CAPILLARY DROPS: CONTACT ANGLE HYSTERESIS AND STICKING DROPS

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Abstract. This paper is concerned with the homogenization of a capillary equation for liquid drops lying on an inhomogeneous solid plane. We show in particular that the homogenization of the Young-Laplace law leads to a contact angle condition of the form $\cos\gamma \in [\beta_1, \beta_2]$, which justifies the so-called contact angle hysteresis phenomenon.

1. Introduction

A liquid drop is resting on a plane Π. We introduce a coordinate system such that Π is the (x, y)-plane. The energy of a drop described by the set E is given by (up to a multiplicative constant)

$$\mathcal{J}(E) = \iint_{z>0} |D\varphi_E| - \int_{z=0} \beta(x,y)\varphi_E(x,y,0)dx\,dy + \frac{1}{\sigma}\int_{z>0} \Gamma\rho\varphi_E dx\,dy\,dz,$$

where φ_E is the characteristic function of E, σ denotes the surface tension, β is the relative adhesion coefficient between the fluid and the solid, Γ denotes the gravitational potential and ρ is the local density of the fluid. The Euler-Lagrange equation for the minimization with volume constrain is the following equation

$$2\mathbf{H} = \frac{\Gamma\rho}{\sigma} - \lambda \tag{1.1}$$

where $\mathbf{H}$ denotes the mean-curvature of the free surface ∂E, and a contact angle condition, known as Young-Laplace's law, which reads:

$$\cos\gamma = \beta(x,y), \tag{1.2}$$

where γ denotes the angle between the free surface ∂E and the support plane $\{z = 0\}$ along the contact line $\partial(E \cap \{z = 0\})$ (measured within the fluid). The coefficient β is determined experimentally, and depends on the properties of the materials (solid and liquid). It is often assumed to be constant, but it is very sensitive to small perturbations in the properties of the solid plane (chemical contamination or roughness). A real solid surface is extremely hard to clean and is never ideal; it always has a small roughness or small spatial inhomogeneities.

These inhomogeneities are responsible for many interesting phenomena, the most spectacular being the contact angle hysteresis (see [JG], [LJ]): In

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the experiments, one almost never measures the equilibrium contact angle given by Young-Laplace's law. Instead, the measured static angle depends on the way in which the drop was formed on the solid. If the equilibrium was reached by advancing the liquid (for example by spreading or condensation), the contact angle has value γ_a larger than the equilibrium value. If on the contrary, the liquid interface was obtained by receding the liquid (evaporation or aspiration of a drop for example), then the contact angle has value γ_r, smaller than the equilibrium value (see L. Hocking [H1], [H2]). In extreme situations (typically when the liquid is not a simple liquid, but a solution), differences of the order of 100 degrees between γ_r and γ_a have been observed (see [LJ]). In [HM], C. Huh and S. G. Mason solve the Young-Laplace equation for some particular type of periodic roughness and explicitly compute the contact angle hysteresis in that case.

Contact angle hysteresis also explains some simple phenomena observed in everyday life, such as the sticking drop on an inclined plane:

If the support plane Π is inclined at angle θ to the horizontal in the y-direction, the potential Γ can be written

$$\Gamma = g(z\cos\theta + y\sin\theta) \tag{1.3}$$

When β is constant and $g, \theta > 0$, no minimizer to (2.4) can exist since any translation in the $y < 0$ direction will strictly decrease the total energy. It can also be shown (see [F2]) that (1.1-1.2) has no solutions when β is constant and $\kappa, \theta \neq 0$. This means that on a perfect surface, the drop should always slide down the plane, no matter how small the inclination. However, a water drop resting on a plane that we slowly inclined will first change shape without sliding, and will only start sliding when the inclination angle reaches a critical value: in the lower parts of the drop, the liquid has a tendency to advance and the contact angle increases until it reaches the advancing contact angle, in the upper parts of the drop, the liquid has a tendency to recede and the contact angle decreases until it reaches the receding contact angle (see [DC]).

In this paper, we address those two issues (contact angle hysteresis and sticking drops) in the case of periodic inhomogeneities, that is when the relative adhesion coefficient reads

$$\beta = \beta(x/\varepsilon, y/\varepsilon),$$

with $(x, y) \mapsto \beta(x, y)$ $\mathbb{Z}^2$-periodic. Naturally, we will need a condition on β to ensure that β is not too close to being a constant in the y direction.

In a previous paper (see [CM]), we investigated the properties of capillary drop for vanishing gravity ($\Gamma = 0$). We showed the existence of global minimizers for $\varepsilon > 0$ and proved that when ε goes to zero, those minimizers converge uniformly to minimizers corresponding to the averaged relative adhesion coefficient

$$\langle\beta\rangle = \int_{[0,1]^2} \beta(x,y)\,dx\,dy.$$

Those minimizers are the intersection of a ball $B_{\rho_o}(z_o)$ in $\mathbb{R}^3$ with the upper half-plane $\{z > 0\}$. This result is recalled in detail in the next section since it will be used throughout the present paper.

As a consequence of this result, there exist viscosity solutions of the free boundary problem (1.1)-(1.2) the limit of which (when $\varepsilon \to 0$) satisfies the following homogenized Young-Laplace law:

$$\cos\gamma = \langle\beta\rangle.$$

This behavior was expected from global minimizers, and it seems to exclude contact angle hysteresis phenomenon. However, in the present paper, we will look for local minimizers of the energy functional. Those minimizers will provide viscosity solutions of the free boundary problem (1.1)-(1.2) for which the homogenized free boundary condition reads

$$\cos\gamma \in [\gamma_1, \gamma_2]$$

(with non trivial interval).

Finally, we point out that when the surface of the drop is a graph $z = u(x,y)$, (1.1)-(1.2) can be rewritten as

$$\begin{cases} \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = -\lambda & \text{in } \{u>0\} \\ \frac{Du}{\sqrt{1+|Du|^2}} \cdot \nu = \beta(x/\varepsilon, y/\varepsilon) & \text{on } \partial\{u>0\}. \end{cases} \tag{1.4}$$

In this case, our result provides viscosity solutions u^ε of (1.4) such that u^ε converges, when $\varepsilon \to 0$, to a solution of

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = -\lambda \quad \text{in } \{u>0\},$$

satisfying

$$\frac{Du}{\sqrt{1+|Du|^2}} \cdot \nu \le \langle\beta\rangle \quad \text{on } \partial\{u>0\}$$

with strict inequality on part of the free boundary. This result has to be compared with that of [CLM] in which we constructed particular solutions of the elliptic free boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \{u>0\} \\ |Du|^2 = f(x/\varepsilon, y/\varepsilon) & \text{on } \partial\{u>0\}, \end{cases} \tag{1.5}$$

and proved that the homogenized free boundary condition in that case read

$$|Du|^2 \in [\gamma_{\min}, \gamma_{\max}].$$

We stress out the fact, however, that solutions of (1.5) were constructed by studying a singular nonlinear equation rather than by minimizing a functional.

The article is organized as follows: In Section 2, we precise our framework, which is that of sets with finite perimeter (or Caccioppoli sets) and recall some results concerning the gravity-free capillary surface. We then state our main results in Section 3. Section 4 is devoted to the construction of constrained minimizers.

In Section 5, we adress the question of the sticking drops and show the existence of local minimizers on an inclined plane for small gravity and small ε (or given gravity and small volume).

Finally, in Section 6, we justify the contact angle hysteresis phenomenon in the gravity-free case.

