The thin film equation with non zero contact angle: A singular perturbation approach

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Abstract

In this paper we prove the existence of weak solutions for the thin film equation with prescribed non zero contact angle and for a large class of mobility coefficients, in dimension 1. The existence of weak solutions for this degenerate parabolic fourth order free boundary problem was proved by F. Otto in [35] when the mobility coefficient is given by $f(u) = u$, using a particular gradient flow formulation which does not seem to generalize to other mobility coefficients. Short time existence (and uniqueness) of strong solutions was recently proved by H. Knüpfer and N. Masmoudi in [33, 34] for $f(u) = u$ and $f(u) = u^2$ and for regular enough initial data (corresponding to a single droplet). In this paper, we use a different approach to prove the global in time existence of weak solutions without condition on the support, by using a diffuse approximation of the free boundary condition. This approach, which can be physically motivated by the introduction of singular disjoining/conjoining pressure forces has been suggested in particular by Bertsch, Giacomelli and Karali in [11]. Our main result is the existence of some weak solutions for the free boundary problem when the mobility coefficient satisfies $f(u) \sim u^n$ as $u \to 0$ for some $n \in [1, 2)$.

1 Introduction

1.1 Regularization of a free boundary problem for capillary surfaces

In this paper, we consider a free boundary problem modeling the motion of thin viscous liquid droplets on a solid surface. Before discussing the evolution equation that we will be studying, let us briefly discuss the equilibrium case, which is considerably simpler. Though the results presented here will only be

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valid in dimension 1, we will discuss the model in any dimension d (the physically relevant cases correspond to $d = 1$ or 2).

Equilibrium capillary surfaces. Consider a small liquid droplet lying on a flat solid surface. We will always assume that the drop can be described as the set

$$E = \{(x, z) \in \mathbb{R}^d \times (0, \infty)\, ;\, 0 < z < u(x)\}$$

for some function $u : \mathbb{R}^d \to [0, \infty)$. The graph of the function $x \mapsto u(x)$ is the free surface of the drop (liquid/air interface), while the support of u is the wetted region (liquid/solid interface). The boundary of the support of u, $\partial\{u > 0\}$, is known as the contact line (this is the triple junction where air/liquid/solid meet). At equilibrium, the shape of the drop is determined by minimizing its energy, which, neglecting gravitational and other body forces, reads

$$\sigma \int_{\{u>0\}} \sqrt{1 + |\nabla u|^2}\, dx - \sigma\beta \int_{\mathbb{R}^d} \chi_{\{u>0\}}\, dx.$$

The first term is the surface tension energy, proportional to the area of the free surface (σ is the surface tension coefficient) and the second term is the wetting energy, proportional to the area of the wetted area ($\beta \in (0, 1)$ is the relative adhesion coefficient). The resulting minimization problem leads to an elliptic free boundary problem involving the mean-curvature operator which has been well studied (see [13, 14, 16, 23] and reference therein).

In the framework of the lubrication approximation, the droplet is assumed to be very thin, so that we can use the approximation

$$\sqrt{1 + |\nabla u|^2} \sim 1 + \frac{1}{2}|\nabla u|^2.$$

Taking $\sigma = 1$ (without loss of generality), the energy of a droplet now reads:

$$\mathscr{J}(u) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u|^2 + (1 - \beta)\chi_{\{u>0\}}\, dx$$

and minimizers of $\mathscr{J}$ (with a volume constraint) are solutions of the following free boundary problem:

$$\begin{cases} \Delta u = -\lambda & \text{in } \{u > 0\} \\ \dfrac{1}{2}|\nabla u|^2 = 1 - \beta & \text{on } \partial\{u > 0\}, \end{cases} \tag{1}$$

where λ is a Lagrange multiplier.

The mathematical analysis of this free boundary problem goes back to the work of Alt and Caffarelli [1] (see also Caffarelli-Friedman [16] for results involving the mean-curvature operator). In particular, it is known that the solutions of (1) are Lipschitz (optimal regularity) and in dimension $d = 2$ and 3, the free boundary $\partial\{u > 0\}$ of the minimizers of $\mathscr{J}$ is smooth (see [1, 17]).

The lubrication approximation. Studying the motion of a liquid drop is a considerably more difficult problem, mainly because one has to take into account the motion of the liquid inside the drop. The lubrication approximation allows to greatly simplify this problem. This approximation is valid for very thin drops of very viscous liquid, and it consists of a depth-averaged equation of mass conservation and a simplified Navier-Stokes equations (see [9, 25] for the derivation of the equation). The evolution of the height of the drop $u(x,t)$ is then described by the so-called thin film equation:

$$\partial_t u + \mathrm{div}_x(f(u)\nabla\Delta u) = 0 \qquad \text{in } \{u > 0\} \tag{2}$$

where the mobility coefficient $f(u)$ is typically given by $f(u) = u^3$ or $f(u) = u^3 + \lambda u^s$, $s \in [1,2]$ depending on the type of boundary conditions imposed on the fluid velocity at the contact with the solid support. Note that the case $f(u) = u$ is also important (it corresponds to the evolution of a thin film at the edge of a Hele-Shaw cell, see [24]), and that the case $f(u) = u^3$ is known to be critical since the motion of the contact line in that case would lead to infinite energy dissipation (see [32])

Equation (2) is a fourth order degenerate parabolic equation often studied when the mobility coefficient is of the form $f(u) = u^n$, $n > 0$. For such mobility coefficients, (2) looks like the porous media equation. However, because it is of order four, none of the classical techniques (which often involve the maximum principle) can be used. Early results were restricted to the one-dimensional case ($\Omega = (0,1)$). In particular Bernis and Friedman [6] proved the existence of non-negative weak solutions for $f(u) = u^n$, $n > 1$. The result was later improved to include $n > 0$ and several regularity results and qualitative properties were established (see in particular [3, 26, 8, 4, 5]). More recently, much of the theory was extended to higher dimensions (see in particular [20, 21, 10, 27, 28, 29] and references therein).

In all of the works mentioned above, equation (2) is assumed to be satisfied on the whole set Ω rather than only in the support of u, and the equation is supplemented with boundary conditions on $\partial\Omega$:

$$\begin{aligned} &f(u)\nabla\Delta u\cdot n = 0 && \text{(null-flux)} \\ &\nabla u \cdot n = 0 && \text{(contact angle condition on } \partial\Omega). \end{aligned}$$

The existence of solutions is then proved via a regularization approach (for instance by replacing $f(u)$ by $f(u) + \varepsilon$ and then passing to the limit $\varepsilon \to 0$). In particular, the boundary of the support $\partial\{u > 0\}$ (contact line) plays no particular role in that construction. It can nevertheless be shown that compactly supported initial data lead to compactly supported solutions (finite speed of propagation of the support, see [4, 5, 27, 28]) so that we recover the existence of a contact line. Furthermore, in dimension $d = 1$ and for $n \in (0,3)$ it has been shown that the solutions constructed by this method are $\mathcal{C}^1$, and therefore satisfy

$$|u_x| = 0 \qquad \text{on } \partial\{u > 0\}.$$

This zero contact angle condition should be compared with the free boundary condition in (1): It correspond to $\beta = 1$ (hydrophilic support), and is usually referred to as the **complete wetting regime** (note that no stationary solution can exist in that case).

This contact angle condition is obtained as a consequence of the regularization method used to construct the solution, rather than as a conscious choice of a free boundary condition (this is similar to what is usually done with the porous media equation). Our goal in this paper is to prove the existence of solutions to the free boundary problem corresponding to the thin film equation (2) with non-zero contact angle condition (as in (1)), in dimension $d = 1$. In other words, we consider the following free boundary problem (we take $\beta = 0$ from now on for the sake of simplicity):

$$\begin{cases} \partial_t u + \partial_x(f(u)\partial_{xxx}u) = 0 & \text{in } \{u > 0\} \\ f(u)\partial_{xxx}u = 0 & \text{on } \partial\{u > 0\} \\ \frac{1}{2}|u_x|^2 = 1 & \text{on } \partial\{u > 0\}. \end{cases} \tag{3}$$

Note that the first free boundary condition imposes null-flux at the contact line, and thus ensures the preservation of the total volume of the drop. The fact that we need an additional boundary condition on $\partial\{u > 0\}$ compared with (1) is natural since we have a fourth order operator. As usual with null-flux conditions, it will be enforced in some weak sense as a consequence of the integral formulation of the equation.

The existence of solutions for (3) is clearly a difficult problem. Short time existence (and uniqueness) of classical solutions is proved by H. Knüpfer [33] for $f(u) = u^2$ and by H. Knüpfer and N. Masmoudi [34] for $f(u) = u$. These results hold for initial data that are in the form of a single droplet (simply connected support) and they do not describe the splitting and merging of droplets. To our knowledge, the only known long time existence result for weak solutions was obtained by F. Otto [35] in the case $f(u) = u$. The proof of this result relies on the fact that when $f(u) = u$, (3) is the gradient flow for the energy

$$\mathscr{J}_0(u) = \int \frac{1}{2}|u_x|^2 + \chi_{u>0}\, dx$$

with respect to the Wasserstein distance. Unfortunately, this particular structure is limited to the case $f(u) = u$ and it does not seem possible to extend this approach to more general mobility coefficients (see [22] for some extension of this framework to the case $f(u) = u^n$, $0 < n < 1$).

In this paper, we prove the existence of weak solutions for this free boundary problem for mobility coefficient satisfying

$$f(u) \sim u^n \qquad \text{as } u \to 0 \text{ for some } n \in [1, 2).$$

Our result thus includes the classical case where f is given by

$$f(u) = u^3 + \Lambda u^s, \qquad s \in [1, 2). \tag{4}$$

Conjoining/Disjoining pressure. To prove the existence of a solution in this case, we consider a different approach, which relies on the regularization of the energy $\mathscr{J}$ (and of the free boundary problem (1)) and which can be physically motivated by the introduction of microscopic scale forces in the form of long range Van der Waals interactions between the liquid and solid surfaces. This approach was suggested in particular by Bertsch, Giacomelli and Karali in [11] who also derive some properties of the singular limit.

From a mathematical point of view, the idea is simply to consider a regularized energy functional $\mathscr{J}_\varepsilon$ defined by

$$\mathscr{J}_\varepsilon(u) = \int \frac{1}{2}|u_x|^2 + Q_\varepsilon(u)\,dx \tag{5}$$

where $u \mapsto Q_\varepsilon(u)$ is a smooth function which converges, as ε goes to zero, to $\chi_{\{u>0\}}$. This is a very classical approach for the study of the elliptic (and parabolic) free boundary problem (1) (see [2, 15, 18] for instance). The minimizers of $\mathscr{J}_\varepsilon$ (without volume constraint) solve the nonlinear equation

$$\partial_{xx}u = P_\varepsilon(u)$$

where $P_\varepsilon = Q'_\varepsilon$ converges to a Dirac mass at 0. The convergence of the solution of this equation to solutions of the free boundary problem (1) has been studied, in particular in [2].

The corresponding regularized thin film equation (3) reads

$$\partial_t u + \partial_x\Big(f(u)\partial_x[\partial_{xx}u - P_\varepsilon(u)]\Big) = 0 \tag{6}$$

and is supplemented with boundary conditions on $\partial\Omega$:

$$\begin{array}{ll} f(u)\partial_x[\partial_{xx}u - P_\varepsilon(u)] = 0 & \text{(null-flux)} \\ u_x = 0 & \text{(contact angle condition on } \partial\Omega). \end{array} \tag{7}$$

Multiplying the equation by $[\partial_{xx}u - P_\varepsilon(u)]$, it is easy to check that smooth solutions of (6) satisfy

$$\frac{d}{dt}\mathscr{J}_\varepsilon(u) + \int_\Omega f(u)\big[\partial_x(\partial_{xx}u - P_\varepsilon(u))\big]^2\,dx = 0. \tag{8}$$

From a physical point of view, equation (6) corresponds to the classical lubrication approximation equation, when the pressure at the free surface of the drop, rather than being given by the surface tension alone (which is proportional to $-\partial_{xx}u$), is given by

$$\Pi(u) = -\partial_{xx}u + P_\varepsilon(u). \tag{9}$$

The additional pressure term $P_\varepsilon(u)$ can be interpreted as modeling the effects of disjoining/conjoining intermolecular forces due to the interactions of the fluid molecules with the solid support. The inclusion of such forces has been proposed by several authors to describe the precursor film phenomena.

