

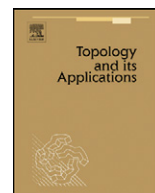


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## On products of isometries of hyperbolic space

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## ABSTRACT

We show that for arbitrary fixed conjugacy classes  $C_1, \dots, C_l$ ,  $l \geq 3$ , of loxodromic isometries of the two-dimensional complex or quaternionic hyperbolic space there exist isometries  $g_1, \dots, g_l$ , where each  $g_i \in C_i$ , and whose product is the identity. The result follows from the properness, up to conjugation, of the multiplication map on a pair of conjugacy classes in rank 1 groups.

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## 1. Statement of the result

There is a deep geometric structure underlying the problem of determining the possible eigenvalues of a product of unitary matrices in prescribed conjugacy classes (cf. [1,2,4,5,9,7,8,13,17,20]). The analogous problem for complex Lie groups was considered by Simpson in [19] and has a different character (see [14] for a survey). In this note we consider another example of this question in the context of isometry groups of symmetric spaces of negative curvature.

Let  $\mathbb{F} = \mathbb{R}, \mathbb{C}$ , or  $\mathbb{Q}$ , the real, complex, or quaternionic fields, and let  $\text{PU}(2, 1, \mathbb{F})$  denote the isometry group of the two-dimensional hyperbolic space over  $\mathbb{F}$ . We will prove the following

**Theorem 1.** Let  $C_1, \dots, C_l$ ,  $l \geq 3$ , be arbitrary conjugacy classes of loxodromic elements of  $\text{PU}(2, 1, \mathbb{F})$ . Then

- (1) there exist  $g_1, \dots, g_l \in \text{PU}(2, 1, \mathbb{F})$ ,  $g_i \in C_i$ , such that  $g_1 \cdots g_l = \mathbf{I}$ ; and
- (2) the set  $\{(g_1, g_2, g_3) : g_i \in C_i, g_1 g_2 g_3 = \mathbf{I}\}$  is compact modulo the diagonal action of conjugation by  $\text{PU}(2, 1, \mathbb{F})$ .

## Notes:

- Compactness (2) does not generalize for products of more than three elements. In fact, Theorem 1(1) implies that for four fixed loxodromic conjugacy classes there are elements  $g_i$  in these classes with arbitrary loxodromic product  $g_1 g_2 = (g_3 g_4)^{-1}$ .
- Theorem 1 also holds for products in  $\text{PU}(1, 1, \mathbb{F})$ ,  $\mathbb{F} = \mathbb{C}, \mathbb{Q}$ . Notice that  $\text{PU}(1, 1, \mathbb{R}) = \mathbb{R}^+$ , and the result clearly does not hold in this case. We do not know if the method presented here extends to prove part (1) for  $\text{PU}(n, 1, \mathbb{F})$ ,  $n \geq 3$ .

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- When the classes  $C_i$  are not loxodromic one does not expect (1) to hold. Indeed, the structure seems to resemble the case of products of unitary matrices. To illustrate this point, below we obtain restrictions of products of three unipotent matrices. Products of elliptic elements have been studied in [16]. Products of loxodromic elements have also been studied in [15] and [21].
- For  $\mathrm{SL}(3, \mathbb{C})$ , Simpson in [18] proves that for  $l$  ( $l \geq 3$ ) regular semisimple (pairwise distinct eigenvalues) conjugacy classes (1) is true. But, for three non-regular semisimple conjugacy classes, (1) fails except in the case of conjugacy classes obtained from three reducible matrices (which preserve a common subspace of  $\mathbb{C}^3$  of dimension 1). Fixing real eigenvalues we obtain counter-examples for the theorem in the case of  $\mathrm{SL}(3, \mathbb{R})$ . On the other hand, observe that for the real form  $\mathrm{SU}(2, 1) \in \mathrm{SL}(2, \mathbb{C})$ , every loxodromic element is regular semisimple. Theorem 1(2) also fails for  $\mathrm{SL}(3, \mathbb{R})$  (see [10] and comments in Section 2.2).