2. Notation

2.1. **Sets of finite perimeter.** We recall here the main facts about sets of finite perimeter and BV functions. The standard reference for BV theory is Giusti [Gi]. Let Ω be an open subset of $\mathbb{R}^{n+1}$; $BV(\Omega)$ denotes the set of all functions in $L^1(\Omega)$ with bounded variation:

$$BV(\Omega) = \left\{ f \in L^1(\Omega) : \int_\Omega |Df| < +\infty \right\},$$

where

$$\int_\Omega |Df| = \sup \left\{ \int_\Omega f(x) \operatorname{div} g(x) dx : \; g \in [C^1_0(\Omega)]^{n+1}, |g| \leq 1 \right\}.$$

If E is a Borel set, and Ω is an open set in $\mathbb{R}^{n+1}$, we recall that the *perimeter of E in* Ω is defined by

$$P(E, \Omega) = \int_\Omega |D\varphi_E|.$$

A *Caccioppoli set* is a Borel set E that has locally finite perimeter (i.e. $P(E, B) < \infty$ for every bounded open subset B of Ω).

Note that sets of finite perimeter are defined only up to sets of measure 0. We shall henceforth normalize E (as in [Gi]) so that

$$0 < |\overline{E} \cap B(x, \rho)| < |B(x, \rho)| \qquad \text{for all } x \in \partial E \text{ and all } \rho > 0.$$

Furthermore, we recall that if the boundary $\partial\Omega$ of Ω is locally Lipschitz, then each function $f \in BV(\Omega)$ has a *trace* f^+ in $L^1(\partial\Omega)$ (see Giusti [Gi]).

From now on, we denote by Ω the upper half space in $\mathbb{R}^3$:

$$\Omega = \mathbb{R}^2 \times (0, +\infty),$$

and we denote by $\mathscr{E}(V)$ the class of closed Caccioppoli sets in Ω with total volume $V > 0$:

$$\mathscr{E}(V) = \left\{ E \subset \Omega : \int_\Omega |D\varphi_E| < +\infty, |E| = V \right\},$$

where $|E| = \int_\Omega \varphi_E \, dx\, dy\, dz$. Since Caccioppoli sets have a trace on $\partial\Omega = \mathbb{R}^2 \times \{z = 0\}$, we can define the following functional for every set $E \in \mathscr{E}(V)$:

$$\begin{aligned} \mathcal{J}_{\varepsilon,\kappa}(E) &= \iint_{z>0} |D\varphi_E| - \int_{z=0} \beta(x/\varepsilon, y/\varepsilon)\varphi_E(x,y,0)dx\, dy \\ &\qquad +\kappa \int (z\cos\theta + y\sin\theta)\varphi_E dx\, dy\, dz \end{aligned}$$

where $\beta(x,y)$ is a continuous function satisfying:

$$-1 < \beta(x,y) < 1, \qquad \text{for all } x, y,$$

and

$$(x,y) \mapsto \beta(x,y) \quad [0,1]^2\text{-periodic.}$$

We denote by $\langle\beta\rangle$ the average of β:

$$\langle\beta\rangle = \int_{[0,1]^2} \beta(x,y)\, dx\, dy.$$

The crucial assumption concerning β is the following:

$$\min_y \max_x \beta(x,y) < \langle\beta\rangle. \tag{2.1}$$

This condition says that there exists a horizontal line ($y = y_o$) along which β is always strictly less than its average. It is trivially satisfied, for example if β is a function of y only (and β non constant).

In this framework, equilibrium liquid drops are solutions of the minimization problem:

$$\mathcal{J}_{\varepsilon,\kappa}(E) = \inf_{F\in\mathscr{E}(V)} \mathcal{J}_{\varepsilon,\kappa}(F) \qquad E \in \mathscr{E}(V). \tag{2.2}$$

2.2. **Gravity-free minimizers.** If we neglect gravity effects, the energy functional becomes:

$$\mathcal{J}_\varepsilon(E) = \int\int_{z>0} |D\varphi_E| - \int_{z=0} \beta(x/\varepsilon, y/\varepsilon)\varphi_E(x,0)dx. \tag{2.3}$$

We also denote by

$$\mathcal{J}_o(E) = \int\int_{z>0} |D\varphi_E| - \int_{z=0} \langle\beta\rangle\varphi_E(x,0)dx, \tag{2.4}$$

the energy functional associated with the average value of the relative adhesion coefficient.

2.2.1. *Constant adhesion coefficient.* The Schwartz symmetrization of E, defined by

$$E^s = \{(x,y,z) \in \Omega\,;\ |(x,y)| < \rho(z)\}, \tag{2.5}$$
$$\rho(z) = \left(\pi^{-1}\int \varphi_E(x,y,z)\,dx\,dy\right)^{\frac{1}{2}}$$

is a Caccioppoli set with volume V satisfying

$$\mathcal{J}_o(E^s) \leq \mathcal{J}_o(E)$$

with equality if and only if E was already axially symmetric. This shows that the minimizer must be axially symmetric and that the wetting surface $E \cap \{z = 0\}$ is a disk.

Considering $B_\rho(z)$ with ρ and z such that

$$|B_\rho(z) \cap \{z > 0\}| = |E|\,, \qquad B_\rho \cap \{z = 0\} = E \cap \{z = 0\},$$

and comparing the perimeter of B_ρ with that of $E \cup (B_\rho \cap \{z < 0\})$, the isoperimetric inequality immediately implies that the unique minimizer for $\mathcal{J}_o$ in $\mathscr{E}(V)$ is a spherical cap

$$B^+_{\rho_o}(z_o) = B_{\rho_o}(0, z_o) \cap \{z > 0\}.$$

Morever, a simple computation (see [CM]) shows that among the spherical cap, the minimum of $\mathcal{J}_o$ is achieved when Young-Laplace's law is satisfied: ρ_o and z_o are such that:

$$\frac{z_o}{\rho_o} = \beta, \qquad \int_{-z_o}^{\rho_o} \pi(\rho_o^2 - r^2)dr = V \tag{2.6}$$

2.2.2. *Periodic adhesion coefficient.* When β depends on (x, y) the method fails, since the rearrangement (2.5) could increase the wetting energy

$$\int \beta(x/\varepsilon, y/\varepsilon)\varphi_E(x,y,0)\,dx\,dy.$$

In [CM] we proved:

Theorem 2.1. *For any volume $V > 0$ and for all $\varepsilon > 0$, there exists a minimizer of $\mathcal{J}_\varepsilon$ in $\mathscr{E}(V)$.*

Moreover, for all $\eta > 0$, there exists ε_o such that if $\varepsilon \leq \varepsilon_o$, then any minimizer E_ε satisfies:

$$B^+_{(1-\eta)\rho_o}(z_o) \subset E_\varepsilon \subset B^+_{(1+\eta)\rho_o}(z_o).$$

where $B^+_{\rho_o}(z_o) = B_{\rho_o}(z_o) \cap \Omega$ is a spherical cap with volume V and contact angle $\cos^{-1}\langle\beta\rangle$.

In particular, when ε is small, the apparent contact angle for the equilibrium drop resting on a inhomogeneous surface satisfies Young-Laplace's law with $\langle\beta\rangle$.