In the literature, P_ε is often a Lennard-Jones type potential with a singularity at 0 (see in particular [30], [7] for a mathematical analysis of the resulting model for fixed $\varepsilon > 0$). For technical reasons however, we consider here only bounded pressure term. More precisely, we assume that P is a given continuous function satisfying

$$P(u) > 0 \text{ for } u \in (0,1), \quad P(u) = 0 \text{ for } u \geq 1, \qquad \int_0^\infty P(u)\, du = 1. \tag{10}$$

and Q is given by

$$Q(u) = \int_0^u P(u)\, du. \tag{11}$$

We then define

$$P_\varepsilon(u) = \frac{1}{\varepsilon} P(u/\varepsilon), \quad Q_\varepsilon(u) = Q(u/\varepsilon). \tag{12}$$

In particular, it is easy to check that $Q_\varepsilon(u) \geq 0$ for all u, $Q_\varepsilon(u) = 1$ for $u \geq \varepsilon$ and $Q_\varepsilon(u) \to \chi_{\{u>0\}}$ as $\varepsilon \to 0$.

Note that the last two conditions in (10) are by no mean necessary. In fact, these conditions could easily be replaced by

$$0 \leq P(u) \leq \frac{C}{u^q} \text{ for all } u \geq 1, \text{ for some } q > 1, \qquad \int_0^\infty P(u)\, du = M > 0$$

but we choose to consider slightly more restrictive assumptions in order to simplify the analysis.

The rest of the paper is organized as follows: In the next section, we give some results concerning the limit $\varepsilon \to 0$ for stationary solutions of (6) (first for energy minimizers, and then for general stationary solutions). In Section 3, we give our main result concerning the behavior of the limit $\varepsilon \to 0$ of the solutions of the singular thin film equation (6). The results for stationary solutions are then proved in Section 4, while our main result (the convergence of solutions of (6) to solutions of the free boundary problem) is proved in Section 5.

2 Stationary solutions - Main results

2.1 Energy minimizers

We first consider energy minimizers, that is solutions of

$$\mathscr{J}_\varepsilon(u^\varepsilon) = \min\{\mathscr{J}_\varepsilon(v)\,;\, v \geq 0 \text{ in } \Omega, \int_\Omega v(x)\, dx = V\} \tag{13}$$

where $\mathscr{J}_\varepsilon$ is given by (5). Note that using the function $v_0(x) = \frac{V}{|\Omega|}$ (which is a local minimizer of $\mathscr{J}_\varepsilon$), we get

$$\mathscr{J}_\varepsilon(u^\varepsilon) \leq \mathscr{J}_\varepsilon(v_0) \leq |\Omega|. \tag{14}$$

Equation (13) is an obstacle problem with volume constraint. Using classical arguments (see [12] for instance), one can show that u^ε satisfies

$$u^{\varepsilon\prime\prime} \leq P_\varepsilon(u^\varepsilon) - \lambda^\varepsilon \quad \text{ in } \Omega \tag{15}$$

and

$$u^{\varepsilon\prime\prime} = P_\varepsilon(u^\varepsilon) - \lambda^\varepsilon \quad \text{ in } \{u^\varepsilon > 0\} \tag{16}$$

for some constant λ^ε (which is the Lagrange multiplier for the volume constraint). Furthermore, u^ε satisfies the usual free boundary condition (see [12])

$$u^\varepsilon = |u^{\varepsilon\prime}| = 0 \quad \text{ on } \partial\{u^\varepsilon > 0\} \cap \Omega \tag{17}$$

and the Neumann boundary condition on $\partial\Omega$:

$$u^{\varepsilon\prime} = 0 \quad \text{ on } \partial\Omega. \tag{18}$$

We are going to show:

Proposition 2.1. *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a sequence of solutions of* (15)-(18) *satisfying* (14) *and*

$$\int_\Omega u^\varepsilon \, dx = V$$

for some fixed volume $V > 0$. Then, the corresponding constants λ^ε are bounded (uniformly in ε) and up to a subsequence, u^ε converges uniformly and $H^1(\Omega)-$strong to u solution of

$$\begin{cases} u'' = -\lambda \text{ in } \{u > 0\}, \\ \frac{1}{2}|u'|^2 = 1 \text{ on } \partial\{u > 0\} \cap \Omega \end{cases} \tag{19}$$

satisfying the following boundary condition for $x \in \partial\Omega$:

$$u'(x) = 0 \text{ if } u(x) > 0, \quad \frac{1}{2}|u'(x)|^2 = 1 \text{ if } u(x) = 0.$$

The condition (17) can be interpreted as a microscopic contact angle condition (with zero contact angle), while (19) gives the macroscopic (or apparent) contact angle condition (which is not zero).

If we fix λ^ε in (15)-(18) (and let the volume vary), then the problem reduces to the well-known (and well studied) Bernouilli problem, for which Proposition 2.1 is a classical result (see [2, 15]). So once we have proved that λ^ε is bounded uniformly with respect to ε, Proposition 2.1 can be proved using classical arguments. We will nevertheless give a complete proof of this result in Section 4.1 because parts of this proof will be used later on (and it introduces some of the ideas that will be used for the other results).

Let us briefly sketch this proof: Multiplying (16) by $u^{\varepsilon\prime}$ and integrating, one finds that the function

$$G^\varepsilon = Q_\varepsilon(u^\varepsilon) - \frac{1}{2}|u^{\varepsilon\prime}|^2 - u^\varepsilon \lambda^\varepsilon$$

must be constant throughout each connected component of $\{u^\varepsilon > 0\}$. If we have $\{u^\varepsilon > 0\} \neq \Omega$ (unfortunately, we will see that this might not be true), then we can take $x_0 \in \partial\{u^\varepsilon > 0\} \cap \Omega$, and using (17) we deduce

$$G^\varepsilon(x) = G^\varepsilon(x_0) = 0 \qquad \text{for all } x \in \{u > 0\}. \tag{20}$$

We will also prove that:

(i) λ^ε is bounded

(ii) u^ε converges strongly in $H^1(\Omega)$,

so that we can pass to the limit in G^ε and obtain

$$1 - \frac{1}{2}|u'|^2 - u\lambda = 0 \text{ in } \{u > 0\},$$

which gives the free boundary condition in (19).

The main difficulty in the proof will be to deal with the case where $\{u^\varepsilon > 0\} = \Omega$ in which case we do not have (20), and we need to show that $G^\varepsilon \to 0$ as $\varepsilon \to 0$. This function G^ε, which plays a crucial role in this proof, as well as in the rest of the paper can also be written as (using (16)):

$$G^\varepsilon(x) = Q_\varepsilon(u^\varepsilon) - \frac{1}{2}(u^{\varepsilon\prime})^2 + u^\varepsilon u^{\varepsilon\prime\prime} - u^\varepsilon P_\varepsilon(u_\varepsilon).$$

In the next section, we discuss the behavior of general stationary solutions which may not be energy minimizers. In this case (and in the evolution case), the main additional difficulty is that neither (i) or (ii) above (the bound of λ^ε and strong convergence of u^ε_x) can be expected to hold.

2.2 General stationary solutions

General solutions of (16)-(18) (which may not be energy minimizers) have been studied in [36] (for fixed $\varepsilon > 0$). It is easy to show that many of the solutions constructed there will not satisfy the expected contact angle condition in the limit $\varepsilon \to 0$. Some of these solutions are depicted in Figure 1 and correspond to ripples. However we will see that such undesirable solutions must cover all of Ω and can thus be discarded with conditions on the support of the drop (for instance by taking Ω large, so that (14) holds). They will nevertheless play an important role in the evolution case in the next section.

But even general solutions of (16)-(17) do not cover all physically relevant configurations. Indeed, when the support of the drop has several connected components, (16) imposes the Lagrange multiplier to be the same on each component. This implies that the droplet can only split into smaller droplets with all the same volume. This is too restrictive, and so we define

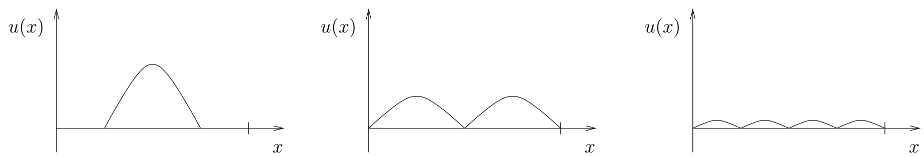

Figure 1: Example of limit of solutions of (16)-(17). Only the first one (global minimizer of $\mathscr{J}_\varepsilon$) satisfies desired contact angle condition.

Definition 2.2. *A continuously differentiable function $u : \Omega \to [0, \infty]$ is a stationary solution if $\mathscr{J}_\varepsilon(u) < \infty$ and for any (a_i, b_i) connected component of $\{u > 0\}$, we have $u \in \mathcal{C}^2((a_i, b_i))$ and there exists a constant λ_i such that*

$$u'' = P_\varepsilon(u) - \lambda_i, \qquad \text{in } (a_i, b_i) \tag{21}$$

and

$$u' = 0 \qquad \text{on } \partial(a_i, b_i) \tag{22}$$

(in particular $u' = 0$ on $\partial\Omega$).

In particular, one can show that if $u : \Omega \to [0, \infty]$ is a continuous function such that $\mathscr{J}_\varepsilon(u) < \infty$ and

$$\int_\Omega u^n [(u'' - P_\varepsilon(u))']^2 \, dx = 0 \tag{23}$$

(we recognize the dissipation of energy for the thin film equation) for some $n \in (0, 3)$ then u is a stationary solution in the sense of Definition 2.2. Indeed (23) implies (21) immediately, and it also implies

$$\int_\Omega u^n [u''']^2 \, dx \le \int_\Omega u^n P'_\varepsilon(u)[u']^2 \, dx \le C \mathscr{J}_\varepsilon(u)$$

which implies (together with the Neumann condition on $\partial\Omega$) that $u \in C^{1,\alpha}(\Omega)$ for some $\alpha > 0$ (depending on n). Hence (22) holds.

Now, given a sequence $\{u^\varepsilon\}_{\varepsilon>0}$ of stationary solutions, we want to pass to the limit $\varepsilon \to 0$. This is more difficult than in Proposition 2.1, because we cannot expect the λ_i^ε to be bounded (the drop may split up into smaller and smaller droplets as ε goes to zero), and in turn, we may not have the strong convergence of u^ε in $H^1(\Omega)$.

Nevertheless, we will show:

Theorem 2.3. *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a sequence of stationary solution such that*

$$\int_\Omega u^\varepsilon \, dx = V$$

and

$$\mathscr{J}_\varepsilon(u^\varepsilon) \le |\Omega|. \tag{24}$$

Then, up to a subsequence, u^ε converges uniformly to a continuous function u satisfying $\int_\Omega u\,dx = V$,

$$\int_\Omega \frac{1}{2}|u'|^2\,dx + |\{u>0\}| \leq \liminf_{\varepsilon\to 0} \mathscr{J}_\varepsilon(u^\varepsilon) \tag{25}$$

and solution of

$$\begin{cases} u'' = -\lambda_i & \textit{constant on each connected component } (a_i,b_i) \textit{ of } \{u>0\} \\ \frac{1}{2}|u'|^2 = 1 & \textit{on } \partial\{u>0\}\cap\Omega \end{cases} \tag{26}$$

satisfying the following condition on $\partial\Omega$:

$$u'(x) = 0 \textit{ if } u(x)>0, \quad \frac{1}{2}|u'(x)|^2 = 1 \textit{ if } u(x) = 0.$$

Note that (24) is used here to discard the undesirable solutions mentioned above (it is not very restrictive since we can assume Ω to be very large). The proof of Theorem 2.3 will rely on the function

$$G^\varepsilon(x) = Q_\varepsilon(u^\varepsilon) - \frac{1}{2}(u^{\varepsilon\prime})^2 + u^\varepsilon u^{\varepsilon\prime\prime} - u^\varepsilon P_\varepsilon(u_\varepsilon).$$

which will be proved to be constant in Ω. The main difficulty (compared with the previous section) is that it is not possible to pass to the limit in G^ε due to the lack of compactness in H^1.

