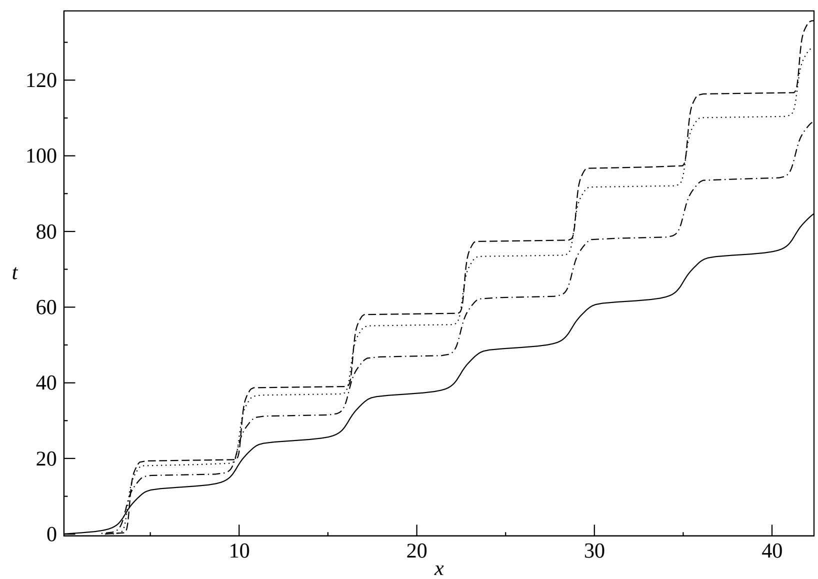

FIGURE 1. Evolution of the free boundary in the rescaled coordinates x/ε, t/ε, for $\varepsilon = 1$ (solid line), 0.1 (dash-dotted line), 0.01 (dotted line) and 0.001 (dashed line).

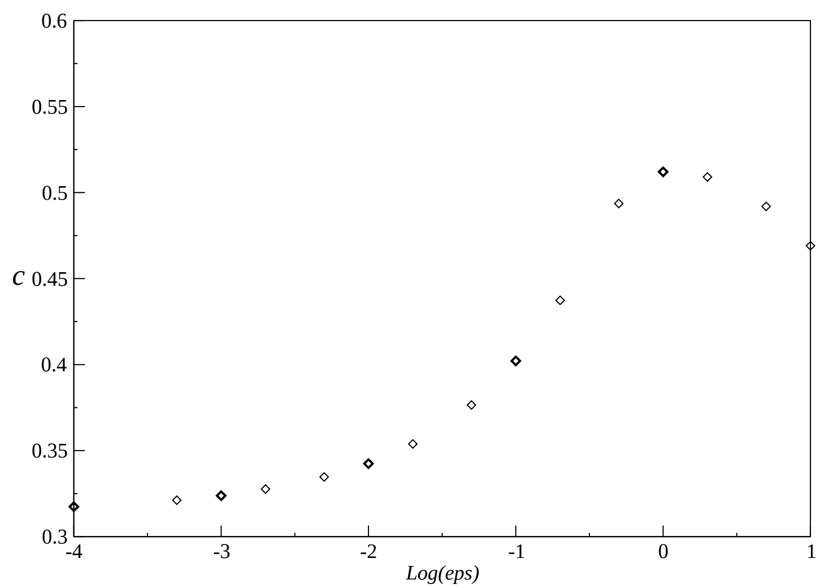

FIGURE 2. Effective speed of propagation with respect to $\log(\varepsilon)$.

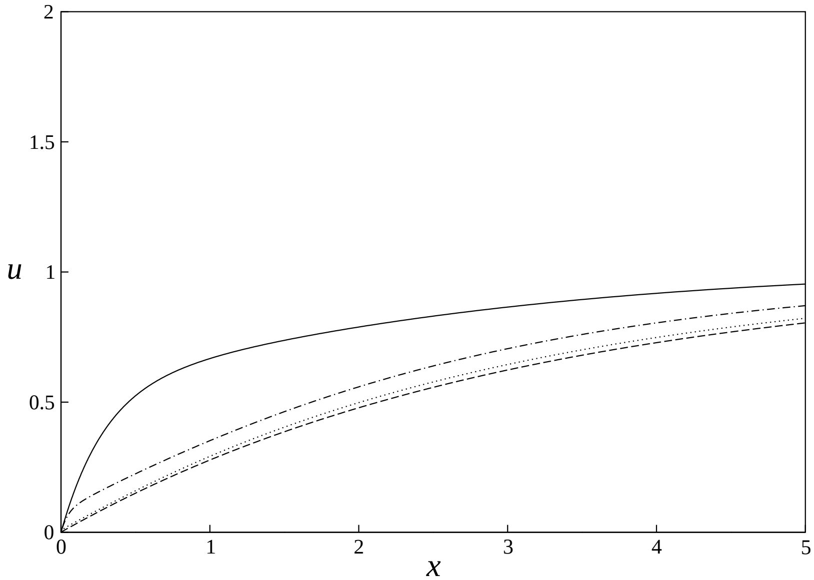

FIGURE 3. Profile of the Pulsating Travelling Front solutions for $\varepsilon = 1$ (solid line), 0.1 (dash-dotted line), 0.01 (dotted line) and 0.001 (dashed line)

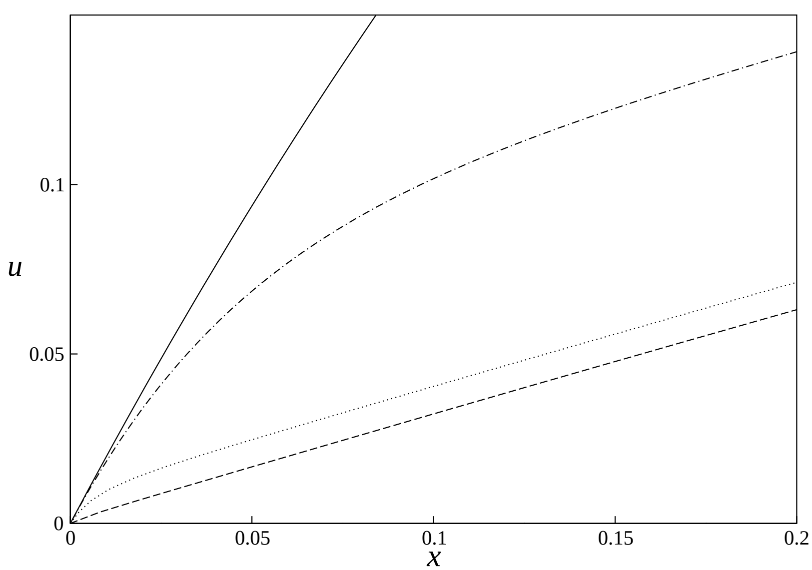

FIGURE 4. The free boundary for $\varepsilon = 1$ (solid line), 0.1 (dash-dotted line), 0.01 (dotted line) and 0.001 (dashed line).

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