We recall that in the proof of Theorem 2.1 a crucial step is to prove that

$$\mathcal{J}_o(E_\varepsilon) \le \mathcal{J}_\varepsilon(E_\varepsilon) + C\varepsilon \le \mathcal{J}_o(B^+_{\rho_o}) + C\varepsilon.$$

The result then follows from the stability result (see [CM] for details):

Theorem 2.2. *Let E be such that*

$$E \in \mathscr{E}(V), \quad E \text{ lies in a bounded subset } B_R \text{ of } \Omega, \tag{2.7}$$

$$\exists \delta > 0 \quad s.t. \; \mathcal{J}_o(E) \le \mathcal{J}_o(F) + \delta \qquad \forall F \in \mathscr{E}(V). \tag{2.8}$$

Then there exists a universal $\alpha > 0$ and a constant C (depending on R) such that

$$|E \Delta B^+_{\rho_o}| \le C\delta^\alpha.$$

3. Main Results

3.1. **Contact angle hysteresis.** Our first result concerns the contact angle hysteresis. We restrict ourself to the gravity-free case. In view of Theorem 2.1, it is clear that the apparent contact angle of global minimizers of $\mathcal{J}_\varepsilon$ satisfies Young-Laplace's law with $\langle\beta\rangle$. However, we will show that the oscillations of $\beta(x,y)$ allows for local minimizers with larger contact angle in some directions.

To that purpose, we recall that the wetting surface corresponding to the asymptotic minimizer $B^+_{\rho_o}$ has radius

$$R_o = \rho_o\sqrt{1 - \langle\beta\rangle^2}.$$

So we introduce

$$\Sigma_t = \{(x,y) \in \mathbb{R}^2\,;\, 0 \le y \le 2R_o - t\},$$

and look for minimizers of $\mathcal{J}_\varepsilon$ whose wetting area stays within the region Σ_t. Clearly, for $t > 0$, E_ε (or $B^+_{\rho_o}$) is not a candidate anymore. We prove the following:

Theorem 3.1. *Assume that the relative adhesion coefficient $\beta(x,y)$ satisfies*

$$\min_y \max_x \beta(x,y) < \langle\beta\rangle. \tag{3.1}$$

Then, for all V, there exists ε_o and t_o such that if $\varepsilon < \varepsilon_o$ and $t < t_o$, then $\mathcal{J}_\varepsilon$ has a local minimizer $\widetilde{E}_\varepsilon$ whose wetting area $\widetilde{E}_\varepsilon \cap \{z = 0\}$ lies in Σ_t.

When ε goes to zero, $\widetilde{E}_\varepsilon$ converges to some set $\widetilde{E} \in \mathscr{E}(V)$. The free surface $\partial\widetilde{E}$ satisfies

$$2\mathbf{H} = -\lambda$$
$$\cos\gamma \le \langle\beta\rangle,$$

and the contact angle of $\widetilde{E}$ is strickly greater than $\cos^{-1}\langle\beta\rangle$ *on parts of the contact line.*

Note that $\partial\widetilde{E}_\varepsilon$ is a viscocity solution of

$$\begin{aligned} &\mathbf{2H} = -\lambda \\ &\cos\gamma = \beta(x/\varepsilon, y/\varepsilon). \end{aligned}$$

The last part of Theorem 3.1 says that the apparent contact angle (or homogenized contact angle) is indeed larger than $\cos^{-1}\langle\beta\rangle$ in some directions. Note that if condition (3.1) is satisfied in other directions, we will observe hysteresis phenomenon in those directions as well. One can actually construct functions β such that (2.1) is satisfied for any finite number of directions. To get infinitely many directions, one would probably have to consider random inhomogeneities.

The so-constructed solution plays the role of barrier during the formation of a liquid drop by slow spreading or condensation. This explains the so-called stick-jump phenomenon: As the volume of the drop increases, the contact line remains unchanged at first, while the contact angle increases. Only when the contact angle reaches a critical value does the contact line jump to the next equilibrium position (see [HM]).

3.2. **Sticking drop on an inclined plane.** Nest, we wish to justify the existence of equilibrium liquid drops on an inclined plane. As we pointed out in the introduction, (2.2) has no global minimizer when $\kappa, \theta > 0$. However, we can look for local minimizers in the following sense:

$$E \Subset \Gamma$$

and

$$\mathcal{J}_{\varepsilon,\kappa}(E) \leq \mathcal{J}_{\varepsilon,\kappa}(F) \text{ for any } F \in \mathscr{E}(V) \text{ with } F \subset \Gamma$$

where Γ is an open subset of Ω. We prove the following result:

Theorem 3.2. *Assume that the relative adhesion coefficient $\beta(x,y)$ satisfies (2.1). Then, for all $V > 0$, there exists an open subset Γ of Ω and two constants κ_o and ε_o such that if $\varepsilon < \varepsilon_o$ and $\kappa < \kappa_o$ the minimizer of $\mathcal{J}_{\varepsilon,\kappa}$ in $\{E \in \mathscr{E}(V)\,;\, E \subset \Gamma\}$ is a local minimizer for $\mathcal{J}_{\varepsilon,\kappa}$.*

Note that such a minimizer gives a viscosity solution of Euler-Lagrange's equation:

$$\begin{aligned} &\mathbf{2H} = -\lambda + \kappa(z\cos\theta + y\sin\theta) \\ &\cos\gamma = \beta(x/\varepsilon, y/\varepsilon). \end{aligned}$$

We recall that R. Finn, [F], proved that this equation has no solutions when β is constant.

4. Constrained minimizer

4.1. **Barrier.** The proof of both Theorem 3.1 and 3.2 relies on the existence of constrained minimizers that turn out to be unconstrained local minimizers for $\mathcal{J}$. This section is devoted to the existence of those constrained minimizers.

First, we need to construct an appropriate barrier. Since we later want to prove that the minimizer stays away from the barrier, it should be a supersolution for (1.1), (1.2), at least near the contact line.

Assume that the minimum value in (2.1) is achieved for $y = 0$, and let

$$\alpha := \max_x \beta(x, 0).$$

We have

$$(1 + 2\eta)\alpha < \langle\beta\rangle,$$

for some small η. Let $\gamma_1 < \gamma_2$ be such that

$$\cos\gamma_1 = \langle\beta\rangle\,, \qquad \cos\gamma_2 = (1 + 2\eta)\alpha.$$

We construct a barrier $\mathcal{S}$ as follows:

- For $z < \lambda$, $\mathcal{S}$ is a portion of cylinder with mean curvature $-\dfrac{1 + 8\eta}{\rho_o}$ (radius $\frac{\rho_o}{2(1+8\eta)}$) which intersects the plane $\{z = 0\}$ along the contact line $y = 0$ with an angle γ_2.
- For $z \geq \lambda$, $\mathcal{S}$ is a plane forming an angle of γ_1 with the plane $\{z = 0\}$.

This leads to an equation $y = S(z)$ with

$$S(z) = \begin{cases} \frac{\rho_o}{2(1+8\eta)}\left(\sin\gamma_2 - \sqrt{1 - (2(1+8\eta)z/\rho_o + \cos\gamma_2)^2}\right) & z \leq \lambda \\ (\text{cotan}\gamma_1)z + \frac{\rho_o}{2(1+8\eta)}\left[\sin\gamma_2 - \sin\gamma_1 - \frac{\cos^2\gamma_1}{\sin\gamma_1} + \frac{\cos\gamma_1\cos\gamma_2}{\sin\gamma_1}\right] & z \geq \lambda \end{cases}$$

where $\lambda = \frac{\rho_o}{2(1+8\eta)}(\cos\gamma_1 - \cos\gamma_2)$.