3 The thin film equation - Main results

We now consider the regularized thin film equation:

$$\begin{cases} \partial_t u + \partial_x(f(u)\partial_x[\partial_{xx}u - P_\varepsilon(u)]) = 0 & \text{for } x\in\Omega,\ t>0 \\ f(u)\partial_x[\partial_{xx}u - P_\varepsilon(u)] = 0, \qquad u_x = 0 & \text{for } x\in\partial\Omega,\ t>0 \\ u(x,0) = u_0(x) & \text{for } x\in\Omega \end{cases} \tag{27}$$

where u_0 is a non-negative function in $H^1(\Omega)$ and the mobility coefficient $f(u)$ is a smooth function satisfying

$$f(u)>0 \text{ for } u>0 \text{ and } f(u)\sim u^n \text{ as } u\to 0 \text{ for some } n\in[1,2). \tag{28}$$

This includes in particular the physical case $f(u) = u^3 + u^n$ with $n\in[1,2)$.

For fixed $\varepsilon>0$, the function P_ε is a smooth bounded function for $u\geq 0$, so the existence of a non-negative solution for (27) is a classical problem (see for instance [6, 8, 11]). Our starting point is thus the following theorem:

Theorem 3.1. *Assume that* (28) *holds. Then for any* $\varepsilon > 0$ *and any non-negative* u_0 *such that* $\mathscr{J}_0(u_0) < \infty$*, there exists a non-negative function* $u^\varepsilon \in \mathcal{C}^{\frac{1}{8},\frac{1}{2}}([0,\infty) \times \overline{\Omega})$ *such that* $u^\varepsilon \in \mathcal{C}^{1,4}(\{u^\varepsilon > 0\})$ *and for all* $T > 0$*:*

$$u^\varepsilon \in L^\infty(0,\infty; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)), \qquad \sqrt{f(u^\varepsilon)}[u^\varepsilon_{xxx}] \in L^2(\{u^\varepsilon > 0\})$$

and satisfying, for all $\varphi \in \mathcal{D}((0,\infty) \times \overline{\Omega})$*:*

$$\begin{cases} \displaystyle\int_0^\infty \int_\Omega u^\varepsilon \varphi_t \, dx + \int_0^\infty \int_{\{u^\varepsilon(t)>0\}} f(u^\varepsilon)[u^\varepsilon_{xx} - P_\varepsilon(u^\varepsilon)]_x \varphi_x \, dx \, dt = 0 \\ u^\varepsilon(x,0) = u_0(x). \end{cases} \tag{29}$$

The Neumann boundary condition is satisfied in the sense that

$$u^\varepsilon_x \in L^2(0,T; H^1_0(\Omega)) \qquad \text{for all } T > 0 \tag{30}$$

and u^ε *satisfies the mass conservation*

$$\int_\Omega u^\varepsilon(x,t)\, dx = \int_\Omega u^\varepsilon_0(x)\, dx \qquad a.e.\ t \geq 0, \tag{31}$$

and the energy inequality:

$$\mathscr{J}_\varepsilon(u^\varepsilon(t)) + \int_0^t \int_\Omega |g^\varepsilon(x,s)|^2 \, dx \, ds \leq \mathscr{J}_\varepsilon(u_0) \qquad a.e.\ t \geq 0 \tag{32}$$

with

$$\mathscr{J}_\varepsilon(u) = \int_\Omega \frac{1}{2} u_x^2 + Q_\varepsilon(u)\, dx.$$

and for some function g^ε *satisfying* $g^\varepsilon = \sqrt{f(u^\varepsilon)}\left[(u^\varepsilon_{xx} - P_\varepsilon(u^\varepsilon))_x\right]$ *on* $\{u^\varepsilon > 0\}$*.*

Furthermore, the function

$$G^\varepsilon = Q_\varepsilon(u^\varepsilon) - \frac{1}{2}(u^\varepsilon_x)^2 + u^\varepsilon u^\varepsilon_{xx} - u^\varepsilon P_\varepsilon(u^\varepsilon) \tag{33}$$

belongs to $L^2(0,\infty; H^1(\Omega))$ *and satisfies*

$$|G^\varepsilon_x(x,t)| \leq |u^\varepsilon(x,t)|^{\frac{n-2}{2}} g^\varepsilon(x,t) \qquad a.e.\ in\ (0,\infty) \times \Omega \tag{34}$$

For the sake of completeness (and because (34) is a straightforward, but not classical inequality), we recall the main steps of the proof of this theorem in Appendix A. Note that equation (27) also holds in the following stronger sense:

$$\int_0^\infty \int_\Omega u^\varepsilon \varphi_t \, dx - \int_0^\infty \int_\Omega f'(u^\varepsilon) u^\varepsilon_x [u^\varepsilon_{xx} - P_\varepsilon(u^\varepsilon)] \varphi_x \, dx \, dt \\ - \int_0^\infty \int_\Omega f(u^\varepsilon)[u^\varepsilon_{xx} - P_\varepsilon(u^\varepsilon)] \varphi_{xx} \, dx \, dt = 0.$$

We can also prove the following result which justifies the notion of stationary solution introduced in the previous section:

Proposition 3.2. *Let u^ε be a solution of* (27) *given by Theorem 3.1. Then there exists a sequence $t_k \to \infty$ such that $u^\varepsilon(x,t_k) \to u^\varepsilon_\infty(x)$ uniformly in Ω where $u^\varepsilon_\infty(x)$ is a stationary solution in the sense of Definition 2.2.*

We include the proof of this proposition in Appendix A.

The main result of this paper is then the following theorem:

Theorem 3.3. *Assume that* (28) *holds and let u^ε be a solution of* (27) *given by Theorem 3.1. Then up to a subsequence, u^ε converges locally uniformly to a function $u \in \mathcal{C}^{\frac{1}{8},\frac{1}{2}}([0,\infty)\times\overline{\Omega})$ satisfying $u(x,0) = u_0(x)$,*

$$u_{xxx} \in L^2_{loc}(\{u>0\}),\ \sqrt{f(u)}u_{xxx} \in L^2(\{u>0\}). \tag{35}$$

Furthermore, u solves the free boundary problem (3) *in the following sense:*

(a) For every test function $\varphi \in \mathcal{D}((0,\infty)\times\overline{\Omega})$

$$\int_0^\infty \int_\Omega u\,\varphi_t\,dx\,dt + \int_0^\infty \int_{\{u(t)>0\}} f(u)u_{xxx}\,\varphi_x\,dx\,dt = 0 \tag{36}$$

(note that this weak formulation implicitly includes the null-flux free boundary condition on $\partial\{u>0\}$).

(b) For all $t>0$, we have

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx. \tag{37}$$

(c) For almost every $t>0$, $x \mapsto u(x,t)$ is a Lipschitz function, and there exists a open set $\mathcal{U}(t)$ such that $\{u(\cdot,t)>0\} \subset \mathcal{U}(t)$ and

(i) For all $a \in \partial\{u(\cdot,t)>0\}$,

$$\frac{1}{2}u_x^2(a^\pm,t) \le 1. \tag{38}$$

(ii) If (a,b) is a connected component of $\{u(\cdot,t)>0\}$ and $a \in \partial\mathcal{U}(t)\cap\Omega$, then

$$\frac{1}{2}u_x^2(a^+,t) = 1 \tag{39}$$

(and similarly with b).

(iii) The following energy inequality holds:

$$\int_\Omega \frac{1}{2}u_x^2(x,t)\,dx + |\mathcal{U}(t)| + \int_0^t \int_{\{u(t)>0\}} f(u)(u_{xxx})^2\,dx\,ds \le \mathscr{J}_0(u_0). \tag{40}$$

The most important part of this result is the last point (c) which requires a little discussion. First, we note that the Lipschitz regularity with respect to x is optimal in view of the non-zero contact angle condition (though the Lipschitz constant is not bounded uniformly in time). Then, (i) claims that u is a supersolution for the free boundary condition at any points of the contact line, while (ii) states that u satisfies the expected free boundary condition only at the boundary points of a set $\mathcal{U}(t)$ which might be bigger than the support of u.

The introduction of such a set $\mathcal{U}(t)$ is fairly classical in such problems. A possible interpretation for the necessity of this set is that when the drop splits (i.e. when u vanishes somewhere in Ω), it may not split up cleanly but instead leave a very thin film behind (corresponding to the complete wetting regime). In fact, we will also prove the following result (which follows from Proposition 5.5 (ii)):

Proposition 3.4. *For almost every $t > 0$, and for any interval $[c, d] \subset \mathcal{U}(t)$ such that $u(x, t) = 0$ for all $x \in [c, d]$, we have*

$$u^\varepsilon(x, t) \geq \kappa\varepsilon \qquad \text{for all } x \in [c, d]$$

for some $\kappa > 0$ and for all ε along some subsequence $\varepsilon_k \to 0$.

This result suggests that the set $\mathcal{U}(t)$ (rather than the support of u) truly describes the "wetted" region.

A similar issue arises in [35] with the thin film equation when $f(u) = u$, and in [31], where the motion of liquid drops is studied in the framework of the quasi-static approximation regime (see also [19]). However it should be noted that the solutions constructed by F. Otto in [35] are stronger since they satisfy $u(\cdot, t) \in C^{1,2/3}(\mathcal{U}(t))$. This implies in particular that if u vanishes in $\mathcal{U}$, it does so with a zero contact angle $u_x = 0$. In our case, we only obtain the inequality (38) instead. A possible interpretation is that while in [35] the "very thin" film left behind by the splitting droplet is always smooth, in our approach we could not eliminate the possibility of some "very thin" ripples, which correspond exactly to the undesirable stationary solutions discussed in Section 2 (note that these stationary solutions are not local energy minimizers). In order to further characterize this behavior, we can prove the following result:

Proposition 3.5. *For almost every $t > 0$, the following holds: Let $a \in \Omega$ be such that $u(\cdot, t) = 0$ in $[a - \delta, a]$ and $u(\cdot, t) > 0$ in $(a, a + \delta)$ for some small δ. Then,*

(i) if $0 < \frac{1}{2}u_x^2(a^+, t) < 1$, then

$$\lim_{\varepsilon \to 0} \int_I |u_x^\varepsilon(x, t)|^2 \, dx > \kappa |I|$$

for some $\kappa > 0$ (depending on $\frac{1}{2}u_x^2(a^+, t)$) and for all interval $I \subset [a - \delta, a]$.

(ii) if

$$\frac{d}{dt} \int_0^{x_0} u^\varepsilon(x, t) \, dx \geq 0 \quad \text{for all } x_0 \in (a - \delta, a + \delta) \tag{41}$$

for all ε along an appropriate subsequence, then

$$\text{either } \tfrac{1}{2}u_x^2(a^+,t) = 0 \text{ or } \tfrac{1}{2}u_x^2(a^+,t) = 1.$$

The first part of this proposition says that the undesirable contact angle values, $\frac{1}{2}u_x^2(a^+,t) \in (0,1)$, can only appear when u^ε goes to zero with a lot of oscillations (in particular u^ε does not converge strongly in H^1 in that case). The second part says that if the quantity of liquid to the left of the free boundary point is increasing, then we must have either complete wetting regime ($\frac{1}{2}u_x^2(a^+,t) = 0$) or the expected contact angle ($\frac{1}{2}u_x^2(a^+,t) = 1$). This condition on the quantity of liquid implies in particular that the contact line is moving to the left. In other words, this result states (in a weak sense) that the undesirable contact angle values can only be the result of a de-wetting process (so the wetting process is a clean process, but the de-wetting process can be messy).

Finally, we note that if $\mathscr{J}_0(u_0) < |\Omega|$ (which for compactly supported initial data can always be assumed by taking Ω large enough), then (40) implies that $\mathcal{U}(t) \neq \Omega$, so u will have at least one free boundary point with the right contact angle.