The proof of Theorem 1 rests on the following general result. Let  $X$  be a Riemannian symmetric space with strictly negative curvature,  $\mathrm{Iso}(X)$  its isometry group, and  $G$  the connected component of the identity in  $\mathrm{Iso}(X)$ . Let  $\mathcal{C}(G)$  denote the space of conjugacy classes of semisimple elements of  $G$ . A topology on  $\mathcal{C}(G)$  is given as follows. Let  $G/G$  denote the quotient of  $G$  acting on itself by conjugation, with the quotient topology. Then  $\mathcal{C}(G)$  is the maximal Hausdorff quotient of  $G/G$ . Hence,  $\mathcal{C}(G)$  is a locally compact Hausdorff space whose points are in 1–1 correspondence with conjugacy classes of semisimple elements of  $G$ . Furthermore, there is an induced continuous surjection  $\pi : G \rightarrow \mathcal{C}(G)$  that is invariant by the action of conjugation on  $G$ . Effectively, this map identifies (non-closed) conjugacy classes of non-semisimple elements with conjugacy classes of semisimple elements appearing in the closure. With this understood, we have the following

**Theorem 2.** *Let  $C_1, C_2$  be conjugacy classes of semisimple elements of  $G$ , and let  $G$  act on  $C_1 \times C_2$  diagonally by conjugation. Then multiplication  $(g_1, g_2) \mapsto g_1 g_2$  descends to a map*

$$p : \{C_1 \times C_2\}/G \longrightarrow \mathcal{C}(G)$$

*that is proper.*

Theorem 2 itself follows easily from the theory of group actions on  $\mathbb{R}$ -trees. A consequence of the result, however, is that the image of  $p$  is closed. The importance of this lies in the fact that if the elements  $\{g_1, g_2\}$  generate a subgroup that acts irreducibly on  $X$ , then there is an open neighborhood of  $\pi(g_1 g_2)$  contained in the image of  $p$ . It follows that the image of  $p$  consists of “chambers” bounded by “walls” corresponding to the image of pairs acting reducibly on  $X$ . In certain cases, such as  $\mathrm{PU}(2, 1)$ , this allows one to completely determine the image and leads to the proof of Theorem 1. We note that for symmetric spaces of higher rank, Theorem 2 is no longer valid. Below we provide a simple counter-example.

## 2. Products in rank 1 groups

### 2.1. Proof of Theorem 2

We start with the example of  $\mathbb{R}$ -trees. For an isometry  $g$  of an  $\mathbb{R}$ -tree  $(T, d_T)$ , let

$$|g|_T = \inf_{x \in T} d_T(x, gx)$$

denote the translation length of  $g$ . Isometries of trees are always *semisimple*, i.e.

$$\min(g) = \{x \in T : d_T(x, gx) = |g|_T\} \neq \emptyset.$$

If  $|g|_T = 0$ , then  $g$  is *elliptic* and has at least one fixed point. If  $|g|_T \neq 0$ ,  $g$  is *hyperbolic* and has a unique axis  $A_g = \min(g) \simeq \mathbb{R}$ . The following is well known (cf. [18, pp. 89–90]).

**Lemma 1.** *Suppose  $g, h \in \mathrm{Iso}(T)$  satisfy  $\min(g) \cap \min(h) = \emptyset$ . Then the isometry  $gh$  is hyperbolic.*

Now let  $G \subset \mathrm{Iso}(X)$  be as in Section 1.