Inclined plane: On the inclined plane, we want to use the barrier $\mathcal{S}$ to prevent the drop from sliding down the slope (see figure (4.1)). This leads to the domain

$$\Gamma = \{(x, y, z) \in \mathbb{R}^3\,;\, z > 0\,,\ y \geq S(z)\}.$$

Since Γ is not bounded in the x and z direction, we will first prove the existence of a minimizer in

$$\Gamma_{R,T} = \{(x, y, z) \in \mathbb{R}^3\,;\, z \in [0, T], x \in [-R, R], S(z) \leq y \leq R + S(z)\}. \tag{4.1}$$

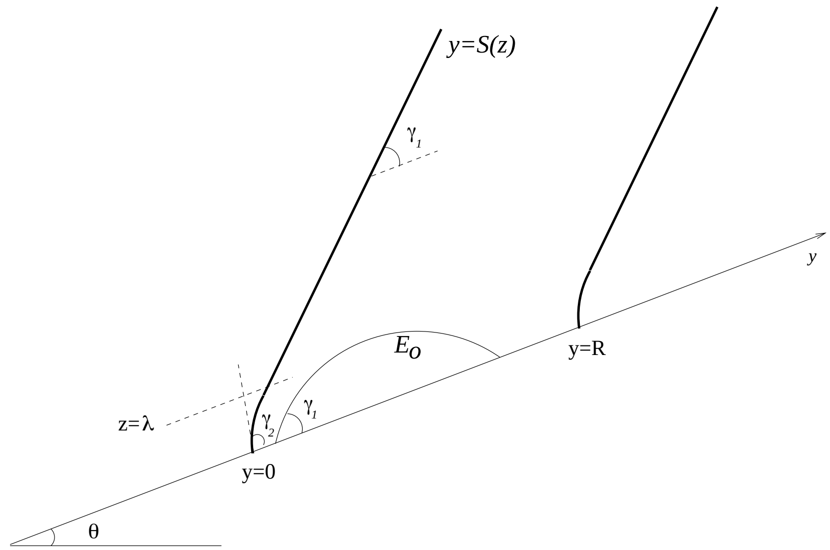

FIGURE 1. Barrier for the inclined plane

Contact angle hysteresis: We want to construct a local minimizer of $\mathcal{J}_\varepsilon$ whose wetting surface stays within

$$\Sigma_t = \{(x,y)\,;\, 0 \leq y \leq 2R_o - t\},$$

where $R_o = \rho_o\sqrt{1-\langle\beta\rangle^2}$ is the radius of the wetting region $B^+_{\rho_o} \cap \{z=0\}$. To that purpose, we consider the domain (see figure (4.1)):

$$\Gamma_t = \{(x,y,z) \in \mathbb{R}^3\,;\, z > 0\,,\ S(z) \leq y \leq 2R_o - S(z) - t\}.$$

Since we want $\beta(x,y)$ to be greater than $\cos\gamma_2$ along the contact line $y = 2R_o - S(z)$, we need to request that t be such that there exists $n \in \mathbb{N}$ such that

$$t = 2R_o - n\varepsilon.$$

Since we aim at describing the behavior of the minimizers when $\varepsilon \to 0$, this condition is not very restrictive and we shall not dwell on it any further. Again, in order to get a bounded domain, we introduce
(4.2)

$$\Gamma_{R,T} = \{(x,y,z) \in \mathbb{R}^3\,;\, z \in [0,T], x \in [-R,R], S(z) \leq y \leq 2R_o - S(z) - t\}.$$

4.2. **Existence of constrained minimizers.** We conclude this section by recalling the main steps of the construction of a constrain minimizer. Here, the bounded domain $\Gamma_{R,T}$ could be either (4.1) or (4.2) and we work with $\mathcal{J}_{\varepsilon,\kappa}$. The gravity-free functional $\mathcal{J}_\varepsilon$ is nothing but a particular case.

Let

$$\mathscr{E}_{R,T}(V) = \{E \in \mathscr{E}(V)\,;\, E \subset \Gamma_{R,T}\}.$$

Then we have:

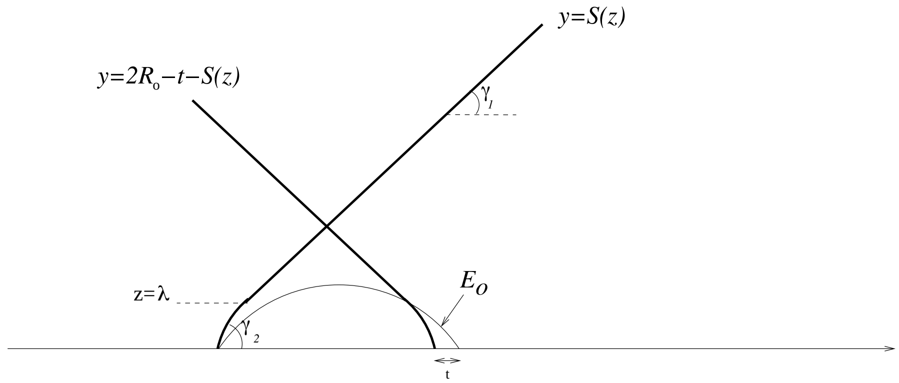

FIGURE 2. Barrier for the contact angle hysteresis

Proposition 4.1. *If R and T are such that there exists a ball $\widetilde{B}$ with volume V in $\Gamma_{R,T}$, then there exists $E_{\varepsilon,\kappa} \in \mathscr{E}_{\Gamma_{R,T}}(V)$ satisfying*

$$\mathcal{J}_{\varepsilon,\kappa}(E_{\varepsilon,\kappa}) = \min_{F \in \mathscr{E}_\Gamma(V)} \mathcal{J}_{\varepsilon,\kappa}(F) \tag{4.3}$$

Moreover, we have

$$P(E_{\varepsilon,\kappa}) \leq CV^{\frac{n}{n+1}} \quad \text{and } \mathscr{H}^n(E_{\varepsilon,\kappa} \cap \{z=0\}) \leq CV^{\frac{n}{n+1}}. \tag{4.4}$$

Proof: After rescaling, we may prove the result when $|E| = V = 1$. First, we recall the following result for function with bounded variation (see [Gi]):

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set, and (f_j) a sequence of functions in $BV(\Omega)$ which converges in $L^1_{loc}(\Omega)$ to a function f. Then*

$$\int_\Omega |Df| \leq \liminf_{j\to\infty} \int |Df_j|.$$

It follows (see [CM] for details) that:

Lemma 4.3. *The functional $\mathcal{J}$ is lower continuous with respect to the L^1 topology: If (E_j) is a sequence of Caccioppoli sets such that $E_j \longrightarrow E$ in L^1 then*

$$\mathcal{J}_{\varepsilon,\kappa}(E) \leq \liminf_{j\to\infty} \mathcal{J}_{\varepsilon,\kappa}(E_j)$$

Next, we recall the following compactness result for functions of bounded variations:

Lemma 4.4. *Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary. Then sets of functions uniformly bounded in BV-norm are relatively compact in $L^1(\Omega)$.*

Finally, the existence of a constrained minimizer is a consequence of the following a priori estimates:

Lemma 4.5. *If* $-1<\beta<1$*, then*

$$\mathcal{J}_{\varepsilon,\kappa}(E) \;\geq\; \frac{1-\beta_{\max}}{2}\int_{z>0}|D\varphi_E| + \frac{1-\beta_{\max}}{2}\int_{z=0}\varphi_E dx\,dy + \kappa\int(z\cos\theta+y\sin\theta)\varphi_E\,dx\,dy\,dz$$

for all $E\in\mathscr{E}(V)$*. In particular*

$$\mathcal{J}_{\varepsilon,\kappa}(E)\geq\frac{1-\beta_{\max}}{2}P(E,\mathbb{R}^3)$$

for all $E\in\mathscr{E}_\Gamma(V)$*.*

Proof of Lemma 4.5: If $g(x)$ is a non-negative function, then

$$\int_{z>0}g(x)|D\varphi_E|\geq\int_{z=0}g(x)\varphi_E dx.$$

Hence

$$\int_{z>0}\frac{1+\beta(x)}{2}|D\varphi_E|\geq\int_{z=0}\frac{1+\beta(x)}{2}\varphi_E dx$$

and therefore

$$\int_{z>0}\left(1-\frac{1-\beta(x)}{2}\right)|D\varphi_E|\geq\int_{z=0}\left(\frac{1-\beta(x)}{2}+\beta(x)\right)\varphi_E dx$$

It follows that

$$\int|D\varphi_E|-\int_{z=0}\beta(\frac{x}{\varepsilon},\frac{y}{\varepsilon})\varphi(x,y,0)\geq\int_{z>0}\frac{1-\beta}{2}|D\varphi_E|+\int_{z=0}\frac{1-\beta}{2}\varphi_E dx$$

which gives the first inequality in Lemma 4.5. It remains to notice that the gravitational energy is positive for $E\in\mathscr{E}_\Gamma(V)$ to get the second inequality. □

Lemma 4.3, 4.4 and 4.5, together with the fact that $\mathscr{E}_\Gamma(V)\neq\emptyset$ gives Proposition 4.1. □

5. Sticking drop on an inclined plane

This section is devoted to the proof of Theorem 3.2. We recall that

$$\Gamma_{R,T}=\{(x,y,z)\in\mathbb{R}^3\,;\,z\in[0,T],x\in[-R,R],S(z)\leq y\leq R+S(z)\},$$

and throughout this section, $E_{\varepsilon,\kappa}$ denotes a minimizer of $\mathcal{J}_{\varepsilon,\kappa}$ in $\Gamma_{R,T}$, as constructed in Section 4. To prove Theorem 3.2, we only need to check that $E_{\varepsilon,\kappa}\Subset\Gamma$ when ε and κ are small enough.

The proof is organized in three steps:

Step 1: We show that $E_{\varepsilon,\kappa}$ converges uniformly to $B^+_{\rho_o}$ when ε and κ go to zero. This implies that $E_{\varepsilon,\kappa}$ stays away from $x=\pm R$, $z=T$, $y=R+S(y)$ and from the linear part of the barrier $\mathcal{S}$. We will also deduce that the mean curvature of the unconstrained part of $\partial E_{\varepsilon,\kappa}$ converges to $1/\rho_o$.

Step 2: Next, we show that $\partial E_{\varepsilon,\kappa}$ stays away from the curved part of the barrier $\mathcal{S}$, using the fact that the mean-curvature is smaller than that of $E_{\varepsilon,\kappa}$. This is the object of Proposition 5.6.

Step 3: Finally, Proposition 5.8 shows that the contact line of $E_{\varepsilon,\kappa}$ cannot touch $y=0$ thanks to the condition on $\beta(x,y)$ and the choice of the contact angle of the barrier.

Note that Step 2 & 3 rely on the fact that the barrier $\mathcal{S}$ is a supersolution for the Euler-Lagrange equation, and make use of a cutting argument first introduced in [CC]. Alltogether, those three steps yield Theorem 3.2.

5.1. **Step1: Stability.** The first step is a consequence of the following result:

Proposition 5.1. *The minimizer $E_{\varepsilon,\kappa}$ of $\mathcal{J}_{\varepsilon,\kappa}$ in $\mathscr{E}_{\Gamma_{R,T}}(V)$ satisfies*

$$\mathcal{J}_o(E_{\varepsilon,\kappa}) \leq \mathcal{J}_o(F) + C\varepsilon + C\kappa \tag{5.1}$$

for all $F \in \mathscr{E}(V)$

In particular, Theorem 2.2 yields:

$$|E_{\varepsilon,\kappa} \cap B^+_{\rho_o}| \leq (C\varepsilon + C\kappa)^\alpha.$$

Proof. We recall that E_ε denotes the minimizer of $\mathcal{J}_\varepsilon$. Then we have

$$\mathcal{J}_\varepsilon(E_{\varepsilon,\kappa}) \leq \mathcal{J}_{\varepsilon,\kappa}(E_{\varepsilon,\kappa}) \leq \mathcal{J}_{\varepsilon,\kappa}(E_\varepsilon)$$

and since E_ε is uniformly bounded, we also have

$$\mathcal{J}_{\varepsilon,\kappa}(E_\varepsilon) \leq \mathcal{J}_\varepsilon(E_\varepsilon) + C\kappa.$$

It follows that

$$\mathcal{J}_\varepsilon(E_{\varepsilon,\kappa}) \leq \mathcal{J}_\varepsilon(E_\varepsilon) + C\kappa.$$

Since we already know (see [CM]) that

$$\mathcal{J}_\varepsilon(E_\varepsilon) \leq J_o(E_\varepsilon) + C\varepsilon,$$

and

$$\mathcal{J}_o(E_\varepsilon) \leq \mathcal{J}_o(F) + C\varepsilon$$

for all $F \in \mathscr{E}(V)$, we will prove Proposition 5.1 if we can prove that

$$\mathcal{J}_o(E_{\varepsilon,\kappa}) \leq \mathcal{J}_\varepsilon(E_{\varepsilon,\kappa}) + C\varepsilon \tag{5.2}$$

This amounts to showing that

$$\int \varphi_{E_{\varepsilon,\kappa}}(x,0)|\beta(x/\varepsilon, y/\varepsilon) - \langle\beta\rangle|\,dx\,dy \leq C\varepsilon.$$

Noticing that this integrale is zero over every cell of $\mathbb{R}^2/\mathbb{Z}^2$ that is fully contains in $E_{\varepsilon,\kappa}$ or $E^c_{\varepsilon,\kappa}$, this inequality will be satisfied if we can prove that the number of cells (of size ε) intersecting the contact line $(\partial(E_{\varepsilon,\kappa} \cap \{z=0\})$ is of order ε^{-1}. This amounts to showing

Proposition 5.2. *The contact line has finite 1-dimensional hausdorff measure in* $\mathbb{R}^2$*:*

$$\mathscr{H}^1(\partial E \cap \{z=0\}) < +\infty.$$

The proof of Proposition 5.2 follows essentially the same ideas as the proof of a similar result established in [CM] for the gravity-free drop. We outline the main step of the proof in Appendix A. □

The non-degeneracy lemma A.2 and the L^1 convergence given by Proposition 5.1 implies uniform convergence:

Proposition 5.3. *For all* $\eta' > 0$*, there exists* ε_o *and* κ_o *such that if* $\varepsilon < \varepsilon_o$ *and* $\kappa < \kappa_o$*, then*

$$B^+_{(1-\eta')\rho_o} \subset E_{\varepsilon,\kappa} \subset B^+_{(1+\eta')\rho_o}$$

Clearly, Proposition 5.3 implies that if R and T are large enough, $E_{\varepsilon,\kappa}$ stays away from $z = T$, and up to a translation with respect to x, it stays also away from $x = \pm R$ and $y = R + S(z)$.