The proof of Theorem 3.3 is detailed in Section 5. It relies on the function

$$G^\varepsilon(x,t) = Q_\varepsilon(u^\varepsilon) - \frac{1}{2}(u_x^\varepsilon)^2 + u^\varepsilon u_{xx}^\varepsilon - u^\varepsilon P_\varepsilon(u_\varepsilon),$$

introduced in Theorem 3.1. In the stationary case, this function was shown to be constant throughout Ω. This is not the case here, but inequality (34) implies that for almost every t, $x \mapsto G^\varepsilon(x,t)$ converges uniformly to a continuous function $x \mapsto G^0(x,t)$ which is constant on each connect component of $\{u = 0\}$ and satisfies

$$G^0(x,t) = 1 - \frac{1}{2}(u_x)^2 + u u_{xx} \qquad \text{in } \{u > 0\}.$$

Formally, we thus expect to have

$$\frac{1}{2}(u_x)^2 = 1 - G^0(x,t) \qquad \text{on } \partial\{u > 0\},$$

so the value of $\frac{1}{2}(u_x)^2$ at the free boundary is related to the value of $G^0(x,t)$ on the zero set of u. In Proposition 5.5, we will show that $G^0 \in [0,1]$ in $\{u = 0\}$ (implying (i)) and that $\lim_{\varepsilon\to 0} Q_\varepsilon(u^\varepsilon) = 1$ a.e. in $\{G^0 \neq 0\}$ (which will give (iii)). The set $\mathcal{U}$ is then defined as $\{u > 0\} \cup \{G^0 \neq 0\}$.

Proposition 3.5 is proved at the end of Section 5.

4 Stationary solutions - Proof of the main results

In this section, we prove our main results concerning stationary solutions. We start with the simpler case of energy minimizers (Proposition 2.1), and then turn to general stationary solutions (Theorem 2.3).

4.1 Proof of Proposition 2.1

We recall that for all $\varepsilon > 0$, u^ε is a solution of (15)-(18) and satisfies

$$J_\varepsilon(u^\varepsilon) = \int_\Omega \frac{1}{2}|u^\varepsilon_x|^2 + Q_\varepsilon(u^\varepsilon)\, dx \le |\Omega|. \tag{42}$$

In particular, u^ε is bounded in $H^1(\Omega) \subset C^{1/2}(\Omega)$ uniformly with respect to ε and thus converges (up to a subsequence) uniformly (and H^1-weak) to some function $u(x) \in H^1(\Omega)$. Furthermore, classical regularity results for the obstacle problem imply that for all $\varepsilon > 0$ we have $u^\varepsilon \in C^{1,1}(\Omega)$ and $u^\varepsilon \in C^\infty(\{u^\varepsilon > 0\})$.

In order to get better (uniform) estimate and pass to the limit in (16), we need the following lemma:

Lemma 4.1. *There exists a constant C independent of ε such that*

$$0 \le \lambda^\varepsilon V \le C \tag{43}$$

and

$$\int_{\{u^\varepsilon > 0\}} P_\varepsilon(u^\varepsilon)\, dx \le \frac{C}{V}. \tag{44}$$

Proof. Multiplying (15) by u^ε and integrating (and using the Neumann boundary conditions on $\partial\Omega$) we get

$$\lambda^\varepsilon V \le \int_\Omega u^\varepsilon P_\varepsilon(u_\varepsilon)\, dx + \int_\Omega |u^{\varepsilon\prime}|^2\, dx.$$

Using the fact that $uP_\varepsilon(u) = \frac{u}{\varepsilon} P\left(\frac{u}{\varepsilon}\right) \le \sup_{v\ge 0} vP(v) \le C$, we deduce (43).

Next, we note that the open set $\{u^\varepsilon > 0\}$ is the countable union of its connected components (a_i, b_i). Integrating (16) on (a_i, b_i) and using (17), we get

$$\int_{a_i}^{b_i} P_\varepsilon(u^\varepsilon)\, dx = \lambda^\varepsilon (b_i - a_i)$$

and so

$$\int_{\{u^\varepsilon > 0\}} P_\varepsilon(u^\varepsilon)\, dx = \sum_i \int_{a_i}^{b_i} P_\varepsilon(u^\varepsilon)\, dx = \lambda^\varepsilon |\{u^\varepsilon > 0\}| \le \lambda^\varepsilon |\Omega|.$$

□

In view of (43), we can assume that λ^ε converges to λ and passing to the limit in (15) and (16), we deduce

$$u_{xx} \le -\lambda \qquad \text{in } \Omega \tag{45}$$

and

$$u_{xx} = -\lambda \qquad \text{in } \{u > 0\}. \tag{46}$$

Next, multiplying (17) by $u^{\varepsilon\prime}$, one finds

$$\frac{1}{2}(|u^{\varepsilon\prime}|^2)' = Q_\varepsilon(u^\varepsilon)' - u^{\varepsilon\prime}\lambda^\varepsilon \quad \text{ in } \{u^\varepsilon > 0\}$$

and so the function

$$G^\varepsilon(x) := Q_\varepsilon(u^\varepsilon) - \frac{1}{2}|u^{\varepsilon\prime}|^2 - u^\varepsilon\lambda^\varepsilon \tag{47}$$

is constant throughout each connected component of $\{u^\varepsilon > 0\}$. The $C^{1,1}$ regularity of u^ε implies that G^ε is continuous in Ω and vanishes whenever $u = 0$. We easily deduce that

$$G^\varepsilon(x) = G^\varepsilon \text{ is constant in } \Omega$$

(and $G^\varepsilon = 0$ if u^ε vanishes at at least one point in Ω).

Finally, (42) and Lemma 4.1 imply that G^ε is bounded in $L^1(\Omega)$, and so there exists a constant C independent of ε such that

$$|G^\varepsilon| \leq C. \tag{48}$$

We immediately deduce the following optimal regularity estimate:

Corollary 4.2. *There exists a constant C independent of ε such that*

$$\sup_{x\in\Omega} |u^{\varepsilon\prime}| \leq C.$$

Proof. Using (48) and Lemma 4.1, we get

$$\frac{1}{2}|u^{\varepsilon\prime}|^2 \leq |G^\varepsilon| + Q_\varepsilon(u^\varepsilon) + u^\varepsilon\lambda^\varepsilon \leq C$$

□

We can now prove the following crucial lemma, which will allow us to pass to the limit in (47):

Lemma 4.3. *Up to a subsequence u^ε converges strongly in $H^1(\Omega)$ to u, and $u^{\varepsilon\prime}$ converges almost everywhere to u'.*

Proof of Lemma 4.3. The open set $\{u^\varepsilon > 0\}$ is the (at most) countable union of its connected component (a_i, b_i). Multiplying (16) by u^ε and integrating over (a_i, b_i), we get (using (17) and the Neumann condition on $\partial\Omega$)

$$\int_{a_i}^{b_i} |u^{\varepsilon\prime}|^2\, dx = \lambda^\varepsilon \int_{a_i}^{b_i} u^\varepsilon\, dx - \int_{a_i}^{b_i} u^\varepsilon P_\varepsilon(u^\varepsilon)\, dx \leq \lambda^\varepsilon \int_{a_i}^{b_i} u^\varepsilon\, dx$$

Summing over all connected components of $\{u^\varepsilon > 0\}$, we deduce

$$\int_\Omega |u^{\varepsilon\prime}|^2\, dx \leq \lambda^\varepsilon \int_\Omega u^\varepsilon\, dx$$

and so

$$\liminf_{\varepsilon\to 0}\int_\Omega |u^{\varepsilon\prime}|^2\,dx \le \lambda\int_\Omega u\,dx$$

Next, we proceed similarly with u, multiplying (46) by u and integrating over each connected component of $\{u>0\}$. We deduce (using the fact that $u=0$ and $|u'|\le C$ on $\partial\{u=0\}\setminus\partial\Omega$):

$$\int_\Omega |u'|^2\,dx = \lambda\int_\Omega u\,dx.$$

It follows that

$$\liminf_{\varepsilon\to 0}\int_\Omega |u^{\varepsilon\prime}|^2\,dx \le \int_\Omega |u'|^2\,dx$$

which implies the lemma. □

We can now conclude the proof of Proposition 2.1: We note that $Q_\varepsilon(u_\varepsilon)$ is bounded in $L^\infty(\Omega)$ and using (44) and Corollary 4.2, we can check that $Q_\varepsilon(u_\varepsilon)' = P_\varepsilon(u^\varepsilon)u^{\varepsilon\prime}$ is bounded in $L^1(\Omega)$. We can thus assume (up to a subsequence) that $Q_\varepsilon(u^\varepsilon)$ converges $L^1(\Omega)$ strong and almost everywhere to a function $\rho(x)$. Passing to the limit in (47) (along that same subsequence), we deduce that there exists a constant G^0 such that

$$G^0 = \rho - \frac{1}{2}(u')^2 - \lambda u \quad \text{in } \Omega \tag{49}$$

and it only remains to show that $G^0=0$.

First, it is readily seen (using the definition of $Q_\varepsilon(u)$) that $\rho=1$ in $\{u>0\}$, and so (using (49)):

$$\rho = \begin{cases} 1 & \text{in } \{u>0\} \\ G^0 & \text{in } \{u=0\}\setminus\partial\{u>0\}. \end{cases} \tag{50}$$

Now, using a classical argument, we can prove that $\rho=0$ or 1 a.e.: For any $0<\delta_1<\delta_2<1$, we have $\inf_{\delta_1\le u\le\delta_2} P(u)=\kappa>0$. So (44) implies

$$|\{\delta_1\varepsilon \le u^\varepsilon \le \delta_2\varepsilon\}| \le \frac{\varepsilon}{\kappa}\int_\Omega P_\varepsilon(u^\varepsilon)\,dx \le C\frac{\varepsilon}{\kappa}$$

and so

$$|\{Q(\delta_1)\le Q_\varepsilon(u^\varepsilon)\le Q(\delta_2)\}| \le C\frac{\varepsilon}{\kappa}.$$

Passing to the limit $\varepsilon\to 0$, we deduce:

$$|\{Q(\delta_1)\le \rho \le Q(\delta_2)\}| = 0.$$

Since this holds for any $0<\delta_1<\delta_2<1$, it implies that $\rho=0$ or 1 almost everywhere.

Finally, (42) implies

$$\int \frac{1}{2}|u'|^2 + \rho\, dx \leq \liminf_{\varepsilon\to 0} \mathscr{J}_\varepsilon(u^\varepsilon) \leq |\Omega|,$$

so either $\int \rho\, dx = \Omega$, in which case we must have u is constant in Ω and we are done, or $\int \rho\, dx < \Omega$ in which case we must have $\rho = 0$ on a set of positive measure. In view of (50), we must then have $G^0 = 0$ and so (49) gives

$$0 = 1 - \frac{1}{2}(u')^2 - \lambda u \quad \text{ in } \{u > 0\}$$

which implies in particular

$$\frac{1}{2}(u')^2 = 1 \quad \text{ on } \partial\{u > 0\}$$

and competes the proof of Proposition 2.1.

4.2 Proof of Theorem 2.3

We assume now that $\{u^\varepsilon\}_{\varepsilon>0}$ is a sequence of stationary solution (in the sense of Definition (2.2)). The main difficulty in the proof of Theorem 2.3 is that we do not expect λ^ε to be bounded uniformly with respect to ε and as a consequence the equivalent of Lemma 4.3 does not hold (that is $u^{\varepsilon\prime}$ might not converge strongly to u').

The first step is to define an equivalent of the function G^ε defined by (47): Let (a_i, b_i) be a connected component of $\{u > 0\}$. Multiplying (21) by u' and integrating, we get that the function

$$x \mapsto Q_\varepsilon(u^\varepsilon(x)) - \frac{1}{2}|u^{\varepsilon\prime}(x)|^2 - \lambda_i u^\varepsilon(x)$$

is constant on (a_i, b_i).

If $\{u^\varepsilon > 0\} = \Omega$, then there is only one such component, and so we define

$$\begin{aligned} G^\varepsilon(x) &:= Q_\varepsilon(u^\varepsilon) - \frac{1}{2}|u^{\varepsilon\prime}|^2 - \lambda u^\varepsilon \\ &= Q_\varepsilon(u^\varepsilon) - \frac{1}{2}(u^{\varepsilon\prime})^2 + u^\varepsilon u^{\varepsilon\prime\prime} - u^\varepsilon P_\varepsilon(u^\varepsilon) \end{aligned}$$

which is constant in Ω.