**Proposition 1.** (Cf. [3, Theorem 3.9].) *Let  $\Gamma$  be a finitely presented group and  $\rho_j : \Gamma \rightarrow G$  a sequence of non-elementary representations (i.e. no fixed points at infinity). Then after passing to a subsequence one of the following holds:*

- (1) *there exist  $g_j \in G$  such that  $g_j \rho_j(\gamma) g_j^{-1}$  converges in  $G$  for all  $\gamma \in \Gamma$ ; or*
- (2) *there exist  $\varepsilon_j \downarrow 0$  and a non-trivial action (i.e. no global fixed points) of  $\Gamma$  on an  $\mathbb{R}$ -tree  $T$  such that  $\varepsilon_j |\rho_j(\gamma)|_X \rightarrow |\gamma|_T$  for all  $\gamma \in \Gamma$ .*

These two results combine to give

**Proposition 2.** *Let  $\{g_j\}, \{h_j\}$  be a pair of sequences of semisimple isometries in  $G$ . Assume there is  $B > 0$  such that  $|g_j|_X \leq B$  and  $|h_j|_X \leq B$  for all  $j$ . Then one of the following must hold:*

- (1) there is a subsequence  $\{j_k\}$  and  $f_k \in G$  such that  $f_k g_{j_k} f_k^{-1}$  and  $f_k h_{j_k} f_k^{-1}$  converge in  $G$ ; or
- (2) the sequence of translation lengths  $\{|g_j h_j|_X\}$  is unbounded.

**Proof.** Suppose first that  $g_j$  and  $h_j$  are loxodromic with axes  $A_{g_j}$  and  $A_{h_j}$  having a common fixed point on the sphere at infinity of  $X$ . Acting by diagonal conjugation on  $(g_j, h_j)$  we may assume the axis  $A_{g_j} = A_0$  is fixed. Then conjugate by elements fixing  $A_0$  so that  $A_{h_j}$  converges (up to a subsequence). The same argument applies to elliptic elements.

It therefore suffices to consider the case where the groups generated by  $(g_j, h_j)$  are non-elementary for all  $j$ . According to Proposition 1, if (1) is not satisfied then there exist  $\varepsilon_j \downarrow 0$ , an  $\mathbb{R}$ -tree  $T$ , and a non-trivial action of a free group  $\Gamma = \langle g, h \rangle$  such that  $\varepsilon_j |g_j|_X \rightarrow |g|_T$ ,  $\varepsilon_j |h_j|_X \rightarrow |h|_T$ . Since  $|g_j|_X$  and  $|h_j|_X$  are bounded,  $g$  and  $h$  must be elliptic. If  $|g_j h_j|_X$  were bounded, then it would follow that  $gh$  was elliptic. By Lemma 1 there would be a global fixed point of  $T$ , contradicting the non-triviality of the action.  $\square$

Theorems 2 and 1(2) follow immediately from Proposition 2.

### 2.2. Higher rank

Here we give some explanation for why the assumption of strict negative curvature is necessary for the result above. Degenerations of representations will no longer necessarily give actions on trees as in Proposition 1. The following example shows that Proposition 2 also cannot hold.

Let  $H = \text{Iso}(\mathbb{E}^n)$ , where  $\mathbb{E}^n$  is  $n$ -dimensional Euclidean space. Hence,  $H$  is a semidirect product of the orthogonal group  $O(n)$  and the group of translations  $\mathbb{R}^n$ . We write an element  $h \in H$  as  $h = (g, t)$  where  $g \in O(n)$  and  $t \in \mathbb{R}^n$ . Fix  $(g_0, t_0), (h_0, s_0) \in H$  in conjugacy classes  $[(g_0, t_0)]$  and  $[(h_0, s_0)]$ . Assume also that  $\mathbf{I} - g_0, \mathbf{I} - h_0$ , and  $\mathbf{I} - g_0 h_0$  are all invertible.

Notice that  $A_j = (g_0, t_0 + (g_0 - \mathbf{I})t_j) \in [(g_0, t_0)]$  for any translation  $t_j \in \mathbb{R}^n$ , and similarly  $B_j = (h_0, s_0 + (h_0 - \mathbf{I})s_j) \in [(h_0, s_0)]$ . The product is

$$A_j B_j = C_j = (g_0 h_0, t_0 + (g_0 - \mathbf{I})t_j + g_0(s_0 + (h_0 - \mathbf{I})s_j)).$$

Choose  $\|s_j\| \rightarrow \infty$ , and let  $t_j$  be determined by the equation

$$(g_0 - \mathbf{I})t_j + g_0(h_0 - \mathbf{I})s_j = \mathbf{0}. \tag{1}$$

This is possible since  $g_0 - \mathbf{I}$  is invertible. Note that  $\|t_j\| \rightarrow \infty$  also, because  $h_0 - \mathbf{I}$  is invertible. With these choices,  $C_j$  are therefore constants. In particular, the class  $[C_j]$  is fixed.

