Asymptotics of determinants from functional integration

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The expression for the determinant of the Laplace operator is used in terms of functional integration to compute the asymptotic behavior on degenerating Riemann surfaces. In the case of the Arakelov metric, the information is sufficiently precise to give a value for the absolute constants appearing in bosonization formulas.

I. INTRODUCTION

The behavior of determinants of Laplace operators on degenerating Riemann surfaces is of particular interest in string theories, where analytic information about the string integrand near the boundary of moduli space is essentially strong enough to determine the integrand uniquely.[1] For the case of the hyperbolic metric on surfaces of genus of at least 2, the determinant may be expressed in terms of the lengths of closed geodesics. These serve as real parameters of the degeneration, and an analysis of this expression gives the asymptotic behavior.[2-4] Alternatively, one may look at Green's functions and exploit the "bosonization" formulas for determinants. This gives especially precise information in the case of the Arakelov metric.[5] In this paper, we would like to show how the expression for the determinant in terms of functional integrals readily yields the asymptotics of the determinant and the next order term. This is a development of ideas presented in Refs. 1 and 6.

Our main results are as follows: For a compact Riemann surface M with metric g, let

$$Z_g = \left[\frac{8\pi^2 \det' \Delta_g}{\mathrm{Area}(M,g)}\right]^{-1/2},$$

where the determinant of the Laplace operator is given by zeta regularization. We construct a family M_t of surfaces degenerating as $t\to 0$ to a surface with a node. There are two possibilities: if the node separates the singular surface, then the two components M_1 and M_2 are compact Riemann surfaces, and the node is the identification of punctures on M_1 and M_2. Let g_1 and g_2 be metrics on the components, and g_t a family of metrics on M_t. Then we show that

$$\frac{Z_{g_t}}{Z_{g_1}Z_{g_2}} = \left(\frac{\det_D \Delta_0(D_\epsilon)}{\det_D \Delta_0(A_\epsilon)}\right) 2\log\frac{\epsilon}{|t|^{1/2}} H_I(t,\epsilon)$$
$$\times \exp\tfrac{1}{2}\{-S_L(\sigma_t,R_\epsilon^1) - S_L(\sigma_t,R_\epsilon^2) + S_L(\sigma_1,D_\epsilon^1) + S_L(\sigma_2,D_\epsilon^1) - S_L(\sigma_t,A_\epsilon^1) - S_L(\sigma_t,A_\epsilon^2)\}.$$

The determinants are with respect to the Euclidean metrics and the Dirichlet problems on the disk D_ϵ of radius ϵ and the annulus A_ϵ of outer radius ϵ and inner radius $|t|^{1/2}$, and they have closed expressions [see (2.16)]; the regions $R_\epsilon^i = M_i - D_\epsilon^i$, where D_ϵ^i and A_ϵ^i are disks and annuli centered about the punctures; $S_L(\sigma)$ is the Liouville action (see Sec. II G), and H_I is continuous with $\lim_{t\to 0} H_I(t,\epsilon) = 1$.

If the node does not separate, then the singular surface may be regarded as a compact surface M with two punctures a,b identified. In this case, we show

$$\frac{Z_{g_t}}{Z_{\hat g}} = \left(\frac{\det_D \Delta_0(D_\epsilon)}{\det_D \Delta_0(A_\epsilon)}\right)(2\pi)^{-1/2}\left[\log\frac{\epsilon}{|t|^{1/2}}\right]^{1/2} H_{II}(t,\epsilon)$$
$$\times\exp\tfrac{1}{2}\{-S_L(\sigma_t,R_\epsilon) + S_L(\hat\sigma,D_\epsilon^a) + S_L(\hat\sigma,D_\epsilon^b) - S_L(\sigma_t,A_\epsilon^a) - S_L(\sigma_t,A_\epsilon^b)\},$$

where $\hat g$ is a metric on the compact surface M. The difference in the power of $\log(\epsilon/|t|^{1/2})$ is related to the creation of an extra zero mode in the case of degeneration to a separating node.

In Sec. III, we consider the example of the Arakelov metric.[7] Combining the above formulas with our previous results,[5] we are able to determine the constant relating the Faltings invariant $\delta(M)$ to the determinant[8,9]

$$\delta(M) = c_h - 6\log[\det' \Delta_g/\mathrm{Area}(M,g)],$$

where g is the Arakelov metric, and

$$c_h = (1-h)c_0 + hc_1,$$
$$c_0 = -24\zeta'(-1) - 6\log 2\pi - 2\log 2 - 5,$$
$$c_1 = -8\log 2\pi.$$

Finally, we note that the constant c_h is related to the absolute constants appearing in the bosonization formulas (see Refs. 9–11).

II. DERIVATION

In this section, we use the functional integral formulation for the determinant of the Laplace operator to obtain an expression from which the asymptotics of the determinant on degenerating Riemann surfaces are easily obtained.