Otherwise, we have $(a_i, b_i) \neq \Omega$ (for any connected component), and so $u = 0$ at either a_i or b_i. Using (22), we deduce that the function defined by

$$\begin{aligned} G^\varepsilon(x) &:= Q_\varepsilon(u^\varepsilon) - \frac{1}{2}|u^{\varepsilon\prime}|^2 - \lambda_i u^\varepsilon \\ &= Q_\varepsilon(u^\varepsilon) - \frac{1}{2}(u^{\varepsilon\prime})^2 + u^\varepsilon u^{\varepsilon\prime\prime} - u^\varepsilon P_\varepsilon(u^\varepsilon) \text{ in } (a_i, b_i) \end{aligned}$$

satisfies $G^\varepsilon = 0$ in (a_i, b_i).

Either way, the function defined by

$$G^\varepsilon(x) := \begin{cases} Q_\varepsilon(u^\varepsilon) - \frac{1}{2}({u^\varepsilon}')^2 + u^\varepsilon {u^\varepsilon}'' - u^\varepsilon P_\varepsilon(u^\varepsilon), & \text{in } \{u^\varepsilon > 0\} \\ 0 & \text{in } \{u^\varepsilon = 0\} \end{cases}$$

is constant in Ω.

We can then show:

Lemma 4.4. *There exists C (independent of ε) such that*

$$|G^\varepsilon| \leq C$$

and

$$|{u^\varepsilon}'(x)| \leq C \quad \textit{for all } x \in \Omega.$$

Theorem 2.3 will now follow from the following lemma:

Lemma 4.5. *We have*

$$\lim_{\varepsilon \to 0} G^\varepsilon = 0 \quad \textit{in } \Omega$$

(along any sequence $\varepsilon \to 0$).

Postponing the proofs of Lemma 4.4 and 4.5, let us now complete the proof of Theorem 2.3:

Proof of Theorem 2.3. Lemma 4.4 implies that, up to a subsequence, u^ε converges uniformly and H^1-weak to a Lipschitz function u. Furthermore,

$$\int |u'|^2 \, dx \leq \liminf_{\varepsilon \to 0} \int |{u^\varepsilon}'|^2 \, dx,$$

and if we define

$$\rho = \liminf_{\varepsilon \to 0} Q_\varepsilon(u^\varepsilon),$$

Fatou's lemma implies

$$\int_\Omega \frac{1}{2}|u'|^2 \, dx + \rho(x) \, dx \leq \liminf_{\varepsilon \to 0} \mathscr{J}_\varepsilon(u^\varepsilon). \tag{51}$$

Note that $\rho(x) \geq 0$ for all x and $\rho = 1$ in $\{u > 0\}$, so (51) implies (25).

Next, in order to make use of Lemma 4.5, we need to pass to the limit in G^ε. We cannot do it directly, since $({u^\varepsilon}')^2$ does not converge to $(u')^2$. However, we really only need to identify the limit of G^ε on the connected components of $\{u > 0\}$, where we can get better regularity for u^ε:

Let (a, b) be a connected component of $\{u > 0\}$ and K be a compact subset of (a, b). First, Lemma 4.4 and the definition of G^ε imply

$$u^\varepsilon {u^\varepsilon}'' \text{ is bounded in } L^\infty(\Omega)$$

(recall that $u \mapsto Q_\varepsilon(u)$ and $u \mapsto uP_\varepsilon(u)$ are bounded functions). The continuity of u and uniform convergence of u^ε to u implies that there exists $\delta > 0$ such that for ε small enough we have $u^\varepsilon \geq \delta \geq \varepsilon$ in K, and so

$$u^\varepsilon \text{ is bounded in } W^{2,\infty}(K)$$

and

$$Q_\varepsilon(u^\varepsilon) = 1 \text{ and } u^\varepsilon P_\varepsilon(u^\varepsilon) = 0 \text{ in } K.$$

It follows that for ε small enough, we have

$$\frac{1}{2}|u^{\varepsilon\prime}|^2 - u^\varepsilon u^{\varepsilon\prime\prime} = 1 + G^\varepsilon \quad \text{ in } K \tag{52}$$

and that $u^{\varepsilon\prime}$ converges strongly in $L^2(K)$ to u' and $u^{\varepsilon\prime\prime}$ converges weakly in $\star - L^\infty(K)$ to u''. We easily deduce

$$\frac{1}{2}(u')^2 - uu'' - 1 = 0 \text{ in } (a,b). \tag{53}$$

This now implies

$$(uu'')' = (\frac{1}{2}(u')^2)' = u'u'' \in L^\infty(K)$$

and so

$$uu'' \in W^{1,\infty}(K)$$

and hence

$$u \in W^{3,\infty}(K).$$

Differentiating (53) once more, we deduce

$$uu''' = 0 \text{ in } K$$

which implies $u''' = 0$ in K. Since this holds for any compact subset of (a,b), we deduce that u is a parabola in (a,b): There exists a constant λ such that

$$u = -\lambda(x-a)(x-b).$$

Plugging this back into (53) yields

$$\frac{1}{2}\lambda^2(a-b)^2 = 1$$

which determines λ and implies in particular

$$\frac{1}{2}|u'(a)|^2 = \frac{1}{2}|u'(b)|^2 = 1.$$

This completes the proof of Theorem 2.3. □

We now turn to the proofs of the lemma:

Proof of Lemma 4.4. First, for every connected component (a_i, b_i) of $\{u^\varepsilon > 0\}$, we have:

$$\int_{(a_i,b_i)} G^\varepsilon \, dx = -\frac{3}{2} \int_{(a_i,b_i)} ({u^\varepsilon}')^2 \, dx + (u^\varepsilon {u^\varepsilon}')|_{\partial(a_i,b_i)} + \int_{(a_i,b_i)} Q_\varepsilon(u^\varepsilon) - u^\varepsilon P_\varepsilon(u^\varepsilon) \, dx.$$

Hence, using (22) and the fact that $Q_\varepsilon(u)$ and $uP_\varepsilon(u)$ are bounded (uniformly in u and ε), we deduce

$$|\Omega||G^\varepsilon| = \left| \int_\Omega G^\varepsilon \, dx \right| \leq 3 \mathscr{J}_\varepsilon(u^\varepsilon) + C|\Omega| \leq C|\Omega|,$$

which gives the first inequality.

Next, if ${u^\varepsilon}'$ is maximum at x_0, then we must have $x_0 \in \{u^\varepsilon > 0\}$ and ${u^\varepsilon}''(x_0) = 0$. We deduce

$$\frac{1}{2}({u^\varepsilon}')^2(x_0) = G^\varepsilon + Q_\varepsilon(u^\varepsilon(x_0)) - u^\varepsilon(x_0) P_\varepsilon(u^\varepsilon(x_0)).$$

which is bounded by a constant C uniformly in ε. Proceeding similarly with a point where ${u^\varepsilon}'$ is minimum, we deduce the result. □

Proof of Lemma 4.5. If $\{u^\varepsilon > 0\} \neq \Omega$, we have already seen that $G^\varepsilon = 0$. So we only have to consider a subsequence $\varepsilon_k \to 0$ such that $\{u^{\varepsilon_k} > 0\} = \Omega$ for all k. In that case, u^{ε_k} is smooth in Ω and there exists a constant λ^k such that

$${u^{\varepsilon_k}}'' = P_{\varepsilon_k}(u^{\varepsilon_k}) - \lambda^k \text{ in } \Omega.$$

In particular, the sequence $\{u^{\varepsilon_k}\}_{k \in \mathbb{N}}$ satisfies the hypotheses of Proposition 2.1, and we can thus proceed as in the proof of Proposition 2.1 to show that

$$\lim_{k \to \infty} G^{\varepsilon_k} = 0$$

and the result follows.

□

5 Thin film equation: Proof of Theorem 3.3

We now turn to the proof of the main result of this paper. We consider a sequence $\{u^\varepsilon(x,t)\}_{\varepsilon > 0}$ of weak solutions of

$$\begin{cases} \partial_t u + \partial_x(f(u)\partial_x[\partial_{xx}u - P_\varepsilon(u)]) = 0 & \text{for } x \in \Omega, \ t > 0 \\ f(u)\partial_x[\partial_{xx}u - P_\varepsilon(u)] = 0, \qquad u_x = 0 & \text{for } x \in \partial\Omega, \ t > 0 \\ u(x,0) = u_0(x) & \text{for } x \in \Omega \end{cases} \tag{54}$$

given by Theorem (3.1). We also fixe $T > 0$ (T can be arbitrarily large).

5.1 Proof of Theorem 3.3 parts (a) and (b)

The energy inequality (32) implies that u^ε is bounded uniformly in ε in $L^\infty(0,T;H^1(\Omega)) \subset L^\infty(0,T;C^{1/2}(\Omega))$. Classically, this Hölder estimate with respect to the space variable implies some Hölder regularity in time. More precisely, we have (see [6] or [11] for instance):

Lemma 5.1. *There exists a constant C independent of ε such that*

$$||u^\varepsilon||_{\mathcal{C}^{1/8,1/2}([0,\infty)\times\overline{\Omega})} \leq C.$$

Since u^ε is also bounded in $L^\infty(0,T;L^1(\Omega))$, we can extract a subsequence which converges uniformly (with respect to x and t) to a continuous function $u(x,t)$.

From now on, we denote by $\{u^k\}_{k\in\mathbb{N}}$ such a subsequence. So for all $k \in \mathbb{N}$, the function u^k is a solution of (54) with $\varepsilon = \varepsilon_k$ and

$$\begin{aligned} \varepsilon_k &\longrightarrow 0 \\ u^k &\longrightarrow u \quad \text{ uniformly in } (0,T)\times\Omega. \end{aligned}$$

as $k \to \infty$.

Our first task is to show that u satisfy (36) (that is u solves the thin film equation in its support).

For that purpose, we note that (32) implies that the function $g^k(x,t)$, which satisfies

$$g^k = \sqrt{f(u^k)}\partial_x(\partial_{xx}u^k - P_{\varepsilon_k}(u^k)) \qquad \text{in } \{u^k > 0\}$$

is bounded in $L^2((0,T)\times\Omega)$ uniformly with respect to k and thus converges weakly to a function $g(x,t)$ satisfying

$$||g||_{L^2((0,T)\times\Omega)} \leq \liminf_{k\to\infty} ||g^k||_{L^2((0,T)\times\Omega)}.$$

Now, let K be a compact set in $\{u > 0\} \cap (0,T)\times\Omega$. The continuity of u and the uniform convergence of u^k implies that there exists $\delta > 0$ such that $u^k \geq \delta$ in K for k large enough. If $\varepsilon_k < \delta$ (which holds for k large enough), then $P_{\varepsilon_k}(u^k) = 0$ and so $g^k = \sqrt{f(u^k)}u^k_{xxx} \geq \sqrt{f(\delta)}u^k_{xxx}$ in K. It follows that u^k_{xxx} converges weakly in $L^2(K)$ to u_{xxx} and thus that g^k converges weakly in $L^2(K)$ to $\sqrt{f(u)}u_{xxx}$. We deduce that $g = \sqrt{f(u)}u_{xxx}$ in $\{u > 0\}$ and (35) follows.

To prove (36), we first rewrite (29) as

$$\int_0^T\int_\Omega u^k\varphi_t\,dx + \int_0^T\int_\Omega \sqrt{f(u^k)}g^k\varphi_x\,dx\,dt = 0$$

and pass to the limit $k \to \infty$. Note that the conservation of mass, (37), clearly follows from (31).

5.2 Proof of Theorem 3.3 part(c)

The remainder of this section is thus devoted to the proof of Theorem 3.3-(c). The main tool is the function

$$G^k(x,t) = Q_{\varepsilon_k}(u^k) - \frac{1}{2}(u^k_x)^2 + u^k[u^k_{xx} - P_{\varepsilon_k}(u^k)]$$

introduced in Theorem 3.1 (recall that for all k, we have $u^k \in L^2(0,T;H^2(\Omega))$, so G^k is defined almost everywhere in $(0,T)\times\Omega$).