Now, it suffices to show that the pair

$$[(A_j, B_j)] \in [(g_0, t_0)] \times [(h_0, s_0)]/H$$

diverges. Let  $D_j = (k_j, r_j) \in H$ , and calculate:

$$D_j^{-1} A_j D_j = (k_j^{-1} g_0 k_j, k_j^{-1} (t_0 + (g_0 - \mathbf{I})t_j + (g_0 - \mathbf{I})r_j)),$$

$$D_j^{-1} B_j D_j = (k_j^{-1} h_0 k_j, k_j^{-1} (s_0 + (g_0^{-1} - \mathbf{I})t_j + (h_0 - \mathbf{I})r_j)).$$

Suppose that the sequence  $\{D_j^{-1} A_j D_j\}$  is bounded. Then  $(g_0 - \mathbf{I})(t_j + r_j)$  are bounded. Since  $g_0 - \mathbf{I}$  is invertible, we conclude that  $t_j + r_j$  is bounded. Now, if we suppose that  $D_j^{-1} B_j D_j$  is also bounded, we find that  $(g_0^{-1} - h_0)t_j$  is bounded, which implies in turn that  $(\mathbf{I} - g_0 h_0)t_j$  is bounded. Finally, the hypothesis that  $\mathbf{I} - g_0 h_0$  is invertible shows that  $\|t_j\|$  is bounded, which is a contradiction.

By way of explanation, note that the  $A_j, B_j$  are elliptic. The distance between their fixed points is of the order  $\|t_j - s_j\|$ . By (1) we have

$$t_j - s_j = (g_0 - \mathbf{I})^{-1} (\mathbf{I} - g_0 h_0) s_j \longrightarrow \infty,$$

so this distance is unbounded. However,  $C_j$  is also elliptic. We conclude that Proposition 2 cannot be valid in this case.

Observe that the hypothesis that the matrices  $\mathbf{I} - h_0, \mathbf{I} - g_0$  and  $\mathbf{I} - g_0 h_0$  are invertible can only be satisfied for even  $n$ . In odd dimensions we can obtain a reducible example using the above construction on a codimension one subset of  $\mathbb{R}^n$ .

In the example above  $H$  is non-reductive. This is not essential. It follows from Goldman [10, Section 4], for example, that compactness also fails for products of regular semisimple conjugacy classes in  $\text{SL}(3, \mathbb{R})$ .

## 3. The hyperbolic spaces $\mathbb{H}^n(\mathbb{F})$

### 3.1. Definitions

Let  $\mathbb{F}$  be one of the fields  $\mathbb{R}, \mathbb{C}, \mathbb{Q}$ . We denote by  $\mathbf{I}$  the identity matrix whose rank is unspecified but will be clear from the context. Consider the vector space  $V^{(n,1)}(\mathbb{F})$  (with scalars acting on the right) with the  $\mathbb{F}$ -Hermitian form of type  $(n, 1)$

$$\langle z, w \rangle = w^* J z$$

where  $w^*$  denotes conjugation followed by transposition of the column vector  $w$ . We will use the following particular Hermitian forms:

$$J_p = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \mathbf{I} & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad J_e = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -1 \end{bmatrix}, \quad J_l = \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The first is useful to describe parabolic elements, the second for semisimple elliptic elements, and the third for loxodromic elements.