Thus $E_{\varepsilon,\kappa}$ is a minimizer for $\mathcal{J}_{\varepsilon,\kappa}$ in the class

$$\mathscr{E}_\Gamma(V) = \{E \in \mathscr{E}(V)\,;\, E \subset \Gamma\}.$$

Moreover, for an appropriate choice of η and λ, $E_{\varepsilon,\kappa}$ stays away from the linear part of the barrier $\mathcal{S}$.

The free surface $\partial E_{\varepsilon,\kappa}$ can therefore only touch $\mathcal{S}$ in the part $\{0 \le z < \lambda\}$, that is in the region where $\mathcal{S}$ has mean curvature $-\frac{1+\eta}{\rho_o}$. To prevent this from happening, we need to characterize the mean-curvature of the ∂E:

Lemma 5.4. *If* η *is small enough, the mean-curvature of* $E_{\kappa,\varepsilon}$ *is such that*

$$\mathbf{H} \ge -\frac{1+\eta}{\rho_o} \tag{5.3}$$

Proof. We first notice that if $x_o \in \partial E_{\varepsilon,\kappa}$, $x_o \in \Gamma$, then for a small ball $B_r(x_o)$ contained in Γ, $E_{\varepsilon,\kappa}$ minimizes the functional

$$\mathcal{F}(F) = \int_\Omega |D\varphi_F| + \kappa \int (z\cos\theta + y\sin\theta)\varphi_F$$

among the Caccioppoli sets which coincide with $E_{\varepsilon,\kappa}$ on $\partial B_r(x_o)$ and satisfy $|F| = |E_{\varepsilon,\kappa} \cap B_r|$. Hence, by E. Gonzalez et al. [GMT], ∂E is analytic in B_r (recall that $n = 3 \le 7$), and we have the following lemma:

Lemma 5.5. *Away from* $\mathcal{S}$*, the free surface is analytic, and satisfies*

$$2\mathbf{H} = \kappa(z\cos\theta + y\sin\theta) - \lambda \tag{5.4}$$

Next, if (5.3) does not hold at some point, (5.4) yields

$$H \le -\frac{1+\eta/2}{\rho_o}$$

for $\kappa \le \kappa_o$, so that a ball with radius

$$\frac{\rho_o}{1+\eta/2}$$

is a subsolution for the free surface (i.e. it cannot touch the free surface from below). The fact that

$$\partial E_{\varepsilon,\kappa} \subset B_{(1+\eta')\rho_o} \setminus B_{(1-\eta')\rho_o}$$

for small ε and κ leads to a contradiction (for a suitable choice of η'). □

5.2. **Step 2: $E_{\varepsilon,\kappa}$ stays away from $\mathcal{S}$.** We can now prove the following proposition:

Proposition 5.6. *For ε and κ small enough, there exists $\delta > 0$ such that*

$$E_{\varepsilon,\kappa} \subset \{(x,y,z) \in \Gamma\,;\, y \ge S(z) + \delta z^2\}$$

Proof. Let $M_o = (x_o, y_o, z_o)$ be a point on $\mathcal{S}$ with $0 < z_o < T$. We introduce a surface $\mathcal{S}'$ with curvature $-\frac{2(1+7\eta)}{\rho_o}$ in the y direction, and $2\eta/\rho_o$ in the x direction. More precisely, if x, y', z' is a system of coordinates with z' normal to S at M_o and y' tangent to $\mathcal{S}$, we define the surface $\mathcal{S}'$ by

$$z' = -\frac{1+7\eta}{\rho_o} y'^2 + \frac{\eta}{\rho_o} x^2 - \delta.$$

Note that in the neighborhood of M_o, $\mathcal{S}'$ has mean curvature close to $-\frac{1+6\eta}{\rho_o}$. Following [CC], we introduce d the distance function, from above to $\mathcal{S}'$. Then one has the formula:

$$\Delta d(x) = -\sum_{j=1}^{2} \frac{\kappa_j}{1-\kappa_j d(x)}$$

where κ_j is the j th-curvature of the surface at the point where x realizes its distance. We deduce:

Lemma 5.7. *Let $\mathcal{U}$ denotes the region enclosed by $\mathcal{S}$ and $\mathcal{S}'$. If $\delta \le \delta_o$ for some critical δ_o, then d is smooth in $\mathcal{U}$ and satisfies*

$$\Delta d \ge \frac{1+4\eta}{\rho_o} \quad \text{in } \mathcal{U}.$$

Together with Green's formula, this lemma yields

$$\frac{1+4\eta}{\rho_o}|E_{\varepsilon,\kappa} \cap \mathcal{U}| \le \int_{\mathcal{U}\cap E_{\varepsilon,\kappa}} \Delta d = \int_{\partial(\mathcal{U}\cap E_{\varepsilon,\kappa})} \nabla d \cdot \nu,$$

where this inequality must be considered in the perimeter sense.

Since $\nabla d \cdot \nu = -1$ on $(\partial\mathcal{U}) \cap E_{\varepsilon,\kappa} = \mathcal{S}' \cap E_{\varepsilon,\kappa}$, and $0 \le \nabla d \cdot \nu \le 1$ on $\mathcal{U} \cap \partial E_{\varepsilon,\kappa}$, we get

$$\int_{\mathcal{U}\cap E_{\varepsilon,\kappa}} \Delta d \le -\mathscr{H}(\partial\mathcal{U} \cap E_{\varepsilon,\kappa}) + P(E_{\varepsilon,\kappa}, \mathcal{U}).$$

Therefore, the set

$$F = E_{\varepsilon,\kappa} \setminus \mathcal{U}$$

satisfies

$$\begin{aligned}
\mathcal{J}(F) &= \mathcal{J}(E_{\varepsilon,\kappa}) - P(E_{\varepsilon,\kappa},\mathcal{U}) + \mathscr{H}(\partial\mathcal{U}\cap E_{\varepsilon,\kappa}) \\
&\qquad -\kappa\int_{\mathcal{U}\cap E_{\varepsilon,\kappa}} y\sin\theta + z\cos\theta\,dx\,dy\,dz \\
&\leq \mathcal{J}(E_{\varepsilon,\kappa}) - \frac{1+4\eta}{\rho_o}|\mathcal{U}\cap E_{\varepsilon,\kappa}|
\end{aligned}$$

We use a similar method to increase the volume of F away from $\mathcal{S}$ by the same quantity $|E_{\varepsilon,\kappa}\cap\mathcal{U}|$: Let M be a point on ∂F away from $\mathcal{S}$ and $\{z=0\}$. Let x', y', z' be a system of local coordinate such that x' and y' are the principale curvature direction (with principale curvature respectively α_x and α_y) and z' is the outward normal to ∂F. Since $E_{\varepsilon,\kappa}$ (and therefore F) has mean curvature less than $(1+\eta)/\rho_o$ (in absolute value) away from $\mathcal{S}$, we have

$$\frac{1}{2}(\alpha_x+\alpha_y) \geq -\frac{1+\eta}{\rho_o}$$

Let $\mathcal{S}''$ be the surface defined by the quadradic polynomial

$$z' = \frac{1}{2}(\alpha_x - \frac{\eta}{\rho_o})x'^2 + \frac{1}{2}(\alpha_y - \frac{\eta}{\rho_o})y'^2 + \delta.$$