Inequalities (34) and (32) imply that G^k_x is bounded in $L^2((0,T)\times\Omega)$ (we recall that $n<2$). Since

$$\int_\Omega G^k(x,t)\,dx = \int_\Omega Q_{\varepsilon_k}(u^k) - u^k P_{\varepsilon_k}(u^k) - \frac{3}{2}(u^k_x)^2\,dx,$$

the energy estimate (32) together with Poincaré inequality gives

Lemma 5.2. *There exists a constant C independent of k such that*

$$||G^k||_{L^2(0,T;H^1(\Omega))} \le C.$$

Sobolev embeddings thus implies

$$||G^k||_{L^2(0,T;\mathcal{C}^{1/2}(\mathbb{R}))} \le C, \tag{55}$$

and we can also prove the following regularity result:

Corollary 5.3 (Lipschitz regularity)**.** *The function u^k_x is bounded in $L^2(0,T;L^\infty(\mathbb{R}))$ uniformly with respect to k (in particular u^k is Lipschitz in x a.e. in t). More precisely, there exists a constant C such that*

$$||u^k_x(\cdot,t)||_{L^\infty(\Omega)} \le C\sqrt{1+||G^k(\cdot,t)||_{H^1(\Omega)}} \qquad a.e.\ t\in[0,T].$$

Note that the Lipschitz regularity with respect to x is optimal since we expect a jump of the derivative at the free boundary. Whether it is possible to obtain a Lipschitz bound in x that is uniform in time is still an open problem at this time.

Proof of Corollary 5.3. We recall that $u^k \in L^2(0,T;H^2(\Omega))$ for all k (but this does not hold uniformly in k) and so for almost every t we have $u^k_x(\cdot,t) \in C^{1/2}(\Omega)$. For such a t, $u^k_x(\cdot,t)$ takes its maximum value at a point $x_0 \in \overline{\Omega}$. If $x_0 \in \partial\Omega$, then $u^k_x(x,t) \le u^k_x(x_0,t) = 0$ for all $x\in\Omega$ (Neumann boundary condition). If $x_0\in\Omega$ and $u^k(x_0,t)=0$, then u^k has a minimum at x_0 and so we again have $u^k_x(x,t) \le u^k_x(x_0,t) = 0$ for all $x\in\Omega$.

Finally, if $x_0\in\Omega$ and $u^k(x_0,t)>0$, then u^k is smooth at x_0 and so $u^k_{xx}(x_0,t)=0$. In that case, we get

$$\begin{aligned}\frac{1}{2}(u^k_x)^2(x_0,t) &= Q_{\varepsilon_k}(u^k(x_0,t)) - u^k(x_0,t)P_{\varepsilon_k}(u^k(x_0,t)) - G^k(x_0,t)\\ &\le 1 + ||G^k(\cdot,t)||_{L^\infty(\Omega)}.\end{aligned}$$

We deduce that for almost every $t \in [0,T]$, we have

$$[\max_\Omega u_x^k(\cdot,t)]^2 \le C + C||G^k(\cdot,t)||_{L^\infty(\Omega)}$$

and we can proceed similarly to bound $[\min_\Omega u_x^k(\cdot,t)]^2$ and obtain the desired inequality. □

In the remainder of the proof, we will fix $t \in [0,T]$. But first, we note that there exists a set $\mathcal{P} \subset [0,T]$ of full measure such that for all $t \in \mathcal{P}$, we have

$$\liminf_{k\to\infty} \int_\Omega |g^k(x,t)|^2\, dx < \infty$$

and

$$u_{xx}^k(\cdot,t) \in H^2(\Omega) \qquad \text{for all } k \in \mathbb{N}.$$

Indeed, let

$$A_m = \left\{ t \in [0,T]\,;\, \liminf_{k\to\infty} \int_\Omega |g^k(x,t)|^2\, dx \ge m \right\}.$$

Fatou's lemma and (32) yield

$$\begin{aligned} m|A_m| &\le \int_{A_m} \liminf_{k\to\infty} \int_\Omega |g^k(x,t)|^2\, dx\, dt \\ &\le \liminf_{k\to\infty} \int_{A_m} \int_\Omega |g^k(x,t)|^2\, dx\, dt \\ &\le \mathscr{J}(u_0) \end{aligned}$$

and so $|\cap_{m\in\mathbb{N}} A_m| = 0$. Similarly, since $u^k \in L^2(0,T;H^2(\Omega))$ for all k, the set

$$B_k = \{t \in [0,T]\,;\, ||u^k(\cdot,t)||_{H^2(\Omega)} = \infty\}$$

has measure zero. We can then check that the set $\mathcal{P} = [0,T]\backslash[(\cap_{m\in\mathbb{N}}A_m) \cup (\cup_{k\in\mathbb{N}}B_k)]$ has the desired properties.

For the remainder of this section, we fix $t \in \mathcal{P}$ and we drop the variable t for the sake of clarity (that is we will write $u^k(x)$, $G^k(x)$... instead of $u^k(x,t)$, $G^k(x,t)$...). By construction of the set $\mathcal{P}$, up to a subsequence, we can assume that there exists a constant C (this constant, like all other constant in the rest of this proof depends implicitly on t) such that

$$\int_\Omega |g^k(x)|^2\, dx \le C \tag{56}$$

for all k. Note also that all subsequences that we will extract from now on will depend on t. This is not a problem since they still all converge to the previously defined function $u(\cdot,t)$, and our goal is only to characterize the behavior of u.

In particular, inequalities (34) and (56) imply that for any interval $I \subset \Omega$

$$\int_I |G^k_x|^2 \, dx \leq C \sup_{x\in I} u^k(x)^{2-n}. \tag{57}$$

As in Lemma 5.2, (57) implies (using the fact that $n < 2$)

$$||G^k||_{H^1(\Omega)} \leq C < \infty \quad \text{ for all } k,$$

and so up to yet another subsequence, we can assume that

$$G^k \longrightarrow G^0 \text{ in } H^1\text{-weak and uniformly}$$

where the function $x \mapsto G^0(x)$ belongs to $\mathcal{C}^{1/2}(\Omega)$. Furthermore, proceeding as in Corollary 5.3, we can show that the function u^k is Lipschitz uniformly with respect to k.

The proof of Theorem 3.3-(c) relies on the properties of this function G^0. First, using (57) and the definition of G^k, we easily get the following lemma:

Lemma 5.4. *The function $G^0 \in H^1(\Omega) \subset C^{1/2}(\Omega)$ satisfies*

$$\int_I |G^0_x|^2 \, dx \leq C \sup_{x\in I} u(x)^{2-n}$$

for all interval $I \subset \Omega$. In particular, if u vanishes in an interval $[c, d]$, then G^0 is constant in $[c, d]$. Furthermore, G^0 satisfies

$$G^0 = 1 - \frac{1}{2}(u_x)^2 + uu_{xx} \qquad \text{in } \{u > 0\}.$$

Formally, Lemma 5.4 implies that

$$\frac{1}{2}(u_x)^2 = 1 - G^0 \quad \text{ on } \partial\{u > 0\} \tag{58}$$

(this is made rigorous in Corollary 5.6 below). Since G^0 is continuous in Ω, (58) implies that the contact angle condition on $\partial\{u > 0\}$ depends on the values of G^0 on the set $\{u = 0\}$. More precisely, in order to prove Theorem 3.3-(c-i), we need to show that $G^0 \in [0, 1]$ whenever $u = 0$ (see Proposition 5.5-(i)) , while Theorem 3.3-(c-ii) requires us to characterize the points where $G^0 = 0$ (see Proposition 5.5-(ii)).

The key result is thus the following:

Proposition 5.5. *The followings hold:*

(i) Assume that $u = 0$ in $[c, d]$ (with possibly $c = d$). Then G^0 is constant in $[c, d]$ and satisfies $0 \leq G^0 \leq 1$.

(ii) Assume that $G^0 \geq \eta > 0$ in an interval (c,d). Then there exists $\kappa > 0$ (depending on η) such that (up to a subsequence)

$$u^k \geq \kappa\varepsilon_k \text{ in } (c,d)$$

and

$$Q_{\varepsilon_k}(u^k) \longrightarrow 1 \quad \text{a.e. and } L^1\text{-strong in } (c,d). \tag{59}$$

Before proving Proposition 5.5, we give the following corollary of Lemma 5.4 and Proposition 5.5 (the proof of which will make use of Lemma 5.7 below):

Corollary 5.6. *Let (a,b) be a connected component of $\{u > 0\}$. Then*

$$\frac{1}{2}u_x^2(a^+) = 1 - G^0(a) \in [0,1] \quad \text{and} \quad \frac{1}{2}u_x^2(b^-) = 1 - G^0(b) \in [0,1].$$

We can now complete the proof of Theorem 3.3 (we briefly restore the variable t for this proof):

Proof of Theorem 3.3-(c). For all $t \in \mathcal{P}$, the set $\mathcal{U}(t)$ is defined as follows:

$$\mathcal{U}(t) = \{u(\cdot,t) > 0\} \cup \{G^0(\cdot,t) \neq 0\}.$$

Since u and G^0 are both continuous function of x, the set $\mathcal{U}(t)$ is open.

Corollary 5.6 immediately implies (38). Furthermore, if (a,b) is a connected component of $\{u(\cdot,t) > 0\}$ and $b \in \partial\mathcal{U} \setminus \partial\Omega$, then the continuity of G^0 implies that $G^0(b) = 0$, so Corollary 5.6 implies (39).

In order to prove (40), we need to pass to the limit in (32). The only difficulty is to show that

$$\lim_{k\to\infty} \int_\Omega Q_{\varepsilon_k}(u^k(x,t))\,dx \geq |\mathcal{U}(t)|.$$

For that purpose, we write

$$\mathcal{U}(t) = \{u(\cdot,t) > 0\} \cup \left[\bigcup_{m\in\mathbb{N}} \{G^0(\cdot,t) > \frac{1}{m}\}\right].$$

On the set $\{u(\cdot,t) > 0\}$, it is easy to show that we have $Q_{\varepsilon_k}(u^k) \to 1$ pointwise. Next, denoting $A_m = \{G^0(\cdot,t) > \frac{1}{m}\}$, Proposition 5.5 (ii) implies that for any connected component (c,d) of A_m, we have $Q_{\varepsilon_k}(u^k) \longrightarrow 1$ a.e. x in (c,d). We deduce that $Q_{\varepsilon_k}(u^k) \longrightarrow 1$ a.e x in A_m. All together, this implies that

$$Q_{\varepsilon_k}(u^k(x,t)) \longrightarrow 1 \quad \text{a.e } x \text{ in } \mathcal{U}(t)$$

and Lebesgue dominated convergence theorem yields

$$\int_{\mathcal{U}(t)} Q_{\varepsilon_k}(u^k(x,t))\,dx \to |\mathcal{U}(t)|.$$

Passing to the limit in (32), we thus get

$$\int_\Omega \frac{1}{2} u_x^2(x,t)\,dx + |\mathcal{U}(t)| + \int_0^t \int_\Omega g^2\,dx\,ds \le \mathscr{J}_0(u_0)$$

for all $t \in \mathcal{P}$ (and so it holds a.e. $t \in [0,T]$), and using the fact that $g = \sqrt{f(u)}u_{xxx}$ in $\{u > 0\}$, we deduce (40). □

It only remains to show Proposition 5.5 and its Corollary 5.6. First, Corollary 5.6 will follow from Proposition 5.5 and the following technical lemma (the proof of which is presented in Appendix B):

Lemma 5.7.