Define the group

$$U(n, 1, \mathbb{F}) = \{g \in GL(n + 1, \mathbb{F}) \mid \langle gz, gw \rangle = \langle z, w \rangle\}$$

whose center  $Z(n, 1, \mathbb{F})$  consists of  $\pm \mathbf{I}$  if  $\mathbb{F} = \mathbb{R}, \mathbb{Q}$  and  $U(1)\mathbf{I}$  if  $\mathbb{F} = \mathbb{C}$ . The hyperbolic space  $H^n(\mathbb{F})$  is the projectivization (on the right) of the space of negative vectors  $V_- = \{z \mid \langle z, z \rangle < 0\}$ . We have  $H^n(\mathbb{F}) = U(n, 1, \mathbb{F})/U(1, \mathbb{F}) \times U(n, \mathbb{F})$  and the action of  $PU(n, 1, \mathbb{F}) = U(n, 1, \mathbb{F})/Z(n, 1, \mathbb{F})$  on  $H^n(\mathbb{F})$  is transitive and effective. In the following it will sometimes be useful to pass to  $U(n, 1, \mathbb{F})$ .

### 3.2. Conjugacy classes of semisimple elements

Conjugacy classes in  $U(n, 1, \mathbb{F})$  are described in [6]. Recall first that elliptic elements have a fixed point in  $H^n(\mathbb{F})$ , loxodromic elements have precisely two fixed points on the boundary at infinity, and parabolic elements have a single fixed point on the boundary at infinity. A unipotent element is an element whose eigenvalues equal to 1. In particular, a unipotent element is parabolic but not all parabolic elements are unipotent.

**Proposition 3.** (Cf. [6, Section 3].) *The conjugacy classes of semisimple elements of  $U(n, 1, \mathbb{F})$  are precisely*

- (1) *the elliptic elements, described as  $U(1, \mathbb{F}) \times \mathcal{C}(n, \mathbb{F})$  where  $\mathcal{C}(n, \mathbb{F})$  is the space of conjugacy classes of the  $\mathbb{F}$ -unitary group  $U(n, \mathbb{F})$ ,*
- (2) *the loxodromic elements, described as  $(1, \infty) \times U(1, \mathbb{F}) \times \mathcal{C}(n - 1, \mathbb{F})$ .*

Note that the set of elliptic elements forms a compact subset of  $\mathcal{C}(U(n, 1, \mathbb{F}))$ , whereas the subset of loxodromic elements is unbounded. Also, in the statement above the inclusion  $\mathcal{C}(n - 1, \mathbb{F}) \hookrightarrow \mathcal{C}(n, \mathbb{F})$  identifies the boundary of the space of loxodromic elements  $\{1\} \times U(1, \mathbb{F}) \times \mathcal{C}(n - 1, \mathbb{F})$ , with classes of elliptic elements with fixed points “on the boundary” of hyperbolic space.

The simplest example is  $PU(1, 1)$ . The space of conjugacy classes  $\mathcal{C}(PU(1, 1))$  is a pointed circle  $S^1$  union an interval  $[1, \infty)$ , identified such that  $S^1 \cap [1, \infty) = 1$ . The circle corresponds to rotations (elliptic elements) to which one attaches the identity, and the interval corresponds to the hyperbolic elements, to which one also adds the identity. To illustrate Theorem 2 in this case, consider the standard action of  $PU(1, 1)$  on the unit disk in  $\mathbb{C}$ . If one fixes two conjugacy classes of elliptic elements  $C_1 = [g_1]$ ,  $C_2 = [g_2]$ , then without loss of generality we may suppose that  $g_1$  is a rotation fixing the origin and  $g_2$  is a rotation with fixed point on the real line. Notice that to show that the multiplication

$$\{C_1 \times C_2\}/PU(1, 1) \longrightarrow \mathcal{C}(PU(1, 1))$$

is proper, it is equivalent to show that if the fixed point of  $g_2$  tends toward the boundary of the disk and the translation length of  $g_1g_2$  tends to infinity. This can be easily seen, either by a direct calculation of the trace, or geometrically by decomposing the two rotations into three reflections. Conjugacy classes of hyperbolic or mixed elements is treated similarly.