If δ is small enought, then $\mathcal{S}''$ and ∂F enclose a bounded subset $\mathcal{V}$, and

$$\Delta d \leq \frac{1+2\eta}{\rho_o} \quad \text{in } \mathcal{V}.$$

Hence

$$\frac{1+\eta}{\rho_o}|\mathcal{V}| \geq P(E,\mathcal{V}) - \mathscr{H}(\partial\mathcal{V}\setminus E),$$

and we can always choose δ such that $|\mathcal{V}| = |E_{\varepsilon,\kappa}\cap\mathcal{U}|$. The resulting set $\widetilde{E} = F\cup\mathcal{V}$ therefore satisfies

$$\begin{aligned}
\mathcal{J}(\widetilde{E}) &\leq \mathcal{J}(E_{\varepsilon,\kappa}) - \frac{2\eta}{\rho_o}|\mathcal{U}\cap E_{\varepsilon,\kappa}| + \kappa\int_{E_{\varepsilon,\kappa}\cap\mathcal{U}}(y\sin\theta + z\cos\theta)\,dx\,dy\,dz \\
&\leq \mathcal{J}(E_{\varepsilon,\kappa}) - \frac{2\eta}{\rho_o}|\mathcal{U}\cap E_{\varepsilon,\kappa}| + C\kappa|E_{\varepsilon,\kappa}\cap\mathcal{U}|
\end{aligned}$$

and $|\widetilde{E}| = |E_{\varepsilon,\kappa}| = V$. The minimality of $E_{\varepsilon,\kappa}$ yields

$$|\mathcal{U}\cap E_{\varepsilon,\kappa}| = 0,$$

if κ is small enough. Proposition 5.6 follows. □

5.3. **Step 3: The contact line stays away from $y = 0$.** Finally, using similar arguments (taking into account the wetting energy), we prove:

Proposition 5.8. *For ε and κ small enough, there exists $\tau > 0$ such that*

$$E_{\varepsilon,\kappa} \subset \{(x, y, z) \in \Gamma\,;\, y \geq \tau\varepsilon + (\cos\gamma)z\}$$

Proof. We recall that $\mathcal{S}$ intersects $\{z = 0\}$ with an angle γ_2 satisfying

$$\cos\gamma_2 > (1 + 2\eta)\alpha \geq (1 + 2\eta)\beta(x, 0)$$

for all $x \in \mathbb{R}$. Moreover, the continuity of β implies the existence of a τ such that

$$\cos\gamma_2 > (1 + \eta)\beta(x, y) \qquad \text{for all } x \in \mathbb{R} \text{ and } y \in [0, \tau],$$

so there exists γ such that

$$\gamma > \gamma_2 \qquad \text{and} \qquad \cos\gamma > \beta(x, y) \qquad \text{for all } x \in \mathbb{R} \text{ and } y \in [0, \tau].$$

For any $x_o \in \mathbb{R}$, let $M_o = (x_o, \tau, 0)$ and consider a system of local coordinates (x', y', z') such that $x' = x - x_o$ and the $x', y'-$plane is inclined at angle γ with respect to the support plane Π. The surface $\mathcal{S}'$ defined by

$$z' = -\frac{1 + 7\eta}{\rho_o} y'^2 + \frac{\eta}{\rho_o} x'^2,$$

has mean curvature close to $-\frac{1+6\eta}{\rho_o}$ in a neigborhood of M_o, and intersect Π with an angle γ. Since $\gamma > \gamma_2$, for τ small enough, $\mathcal{S}$, $\mathcal{S}'$ and Π enclose a small region $\mathcal{U}$, in a neighborhhod of M_o. Proceeding as in the previous section, we have:

$$\frac{1 + 4\eta}{\rho_o} |E_{\varepsilon,\kappa} \cap \mathcal{U}| \leq \int_{\mathcal{U} \cap E_{\varepsilon,\kappa}} \Delta d = \int_{\partial(\mathcal{U} \cap E_{\varepsilon,\kappa})} \nabla d \cdot \nu.$$

We notice that $\nabla d \cdot \nu = -1$ on $\partial\mathcal{U} \cap E_{\varepsilon,\kappa} = \mathcal{S}' \cap E_{\varepsilon,\kappa}$, and $0 \leq \nabla d \cdot \nu \leq 1$ on $\mathcal{U} \cap \partial E_{\varepsilon,\kappa}$. Moreover, it is easy to check that $d \sim (x' \sin\gamma + y' \cos\gamma)$ in the neighborhood of M_o, and so

$$\nabla d \cdot \nu \sim -\cos\gamma$$

along $\{z = 0\} \cap (\mathcal{U} \cap E_{\varepsilon,\kappa})$. We deduce

$$\int_{\mathcal{U} \cap E_{\varepsilon,\kappa}} \Delta d \leq -\mathscr{H}(\mathcal{S}' \cap E_{\varepsilon,\kappa}) + P(E_{\varepsilon,\kappa}, \mathcal{U}) - \cos\gamma \mathscr{H}(\partial E_{\varepsilon,\kappa} \cap \Pi \cap U).$$

Therefore, the set

$$F = E_{\varepsilon,\kappa} \setminus \mathcal{U}$$

satisfies

$$\begin{aligned} \mathcal{J}(F) &= \mathcal{J}(E_{\varepsilon,\kappa}) - P(E_{\varepsilon,\kappa}, \mathcal{U}) + \mathscr{H}(\partial\mathcal{U} \cap E_{\varepsilon,\kappa}) + \int_{\mathcal{U} \cap \{z=0\}} \beta\varphi_{E_{\varepsilon,\kappa}}\, dx\, dy \\ &\leq \mathcal{J}(E_{\varepsilon,\kappa}) - \frac{1 + 4\eta}{\rho_o} |\mathcal{U} \cap E_{\varepsilon,\kappa}| + \int_{\mathcal{U} \cap \{z=0\}} (\beta - \cos\gamma)\varphi_{E_{\varepsilon,\kappa}}\, dx\, dy \end{aligned}$$

Since $\cos\gamma > \beta(x,y)$ in $\mathcal{U}$, we deduce

$$\mathcal{J}(F) \quad \le \quad \mathcal{J}(E_{\varepsilon,\kappa}) - \frac{1+4\eta}{\rho_o}|\mathcal{U}\cap E_{\varepsilon,\kappa}|$$

The rest of the proof is similar to that of the previous section, adding a piece with volume $|E_{\varepsilon,\kappa}\cap\mathcal{U}|$ to $E_{\varepsilon,\kappa}$ in a region where the mean curvature is known. We deduce that

$$|E_{\varepsilon,\kappa}\cap\mathcal{U}| = 0,$$

and the proof is complete. □

6. Contact angle hysteresis

We now turn to the proof of Theorem 3.1. Throughout this section, we have

$$\Gamma_t = \{(x,y,z)\in\mathbb{R}^3\,;\, z>0\,,\; S(z)\le y\le 2\rho_o\sqrt{1-\langle\beta\rangle^2}-S(z)-t\},$$

and we denote by $E_{\varepsilon,t}$ the constrained minimizer for the gravity-free functional $\mathcal{J}_\varepsilon$, as constructed in section 4.

The proof of Theorem 3.1 follows the same steps as the proof of Theorem 3.2. The main difference is in Proposition 5.1, which becomes

Proposition 6.1. *The minimizer $E_{\varepsilon,t}$ of $\mathcal{J}_\varepsilon$ in $\mathscr{E}_{\Gamma_{R,T}}(V)$ satisfies*

$$\mathcal{J}_o(E_{\varepsilon,t}) \le \mathcal{J}_o(F) + C\varepsilon + Ct \tag{6.1}$$

for all $F\in\mathscr{E}(V)$

Therefore, Theorem 2.2 yields:

$$|E_{\varepsilon,t}\cap B^+_{\rho_o}| \le (C\varepsilon+Ct)^\alpha.$$

This proposition allows us to prove, following the same three steps as in the previous section, that $E_{\varepsilon,t}$ is indeed a local minimizer of $\mathcal{J}_\varepsilon$, that is

$$E_{\varepsilon,t}\Subset\Gamma_t$$

if $t<t_o$ and $\varepsilon<\varepsilon_o$.