(i) Let $x \mapsto v(x)$ be a Lipschitz function on an interval (a,b) satisfying $v(a) = 0$ and

$$-vv'' + \frac{1}{2}|v'|^2 \le h \quad \text{in } (a,b) \tag{60}$$

where h is a non-negative constant. Then

$$v(x) \le \sqrt{2h}(x-a) + \mathcal{O}(|x-a|^2) \quad \text{in } [a,b].$$

(ii) Simlary, if $x \mapsto v(x)$ be a Lipschitz function on an interval (a,b) satisfying $v(a) = 0$ and

$$-vv'' + \frac{1}{2}|v'|^2 \ge h \quad \text{in } (a,b)$$

where h is a non-negative constant. Then

$$v(x) \ge \sqrt{2h}(x-a) + \mathcal{O}(|x-a|^2) \quad \text{in } (a,b).$$

Proof of Corollary 5.6. Let (a,b) be a connected component of $\{u > 0\}$. Lemma 5.4 implies that

$$-uu_{xx} + \frac{1}{2}(u_x)^2 = 1 - G^0$$

in (a,b). Lemma 5.7 and the Hölder continuity of G^0 thus implies

$$u(x) = \sqrt{2(1-G^0(a))}(x-a) + o(|x-a|)$$

which yields

$$\frac{1}{2}u_x^2(a^+) = 1 - G^0(a).$$

The result now follows by using Proposition 5.5-(i) which implies that $1 - G^0(a) \in [0,1]$. □

Finally, we complete this section (and the proof of Theorem 3.3) with the proof of Proposition 5.5:

Proof of Proposition 5.5. In this proof, we use the notation

$$F^k = u^k_{xx} - P_{\varepsilon_k}(u^k),$$

(recall that F^k is in $L^2(\Omega)$ for all k, though not uniformly in k, and it is smooth whenever $u^k > 0$), so that

$$G^k(x) = Q_{\varepsilon_k}(u^k) - \frac{1}{2}(u^k_x)^2 + u^k F^k.$$

First, we check that $G^k(x) = 0$ whenever $u^k(x) = 0$: Assume that $u^k(x_0) = 0$ and $G^k(x_0) = \eta \neq 0$. Then the continuity of the functions G^k, $Q_{\varepsilon_k}(u^k)$ and $\frac{1}{2}(u^k_x)^2$ implies that $|u^k F^k(x)| \geq |\eta|/2$ in a neighborhood of x_0. Using the fact that u^k is Lipschitz, we deduce

$$|F^k(x)| \geq \frac{|\eta|}{2|x - x_0|}$$

in a neighborhood of x_0, which contradicts the fact that F^k is in $L^2(\Omega)$.

We already saw (Lemma 5.4) that G^0 was constant in any interval in which u is identically zero. So to prove Proposition 5.5-(i), we only need to show that this constant is in the interval $[0, 1]$. The proof relies on the following idea: At a point where u^k is minimum in (c, d), we have $u^k_x = 0$ and using the dissipation of energy (and crucially, the fact that $n < 2$) we will show that $u^k F^k$ goes to zero at that point. At such a point, we thus have $G^k \sim Q_{\varepsilon_k}(u^k) \in [0, 1]$.

To make this argument precise, we first replace $[c, d]$ by the largest closed interval containing $[c, d]$ in which u vanishes. Still denoting this interval $[c, d]$, we have that one of the followings hold:

(1) either for all $\delta > 0$, there exists $c_\delta \in (c - \delta, c)$ such that $u(c_\delta) > 0$

(2) or $c \in \partial\Omega$

and similarly for d.

Clearly, (1) must hold for either c or d (and possibly both of them). We assume that (1) holds for c and (2) for d (all other cases can be done similarly). We fix $\delta > 0$ and let $x^k \in [c_\delta, d]$ be such that

$$u^k(x^k) = \min_{[c_\delta, d]} u^k. \tag{61}$$

First, we note that if $u^k(x^k) = 0$, we already checked that we must have $G^k(x^k) = 0 \in [0, 1]$ and we are done.

We can thus assume that $u^k(x^k) \neq 0$. Since $\lim_{k\to\infty} u^k(x) = 0$ in $[c, d]$ and $\lim_{k\to\infty} u^k(c_\delta) = u(c_\delta) > 0$, it is readily seen that for k large enough we have $x^k \in (c_\delta, d]$ and so (using the Neumann boundary condition on $\partial\Omega$ if $x^k = d \in \partial\Omega$)

$$u^k_x(x^k) = 0. \tag{62}$$

Using (56) (note that $[c_\delta, d] \subset \{u^k > 0\}$, so u^k is smooth and $g^k = \sqrt{f(u^k)}F_x^k$ in (c_δ, d)), we get

$$\int_{(c_\delta,d)} |F_x^k|^2\, dx \leq \frac{C}{f(\min_{[c_\delta,d]} u^k)} = \frac{C}{f(u^k(x^k))}.$$

For k large enough, we can assume that $u^k(c_\delta) > u(c_\delta)/2 > \varepsilon_k$ and so $F^k(c_\delta) = u^k_{xx}(c_\delta)$ is bounded uniformly with respect to k. Poincaré inequality and Sobolev embedding thus yield

$$||F^k||_{L^\infty(c_\delta,d)} \leq C + \frac{C}{(f(u^k(x^k)))^{1/2}}. \tag{63}$$

In particular, we deduce

$$|u^k(x^k)F^k(x^k)| \leq Cu^k(x^k) + C\frac{u^k(x^k)}{(f(u^k(x^k)))^{1/2}},$$

and condition (28) (here the fact that $n < 2$ is crucial) thus implies

$$\lim_{k\to\infty} |u^k(x^k)F^k(x^k)| = 0.$$

Together with (62), this gives

$$\lim_{k\to\infty} |G^k(x^k) - Q_{\varepsilon_k}(u^k(x^k))| = 0. \tag{64}$$

Up to another subsequence, we can assume that x^k converges to $x^\delta \in [c_\delta, d]$. Since G^k converges uniformly to G^0, we have proved that for all $\delta > 0$, there exists $x^\delta \in [c - \delta, d]$ such that

$$G^0(x^\delta) = \lim_{k\to\infty} Q_{\varepsilon_k}(\min_{[c_\delta,d]} u^k) \in [0, 1]. \tag{65}$$

Together with the continuity of G^0 and the fact that G^0 is constant in $[c, d]$, this implies that $G^0(x) \in [0, 1]$ for all $x \in [c, d]$.

We now prove (ii): First, we can assume that (c, d) is as large as possible in the sense that one of the following holds:

(1) either for all $\delta > 0$, there exists $c_\delta \in (c - \delta, c)$ such that $G^{0,t}(c_\delta) < \eta$

(2) or $c \in \partial\Omega$

and similarly for d.

If $(c, d) = \Omega$, then there exists $c_0 \in (c, d)$ such that $u(c_0) > 0$. Otherwise, we can assume that (1) holds for c, and using the fact that G^0 is constant in any interval where $u = 0$, we see that we must have that for all $\delta > 0$, there exists $c'_\delta \in (c - \delta, c)$ such that $u(c'_\delta, t) > 0$. Let now x^k be such that

$$u^k(x^k) = \min_{[c'_\delta,d]} u^k$$

(take $c'_\delta = c$ in the case $(c,d) = \Omega$). If $\lim_{k\to\infty} u^k(x^k) \neq 0$, then the result is trivially true. If $\lim_{k\to\infty} u^k(x^k) = 0$, then we can proceed as in the first part of the proof to show that

$$\lim_{k\to\infty} |G^k(x^k) - Q_{\varepsilon_k}(u^k(x^k))| = 0.$$

In particular, for k large enough, we have

$$Q_{\varepsilon_k}(u^k(x^k)) \geq \eta/2.$$

Using the fact that $u \mapsto Q(u)$ is strictly increasing and the definition of Q_ε (see (10)-(12)), we deduce that there exists $\kappa > 0$ (defined by $Q(\kappa) = \eta/2$) such that for k large enough

$$u^k \geq \min_{[c,d]} u^k \geq u^k(x^k) \geq \kappa\varepsilon_k \quad \text{ in } (c,d). \tag{66}$$

It now remains to prove the convergence of $Q_{\varepsilon_k}(u^k)$ to 1. First, Inequality (63) now gives

$$||F^k||_{L^\infty(c,d)} \leq C + \frac{C}{(f(\kappa\varepsilon_k)))^{1/2}},$$

and we can write

$$\begin{aligned}
\int_c^d P_{\varepsilon_k}(u^k)\,dx &\leq ||F^k||_{L^1(c,d)} + \int_c^d u^k_{xx}\,dx \\
&\leq ||F^k||_{L^1(c,d)} + u^k_x(d) - u^k_x(c) \\
&\leq C + \frac{C}{(f(\kappa\varepsilon_k)))^{1/2}}
\end{aligned}$$

where we used the Lipschitz regularity of u^k. Using the definition of P_{ε_k}, we deduce:

$$\int_c^d P\left(\frac{u^k}{\varepsilon_k}\right) dx \leq C\varepsilon_k + C\frac{\varepsilon_k}{(f(\kappa\varepsilon_k)))^{1/2}} \longrightarrow 0 \qquad \text{as } k \to \infty. \tag{67}$$

Finally, for any $\delta > 0$, let $\bar\kappa$ be such that $Q(\bar\kappa) = 1 - \delta$. We then have

$$\begin{aligned}
\int_c^d |1 - Q_{\varepsilon_k}(u^k)|\,dx &\leq \int_{\{u^k \geq \bar\kappa\varepsilon_k\}} 1 - Q_{\varepsilon_k}(u^k)\,dx + \int_{\{u^k \leq \bar\kappa\varepsilon_k\}} 1\,dx \\
&\leq \delta(d-c) + |\{u^k < \bar\kappa\varepsilon_k\} \cap (c,d)|
\end{aligned}$$

where, using (66) and (67), we find

$$|\{u^k < \bar\kappa\varepsilon_k\} \cap (c,d)| \leq \frac{1}{\min_{\kappa \leq u \leq \bar\kappa} P(u)} \int_c^d P\left(\frac{u^k}{\varepsilon_k}\right) dx \longrightarrow 0 \qquad \text{as } k \to \infty$$

(note that $\min_{\kappa \leq u \leq \bar\kappa} P(u) > 0$ by (10)). We deduce

$$\limsup_{k\to\infty} \int_c^d |1 - Q_{\varepsilon_k}(u^k)|\,dx \leq \delta(d-c)$$

for all $\delta > 0$, and the result follows. □

5.3 Proof of Proposition 3.5

We now compete this section with the proof of Proposition 3.5.

To prove the first part we can assume that $G^0 \neq 0$ and $G^0 \neq 1$ in $(a-\delta, a)$. In particular, we can use (67), which implies (since $P_\varepsilon(u) = 0$ for $u > \varepsilon$),

$$\int_I u^k P_{\varepsilon_k}(u^k)\, dx \leq \int_I P\left(\frac{u^k}{\varepsilon_k}\right)\, dx \longrightarrow 0$$

Furthermore, we have

$$\int_I G^k(x)\, dx = \int_I Q_{\varepsilon_k}(u^k)\, dx - \frac{3}{2}\int_I |u_x^k|^2\, dx + u^k u_x^k|_{\partial I} - \int_I u^k P_{\varepsilon_k}(u^k)\, dx$$

We deduce (using the fact that $u^k u_x^k \to 0$ whenever $u^k \to 0$)

$$\lim_{k\to\infty} \frac{3}{2}\int_I |u_x^k|^2\, dx = \lim_{k\to\infty} \int_I Q_{\varepsilon_k}(u^k) - G^k\, dx = |I|(1 - G^0)$$

which gives the result (the last equality follows from (59)).

In order to prove the second part of Proposition 3.5, we assume again that $G^0(a) \in (0,1)$ and derive a contradiction. Since $x \mapsto G^0(x)$ is $C^{1/2}$, we can assume that $G^0(x) \geq \eta > 0$ in $(a-\delta, a+\delta)$ and so Proposition 5.5 implies $u^k \geq \kappa\varepsilon_k$ in $(a-\delta, a+\delta)$. In particular, u^k is smooth in a neighborhood of the point (a,t). Using Equation (41) and the null-flux condition on $\partial\Omega$, the condition (41) is equivalent to

$$f(u^k)F_x^k(x_0) \leq 0 \qquad \text{for all } x_0 \in (a-\delta, a+\delta)$$

where we recall the notation $F^k = u_{xx}^k - P_{\varepsilon_k}(u^k)$. This inequality makes sense since u^k is smooth in $(a-\delta, a+\delta)$. It is proved by taking some test functions which converge to $H(x_0 - x)$, where H denotes the Heaviside function, in (29). Since $f(u^k) > 0$ in $(a-\delta, a+\delta)$, we deduce that $x \mapsto F^k(x)$ is decreasing in $(a-\delta, a+\delta)$ and so

$$-F^k(x) \leq -F^k(a+\delta) \qquad \text{for all } x \in (a-\delta, a+\delta).$$

Finally, this implies

$$-\int_{a-\delta}^a u^k F^k = \int_{a-\delta}^a |u_x^k|^2 + u^k P_{\varepsilon_k}(u^k)\, dx - u^k u_x^k|_{a-\delta}^a \leq -F^k(a+\delta)\int u^k\, dx.$$

Since $u(a+\delta) > 0$, we can prove that $F^k(a+\delta)$ is bounded uniformly in k (and it converges to $u_{xx}(a+\delta)$). It follows that

$$\lim_{k\to\infty} \int_{a-\delta}^a |u_x^k|^2\, dx = 0$$

and the first part of this Proposition gives the expected contradiction.