Using the Hermitian form  $J_e$ , any elliptic element is conjugate to a matrix of the form

$$\begin{bmatrix} U & 0 \\ 0 & \lambda \end{bmatrix},$$

with  $U \in U(n, \mathbb{F})$  and  $\lambda \in U(1, \mathbb{F})$ . Using the Hermitian form  $J_l$ , any loxodromic element is conjugate to a matrix of the form

$$\begin{bmatrix} \lambda U & 0 \\ 0 & \lambda H \end{bmatrix},$$

with  $U \in U(n - 1, \mathbb{F})$ ,  $\lambda \in U(1, \mathbb{F})$  and

$$H = \begin{bmatrix} r & 0 \\ 0 & 1/r \end{bmatrix},$$

with  $r > 1$ .

### 3.3. Proof of Theorem 1

We say that the pair  $(g_1, g_2)$  with  $g_1, g_2 \in U(n, 1, \mathbb{F})$  is reducible if the subgroup generated by these two elements has a proper invariant subspace of  $\mathbb{F}^{n+1}$  and we call it irreducible otherwise. Theorem 1 is a consequence of the following result. Note that it suffices to prove the case  $l = 3$ .

**Proposition 4.** For  $U(n, 1, \mathbb{F})$ ,  $n = 1, 2$ , and  $\mathbb{F} = \mathbb{C}, \mathbb{Q}$ , the image in the space of conjugacy classes of the product of two loxodromic conjugacy classes contains the entire set of loxodromic elements.

**Proof.** We will prove the case  $n = 2$ , since  $n = 1$  is elementary. The image of the product is closed by Theorem 2. It is also open near the image of irreducible pairs  $(g_1, g_2)$  (see for instance Proposition 4.2 [8] or Proposition 2.5 in [16]). This follows by a simple computation showing that the product map to the conjugacy classes has maximal rank at irreducible pairs. It therefore suffices to show that complement of the image of reducible pairs is connected. We follow the same calculation with the form  $J_1$  as [16]. Consider loxodromic elements  $[g_1]$  and  $[g_2]$  of the form

$$g_1 \simeq \begin{bmatrix} A_1 & 0 & 0 \\ 0 & \lambda_1 r_1 & 0 \\ 0 & 0 & \lambda_1 r_1^{-1} \end{bmatrix}, \quad g_2 \simeq \begin{bmatrix} A_2 & 0 & 0 \\ 0 & \lambda_2 r_2 & 0 \\ 0 & 0 & \lambda_2 r_2^{-1} \end{bmatrix}$$

where  $A_1$  and  $A_2$  are in fixed conjugacy classes  $C_1$  and  $C_2$  of  $U(1, \mathbb{F})$ .

If the group generated by  $\{g_1, g_2\}$  is reducible (with non-elliptic product), then without loss of generality we can write

$$g_1 = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & \lambda_1 r_1 & 0 \\ 0 & 0 & \lambda_1 r_1^{-1} \end{bmatrix}, \quad g_2 = \begin{bmatrix} A_2 & 0 & 0 \\ 0 & \lambda_2 a & \lambda_2 b \\ 0 & \lambda_2 c & \lambda_2 d \end{bmatrix}.$$

The product  $g_1 g_2$  is

$$g_1 g_2 = \begin{bmatrix} A_1 A_2 & 0 \\ 0 & \lambda_1 \lambda_2 H \end{bmatrix}$$

where  $H \in SL(2, \mathbb{R})$  is a product of two matrices in  $SL(2, \mathbb{R})$  with translation lengths  $r_1$  and  $r_2$ . Clearly, the image of these reducible pairs with loxodromic product coincides with

$$(1, \infty) \times \{\lambda_1 \lambda_2\} \times \{A_1 A_2\} \subset (1, \infty) \times U(1, \mathbb{F}) \times U(1, \mathbb{F}).$$