Proof of Proposition 6.1. Let B^+_ρ be the spherical cap with volume V and wetting area of radius R_o-t. It is readily seen that $B^+_\rho\subset\Gamma_{R,T}$ and $|\rho-\rho_o|\le Ct$. So

$$\mathcal{J}_o(B^+_\rho)\le\mathcal{J}_o(B^+_{\rho_o})+Ct,$$

which implies

$$\min_{F\in\mathscr{E}_{\Gamma_{R,T}}}\mathcal{J}_\varepsilon(F)\le\mathcal{J}_\varepsilon(B^+_\rho)\le\mathcal{J}_o(B^+_{\rho_o})+C\varepsilon+Ct.$$

It remains to prove that

$$\mathcal{J}_o(E_{\varepsilon,t})\le\mathcal{J}_\varepsilon(E_{\varepsilon,t})+C\varepsilon,$$

but this is again a consequence of the following proposition, the proof of which is similar to that of Proposition 5.2:

Proposition 6.2. *The contact line has finite 1-dimensional hausdorff measure in* $\mathbb{R}^2$*:*

$$\mathscr{H}^1(\partial(E_{\varepsilon,t} \cap \{z=0\}) < +\infty$$

6.1. **Apparent contact angle.** It remains to show the last part of Theorem 3.1, that is we have to prove that the apparent contact angle is indeed greater than that of $B^+_{\rho_o}$, at least on part of the contact line.

We recall that the apparent contact angle is the measured contact angle when $\varepsilon \to 0$. To characterize it, we first prove the following result:

Lemma 6.3. *When* ε *goes to zero* $E_{\varepsilon,t}$ *converges in* L^1 *and uniformly to* $\widetilde{E}$*, minimizer of* $\mathcal{J}_o$ *in* Γ_t*.*

Proof. Note that $E_{\varepsilon,t}$ is bounded uniformly with respect to ε in BV-norm. Therefore, $E_{\varepsilon,t}$ converges in L^1 to a set $\widetilde{E} \in \mathscr{E}(V)$. Moreover

$$\mathcal{J}_o(\widetilde{E}) \leq \liminf_{\varepsilon\to 0} \mathcal{J}_o(E_{\varepsilon,t})$$

Finally, if F is a set in $\mathscr{E}(V)$ with $F \subset \Gamma_t$, then we have

$$\mathcal{J}_o(\widetilde{E}) \leq \mathcal{J}_\varepsilon(E_{\varepsilon,t}) + C\varepsilon \leq \mathcal{J}_\varepsilon(F) + C\varepsilon.$$

It follows that

$$\mathcal{J}_o(\widetilde{E}) \leq \liminf_{\varepsilon\to 0} (\mathcal{J}_\varepsilon(F) + C\varepsilon) = \mathcal{J}_o(F)$$

for all $F \subset \Gamma_t$, which prove that $\widetilde{E}$ is indeed the minimizer of $\mathcal{J}_o$ in Γ_t. Non-degeneracy inequalities for $E_{\varepsilon,t}$ then yields the uniform convergence. □

We deduce:

Lemma 6.4. *The free surface* $\partial\widetilde{E}$ *satisfies*

$$2\mathbf{H} = -\lambda$$
$$\cos\gamma \leq \langle\beta\rangle$$

Moreover, there exists at least one point along the contact line at which the contact angle is strictly greater than $\cos^{-1}\langle\beta\rangle$*.*

Proof. First of all, the previous lemma implies

$$\mathcal{J}_o(\widetilde{E}) \leq \mathcal{J}_o(F) + Ct$$

for all $F \in \mathscr{E}(V)$. In particular, Theorem 2.2 gives

$$|\widetilde{E}\Delta B^+_{\rho_o}| \leq Ct^\alpha$$

and following Step 1 and 2 of Section 5, we can prove that $\widetilde{E}$ does not touch the barrier $\partial\Gamma_t$, except maybe along the contact line. We also deduce that

$\partial\widetilde{E}$ has constant mean-curvature. Moreover, an argument similar to that of Step 3 yields $\cos\gamma \leq \langle\beta\rangle$ along the contact line.

Next, if $\cos\gamma = \langle\beta\rangle$ everywhere along the contact line, then H. Wente [W] proved that there is a vertical line about which $\widetilde{E}$ is axially symmetric. In particular, the wetting surface is a disk, and proceeding as in section 2.2.1, we easily prove that $\widetilde{E}$ is a spherical cap. However, since the volume is the same as $B^+_{\rho_o}$ but with a smaller wetting area, the contact angle must be larger, which contradicts $\cos\gamma = \langle\beta\rangle$. The lemma follows. □

Appendix A. Proof of Proposition 5.2

In this appendix, we give the main steps of the proof of Proposition 5.2. Details can be found in [CM] where similar results are proved for the gravity-free case.

The key results are the following non-degeneracy lemmas:

Lemma A.1. *Let $\Gamma_r(x_o, y_o)$ denote the cylinder*

$$\Gamma_r(x_o, y_o) = B^2_r(x_o) \times \mathbb{R} = \{(x, y, z) \in \mathbb{R}^{n+1}\,;\, |(x, y) - (x_o, y_o)| < r\}.$$

Then there exists c_1, $c_o > 0$ such that for any minimizer E of $\mathcal{J}$ in $\mathscr{E}_R(V)$, if x_o, y_o lies in the projection of E onto $\{z = 0\}$ (i.e. there exists z_o such that $(x_o, y_o, z_o) \in E$) then

$$|E \cap \Gamma_r(x_o)| > c_o r^3,$$

for all r such that $|E \cap \Gamma_r(x_o, y_o)| \leq c_1|E|$.

and

Lemma A.2. *Let $(x_o, z_o) \in \partial E$ with $z_o > 0$. There exists c, universal constant, such that for all $r \leq z_o$ we have*

$$|B_r(x_o, z_o) \cap E| \geq cr^3$$
$$|B_r(x_o, z_o) \setminus E| \geq cr^3$$

Lemma A.2 allows us to deduce uniform convergence from L^1 convergence for any sequence of minimizers.

Lemma A.1 gives the non-degeneracy of the minimizers on the contact line. It allows us to control the perimeter of E in the neighborhood of the contact line:

Corollary A.3. *If $(x_o, 0) \in \partial E$, then for every r,*

$$P(E, B^+_r(x_o, 0)) \geq r^2$$

Finally, by a cutting argument, we prove

Lemma A.4. *There exists a constant C such that*

$$P(E, \{0 < z < t\}) \leq C(V + V^{\frac{1}{3}})t$$

Proposition 5.2 now follws: Let $\cup_j B_\delta(x_j)$ be a covering of $\partial\{E\cap\{z=0\}\}$ with finite overlapping. Then by Corollary A.3, we have

$$P(E, B_\delta(x_j)) \geq \delta^2.$$

But thanks to the finite overlapping property,

$$\sum P(E, B_\delta(x_j)) \leq CP(E, \{0<z<\delta\}) \leq CV^{\frac{1}{3}}\delta,$$

and therefore the number of balls is less than $CV^{\frac{1}{3}}\delta$, hence the result. □

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