A Proof of Theorem 3.1

Throughout this section, ε is fixed, so we can assume that $\varepsilon = 1$ and drop the ε subscript everywhere. The proof of Theorem 3.1 follows classical arguments first introduced by Bernis and Friedman [6]. First, we regularize the equation by introducing, for $\delta > 0$,

$$f_\delta(u) = f(|u|) + \delta$$

Since $f_\delta(u) > \delta$ for all u, it is easy to show (see [6] for details) that the equation

$$\begin{cases} \partial_t u + \partial_x (f_\delta(u)\partial_x[\partial_{xx} u - P(u)]) = 0 & \text{in } \Omega \times (0,\infty) \\ f_\delta(u)\partial_x[\partial_{xx} u - P(u)] = 0, \qquad \partial_x u = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = u_0^\delta(x) & \text{in } \Omega \end{cases} \tag{68}$$

has a unique classical solution $u^\delta(x,t)$. Here u_0^δ is a smooth approximation of u_0. Note that because (68) is a fourth order equation, the solution u^δ may take negative values so the functions $P(u)$ and $Q(u)$ have to be extended by 0 for $u \leq 0$.

The function u^δ satisfies the conservation of mass, and the energy equality:

$$\mathscr{J}(u^\delta(t)) + \int_0^t \int_\Omega f_\delta(u^\delta)\big[(u^\delta_{xx} - P(u^\delta))_x\big]^2\, dx\, ds = \mathscr{J}(u_0^\delta) \quad \text{a.e. } t \geq 0 \tag{69}$$

where

$$\mathscr{J}(u) = \int_\Omega \frac{1}{2} u_x^2 + Q(u)\, dx.$$

In particular, u^δ is bounded in $L^\infty(0,T;H^1(\Omega)) \subset L^\infty(0,T;C^{1/2}(\Omega))$ and satisfies

$$u_t^\delta + h_x^\delta = 0$$

where $h^\delta = f_\delta(u^\delta)(u^\delta_{xx} - P(u^\delta))_x$ is bounded in $L^2((0,T)\times\Omega)$. Classical arguments imply that u^δ is bounded uniformly in $C^{1/8,1/2}((0,T)\times\Omega)$ and thus converges (up to a subsequence) uniformly (and $L^\infty(0,T;H^1(\Omega))$ $\star$-weak) to some function $u \in L^\infty(0,T;H^1(\Omega))$. Furthermore, h^δ converges weakly to h in $L^2((0,T)\times\Omega)$, and we have

$$u_t + h_x = 0 \quad \text{in } \mathcal{D}'.$$

Next, (69) implies that $g^\delta = \sqrt{f_\delta(u^\delta)}(u^\delta_{xx} - P(u^\delta))_x$ is bounded in $L^2((0,T)\times\Omega)$ and thus converges weakly in $L^2((0,T)\times\Omega)$ to a function g. Furthermore, passing to the limit in (69), we get

$$\mathscr{J}(u(t)) + \int_0^t \int_\Omega g^2\, dx\, ds \leq \mathscr{J}(u_0).$$

It is readily seen that $g = \sqrt{f(u)}[u_{xx} + P(u)]_x$ in $\{u > 0\}$, and we deduce (32).

Assuming that $u \geq 0$ (which we will prove shortly), we can also derive (29): The function $h^\delta = \sqrt{f_\delta(u_\delta)}\, g^\delta$ converges weakly in $L^2((0,T)\times\Omega)$ to a function h, and we have

$$\int_{\{u\leq\eta\}} |h^\delta|\, dx \leq C \left(\int_{\{u\leq\eta\}} f_\delta(u_\delta)\, dx \right)^{1/2}$$

and so

$$\int_{\{u\leq\eta\}} |h|\, dx \leq C \left(\int_{\{u\leq\eta\}} f(u)\, dx \right)^{2} \leq C \sup_{0\leq s\leq\eta} f(s)^{1/2}.$$

It follows that

$$h = \begin{cases} f(u)[u_{xx} - P(u)]_x & \text{in } \{u>0\} \\ 0 & \text{a.e. in } \{u=0\} \end{cases}$$

and so (29) holds.

We now need to show that $u \geq 0$. This follows from the following entropy inequality: Let H_δ be such that $H_\delta(s) \geq 0$ for all s and $H_\delta''(s) = \frac{1}{f_\delta(s)}$. Then a straightforward computation yields

$$\frac{d}{dt}\int_\Omega H_\delta(u^\delta)\, dx + \int_\Omega |u^\delta_{xx}|^2\, dx = -\int P'(u^\delta)|u^\delta_x|^2\, dx. \tag{70}$$

Under condition (28), we have $H_\delta(s) \to \infty$ as $\delta \to 0$ for $s < 0$ (because $n \geq 1$) and $H_\delta(s) \leq C$ for $s \geq 0$ (because $n < 2$). In particular, we deduce that $\int_\Omega H_\delta(u^\delta_0)\, dx$ is bounded with respect to δ, and so (using the energy inequality to bound the right hand side in (70)):

$$\int_\Omega H_\delta(u^\delta(t,x))\, dx < C \qquad \text{for all } t > 0,$$

and

$$\int_0^T \int_\Omega |u^\delta_{xx}|^2\, dx\, dt < C(1+T) \qquad \text{for all } T > 0.$$

The first inequality (using the fact that $H_\delta(s) \to \infty$ for $s < 0$) yields

$$u(x,t) \geq 0 \quad \text{in } [0,T]\times\Omega$$

(see [6] for details). The second inequality gives $u \in L^2(0,T;H^2(\Omega))$. Together with the Neumann boundary conditions, it also gives that u^δ_x weakly converges to u_x in $L^2(0,T;H^1_0(\Omega))$, which gives (30).

It only remains to study the properties of the function G and to establish (34). We denote

$$G^\delta = Q(u^\delta) - \frac{1}{2}(u^\delta_x)^2 + u^\delta u^\delta_{xx} - u^\delta P(u^\delta)$$

We recall that u_x^δ is bounded in $L^2(0,T;H_0^1(\Omega))$ and we can see that $u_{xt}^\delta = -h_{xx}^\delta$ is bounded $L^2(0,T;H^{-2}(\Omega)$. Lions-Aubin compactness lemma thus implies that u_x^δ converges strongly in $L^2((0,T)\times\Omega)$ to u_x. Since u_{xx}^δ converges weakly to u_{xx} in $L^2((0,T)\times\Omega)$ and u^δ converges uniformly to u, we deduce that G^δ converges weakly in L^2 to

$$G = Q(u) - \frac{1}{2}(u_x)^2 + uu_{xx} - uP(u)$$

Now, a simple computation gives

$$G_x^\delta = u^\delta[u_{xx}^\delta - P(u^\delta)]_x = \frac{u^\delta}{\sqrt{f_\delta(u^\delta)}} g^\delta.$$

Condition (28) implies that $\frac{s}{\sqrt{f_\delta(s)}} \leq C|s|^{\frac{2-n}{2}}$ (with $n < 2$), and so the right hand side converges weakly in $L^2((0,T)\times\Omega)$ to $\frac{u}{\sqrt{f(u)}}g$. Since the left hand side converges (in $\mathcal{D}'$) to G_x, we deduce

$$G_x = \frac{u}{\sqrt{f(u)}} g \in L^2((0,T)\times\Omega),$$

and (34) follows.

Proof of Proposition 3.2. Again, we can fix $\varepsilon = 1$ and drop the ε dependence in this proof. By (32) and (34), there exists a sequence $t_k \to \infty$ such that

$$\int_\Omega |G_x(x,t_k)|^2\,dx \longrightarrow 0 \qquad \text{as } k \to \infty$$

and

$$\int_\Omega u(x,t_k)\,dx = \int_\Omega u_0(x)\,dx, \qquad \int_\Omega |u_x(x,t_k)|^2\,dx \leq C \quad \text{ for all } k.$$

In particular $u(\cdot,t_k)$ is bounded uniformly in $C^{1/2}(\Omega)$ and we deduce that up to a subsequence (still denoted t_k), we have

$$u(x,t_k) \longrightarrow u^\infty(x) \quad \text{ as } k\to\infty, \text{ uniformly w.r.t. } x\in\Omega.$$

for some function $u^\infty \in H^1(\Omega)$ and

$$G(x,t_k) \longrightarrow G^\infty \text{ as } k\to\infty, \text{ uniformly w.r.t. } x\in\Omega.$$

for some constant $G^\infty \in \mathbb{R}$.

In view of (70), we can also assume that

$$u(\cdot,t_k) \longrightarrow u^\infty \quad \text{ in } H^2(\Omega)\text{−weak}.$$

In particular, passing to the limit in the definition of the function G, we deduce

$$G^\infty = Q(u^\infty) - \frac{1}{2}(u_x^\infty)^2 + u^\infty u_{xx}^\infty - u^\infty P(u^\infty).$$

Differentiating this equality in $\{u^\infty > 0\}$, we get

$$u^\infty [u^\infty_{xx} - P(u^\infty)]_x = 0 \qquad \text{in } \{u^\infty > 0\}$$

which implies that u^∞ satisfies (21) for every connected component of $\{u^\infty > 0\}$. It only remains to show that u^∞ satisfies the zero contact angle condition (22), but this is an immediate consequence of the fact that $u^\infty \in H^2(\Omega)$ (and so $u^\infty_x \in C^{1/2}(\Omega)$).

□

B Proof of Lemma 5.7

Proof of Lemma 5.7. We only prove (i) since (ii) can be proved similarly, and without loss of generality (consider the function $\frac{1}{(b-a)}u(a+(b-a)x)$), we can assume that $[a,b] = [0,1]$. Denote $m = u(1)$.

The polynomial

$$w_\lambda(x) = (m-\lambda)x^2 + \lambda x$$

solves (for all λ):

$$w(0) = 0, \quad w(1) = m, \quad \text{ and } -ww'' + \frac{1}{2}|w'|^2 = \frac{1}{2}\lambda^2$$

We are going to show that

$$\text{if } \frac{1}{2}\lambda^2 \geq h, \text{ then } u(x) \leq w_\lambda(x) \text{ in } [0,1]. \tag{71}$$

Since $w_\lambda(x) \leq \lambda x + Cx^2$, the result follows.

In order to prove (71), we first lift w by defining $w^\delta_\lambda = w_\lambda + \delta$, which solves

$$w(0) = \delta, \quad w(1) = m + \delta, \quad \text{ and } -ww'' + \frac{1}{2}|w'|^2 = \frac{1}{2}\lambda^2 + \mathcal{O}(\delta). \tag{72}$$

Since u is Lipschitz and $w^\delta_\lambda(x)$ goes to infinity as λ goes to infinity for all $x \in (0,1)$, it is easy to show that for λ very large, we have $w^\delta_\lambda(x) \geq u(x)$ in $[0,1]$. Let now λ_* be the smallest λ for which $w^\delta_\lambda(x) \geq u(x)$ in $[0,1]$ and denote $w_* = w^\delta_{\lambda_*}$. The boundary conditions in (72) implies that there exists x_0 in $(0,1)$ such that $w_*(x_0) = u(x_0)$ and $w_* - u$ has a minimum at x_0. In particular

$$w'_*(x_0) = u'(x_0) \text{ and } w''_*(x_0) \geq u''(x_0).$$

So (60) and (72) imply

$$\frac{1}{2}\lambda_*^2 + \mathcal{O}(\delta) \leq h.$$

We deduce

$$\text{if } \frac{1}{2}\lambda^2 + \mathcal{O}(\delta) \geq h, \text{ then } w^\delta_\lambda \geq u \text{ in } [0,1]$$

and (71) follows by letting δ go to zero.

□

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