Hence, the complement of the image of the set of reducible pairs in the loxodromic conjugacy classes is connected because the complement of a point in  $U(1, \mathbb{F})$  (which is, respectively, a 0-sphere, a 1-sphere and a 2-sphere for  $\mathbb{R}, \mathbb{C}$  and  $\mathbb{Q}$ ) is connected. Since the image of the irreducible pairs is open and closed, we conclude that the intersection of the image of irreducible pairs with the loxodromic conjugacy classes coincides with the cylinder with base  $U(1, \mathbb{F}) \times U(1, \mathbb{F})$ .  $\square$

**Remark 1.** The cases  $SO_0(1, 1, \mathbb{R}) \simeq PU(1, 1, \mathbb{R})$  and  $SO_0(2, 1, \mathbb{R}) \simeq PU(2, 1, \mathbb{R}) \simeq PU(1, 1, \mathbb{C})$  are very easy. The same proof gives the well-known result that the product of two loxodromic classes in  $SO_0(3, 1, \mathbb{R}) \simeq PSL(2, \mathbb{C})$  contain the entire loxodromic part of the space of conjugacy classes (see [12] for a recent account). Similarly, we note that since  $SO_0(4, 1, \mathbb{R}) \simeq PU(1, 1, \mathbb{Q})$ , the same argument allows one to conclude the analogous result for  $SO_0(4, 1, \mathbb{R})$ .

### 3.4. Unipotent parabolic elements

There is a single conjugacy class of unipotent parabolic elements in  $U(n, 1, \mathbb{R})$  for  $n \geq 2$  (there are none for  $n = 1$ ). We therefore assume for the rest of this section that  $\mathbb{F} = \mathbb{C}$  or  $\mathbb{Q}$ . In these cases, there is a single conjugacy class of unipotent parabolic elements in  $U(1, 1, \mathbb{F})$ , so we also will suppose  $n \geq 2$ .

To describe the parabolic elements we use the Hermitian form  $J_p$ . Fix a distinguished point on the boundary of  $H^n(\mathbb{F})$

$$q_\infty = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Unipotent elements fixing  $q_\infty$  are (for  $z = (z_1, \dots, z_{n-1})$ )

$$\begin{bmatrix} 1 & -\bar{z} & (-|z|^2 + t)/2 \\ 0 & \mathbf{I} & z^T \\ 0 & 0 & 1 \end{bmatrix}$$

with  $z \in \mathbb{F}^{n-1}$  and  $t \in \mathbb{I}$ , and where we denote by  $\mathbb{I}$  the purely imaginary elements of  $\mathbb{F}$ . The coordinates  $(z, t)$  may be interpreted as coordinates on a nilpotent group  $(\mathbb{F}^{n-1} \rtimes \mathbb{I})$  and the above matrix corresponds to a translation in the group, i.e.

$$(z, t) * (z_0, t_0) = (z + z_0, t + t_0 + 2\Im(z_0^*z))$$

where  $\Im$  denotes the imaginary part. Now there exist two conjugacy classes of unipotent parabolic elements in  $U(n, 1, \mathbb{F})$ :

**Proposition 5.** ([6, Section 3]) Use the notation  $z = (z_1, \dots, z_{n-1})$ ,  $\mathbf{1} = (1, 0, \dots, 0) \in \mathbb{F}^{n-1}$ . The conjugacy classes of unipotent parabolic elements in  $U(n, 1, \mathbb{F})$ ,  $\mathbb{F} = \mathbb{C}, \mathbb{Q}$  and  $n \geq 2$  are:

- ( $\mathbb{R}$ -parabolic) the class represented by

$$\begin{bmatrix} 1 & -\mathbf{1} & -1/2 \\ 0 & \mathbf{I} & \mathbf{1}^T \\ 0 & 0 & 1 \end{bmatrix},$$

- ( $\mathbb{C}$ -parabolic) the class represented by

$$\begin{bmatrix} 1 & 0 & i/2 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**Remark 2.** In the case  $\mathbb{F} = \mathbb{C}$ , this dichotomy corresponds to unipotent parabolic elements preserving a totally geodesic subspace of complex hyperbolic space that is either Lagrangian or complex.

**Proposition 6.** Let  $g_i$ ,  $i = 1, 2, 3$ , be three unipotent elements of  $U(n, 1, \mathbb{F})$  such that  $g_1 g_2 g_3 = \mathbf{I}$ . Then the number of  $\mathbb{C}$ -parabolic unipotents among  $\{g_i\}$  is 0, 1, or 3.

**Proof.** To analyze a product of parabolic elements  $g_1, g_2$ , consider the following two cases:

- (1) the  $g_i$  have the same fixed point at infinity, or
- (2) the  $g_i$  have distinct fixed points at infinity.

In the first case, the product  $g_1 g_2$  also fixes a point at infinity. Write

$$g_1 g_2 = \begin{bmatrix} 1 & -\bar{z}_1 & (-|z_1|^2 + t_1)/2 \\ 0 & \mathbf{I} & z_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\bar{z}_2 & (-|z_2|^2 + t_2)/2 \\ 0 & \mathbf{I} & z_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that the product is either the identity or unipotent. One sees that the product of two  $\mathbb{R}$ -parabolic elements can either be  $\mathbb{R}$ -parabolic or  $\mathbb{C}$ -parabolic, whereas the product of two  $\mathbb{C}$ -parabolic elements is always either  $\mathbb{C}$ -parabolic or the identity. Indeed, the product of two unipotent  $\mathbb{C}$ -parabolic elements is

$$g_1 g_2 = \begin{bmatrix} 1 & 0 & t_1/2 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_2/2 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & (t_1 + t_2)/2 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In the second case, we may suppose that the fixed point of  $g_1$  is  $\infty$ , and the fixed point of  $g_2$  is  $\mathbf{0} = (0, \dots, 1) \in \mathbb{F}^{n-1}$ . Without loss of generality we may write (where  $g_2$  is obtained from the matrix above after conjugating by  $J_e$ )

$$g_1 g_2 = \begin{bmatrix} 1 & -\bar{z}_1 & (-|z_1|^2 + t_1)/2 \\ 0 & \mathbf{I} & z_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ z_2 & \mathbf{I} & 0 \\ (-|z_2|^2 + t_2)/2 & -\bar{z}_2 & 1 \end{bmatrix}.$$

**Lemma 2.** If the product of two unipotent  $\mathbb{C}$ -parabolic elements is unipotent, it must be  $\mathbb{C}$ -parabolic.

**Proof.** We have already considered the case of a common fixed point. Suppose the fixed point are distinct. Then the special case of unipotents gives

$$g_1 g_2 = \begin{bmatrix} 1 & -\bar{z}_1 & (-|z_1|^2 + t_1)/2 \\ 0 & \mathbf{I} & z_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ z_2 & \mathbf{I} & 0 \\ (-|z_2|^2 + t_2)/2 & -\bar{z}_2 & 1 \end{bmatrix}.$$

If the two elements are  $\mathbb{C}$ -parabolic,  $z_1 = z_2 = 0$ , and one therefore obtains a parabolic as the product if and only if either  $t_1$  or  $t_2$  vanish. In other words, one of the two elements is the identity.  $\square$

Proposition 6 follows immediately from Lemma 2.  $\square$

**Remark 3.** In the case  $\mathbb{F} = \mathbb{C}$ ,  $n = 2$ , one may give a precise expression for the product of  $\mathbb{C}$ -parabolic elements. Indeed, in this case  $z_1 = z_2 = 0$  and so

$$\mathrm{Tr}(g_1 g_2) = 3 + t_1 t_2 / 4.$$

By [11, Theorem 6.2.4], the isometry  $g_2 g_1$  is hyperbolic if  $t_1 t_2 < 0$  or  $t_1 t_2 > 16$ , and elliptic if  $0 < t_1 t_2 < 16$ .

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