A. Degeneration to a separating node

Fix two compact Riemann surfaces M_1,M_2 of genus $h_1,h_2 > 0$. Choose points p_1,p_2 (henceforth both denoted simply by p) and local coordinates z_1,z_2 centered at p. When M_1 and M_2 are endowed with metrics, we will always assume the coordinates to be normalized. By this we mean that if the metric is expressed in the coordinates z_i,

$$d^2s = 2g_{z_i\bar z_i}\,dz_i\,d\bar z_i,$$

then $g_{z_i\bar z_i}(0) = 1$.

The degenerating family M_t is constructed as follows: From M_1 and M_2 remove the disks $|z_i| < |t|$ and identify the annuli $|t| < |z_i| < 1$ by the equation, $z_1z_2 = t$. M_t is then an

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analytic family of Riemann surfaces, compact of genus $h = h_1 + h_2$ for $t \neq 0$, and "stable" in the sense of Deligne–Mumford over the fiber $t = 0$ (see Ref. 12).

B. Degeneration to a nonseparating node

Fix a compact Riemann surface M of genus h. Choose two points a,b and local coordinates z_a and z_b, respectively. As above, we construct a family M_t by cutting out the disks $|z_a| < |t|$ and $|z_b| < |t|$ and identifying the remaining regions $|t| < |z| < 1$ by $z_a z_b = t$. For $t \neq 0$, M_t is then compact of genus $h + 1$. M_0 is the surface M with the "punctures" a,b identified. Notice that in this case, the node does not separate the surface.

C. Definition of determinants from functional integration

Given a metric g on M compatible with the complex structure, we define the L^2 norm on real-valued functions ϕ on M with respect to g,

$$\|\phi\|^2 = \int_M d^2\xi \sqrt{g} |\phi|^2.$$

We then fix a path integral measure $\mathscr{D}\phi$ by,

$$1 = \int \mathscr{D}\phi \, e^{-(1/8\pi)\|\phi\|^2}. \tag{2.1}$$

We will denote by ϕ^* the projection onto the orthogonal complement of the constant functions. Hence,

$$\phi^* = \phi - \frac{1}{\text{Area}(M,g)} \int_M d^2\xi \sqrt{g}\phi.$$

We can now derive the normalization of $\mathscr{D}\phi^*$ from (2.1). Let $A = \text{Area}(M,g)$. Then,

$$\begin{aligned} 1 &= \int \mathscr{D}\phi \, e^{-(1/8\pi)\|\phi\|^2} \\ &= \int dc \, \mathscr{D}\phi^* \, e^{-(A/8\pi)c^2 - (1/8\pi)\|\phi^*\|^2} \\ &= \sqrt{\frac{8\pi^2}{A}} \int \mathscr{D}\phi^* \, e^{-(1/8\pi)\|\phi^*\|^2}, \end{aligned}$$

or

$$\int \mathscr{D}\phi^* \, e^{-(1/8\pi)\|\phi^*\|^2} = \left(\frac{8\pi^2}{A}\right)^{-1/2}. \tag{2.2}$$

Choosing zeta function regularization, a similar argument shows

$$Z_g = \int \mathscr{D}\phi^* \, e^{-I(\phi^*)} = \left[\frac{8\pi^2 \det{}' \Delta_g}{\text{Area}(M,g)}\right]^{-1/2}, \tag{2.3}$$

where the action I is defined by

$$I(\phi) = \frac{1}{8\pi} \int_M d^2\xi \sqrt{g} |\nabla\phi|^2. \tag{2.4}$$

D. Factorization of the path integral—Separating node case

We now localize the right-hand side of (2.3). Recall the construction of Sec. II A. For small ϵ, $|t| < \epsilon < 1$, we define three curves:

$$\begin{aligned} C_\epsilon^1 &= \{|z_1| = \epsilon\}, \\ C_\epsilon^2 &= \{|z_2| = \epsilon\}, \\ C_t &= \{|z_1| = |z_2| = |t|^{1/2}\}. \end{aligned}$$

Furthermore, let A_ϵ^j be the annulus bound by the curve $\{|z_j| = |t|^{1/2}\}$ and C_ϵ^j, and let D_ϵ^j be the disk in M_j bound by the curve C_ϵ^j.

By the "sewing" property of functional integration, (2.3) becomes

$$\int \mathscr{D}\phi^* \, e^{-I(\phi^*)} = \int \mathscr{D}\phi_1 \, \mathscr{D}\phi_2 \, \mathscr{D}\psi_1 \, \mathscr{D}\psi_2 \, \mathscr{D}\beta_1 \, \mathscr{D}\beta_2 \times \mathscr{D}\alpha^* \times e^{-I(\phi_1) - I(\phi_2) - I(\psi_1) - I(\psi_2)}, \tag{2.5}$$

where ϕ_1 maps $R_\epsilon^1 = M_1 - D_\epsilon^1$ to the reals with boundary values

$$\phi_1|_{\partial D_\epsilon^1} = \beta_1.$$

Likewise for ϕ_2, ψ_1 maps A_ϵ^1 to $\mathbb{R}$ with boundary values β_1 and α^*, respectively. Likewise for ψ_2. While no satisfactory proof of the "sewing" property (2.4.1) for functional integrals on Riemann surfaces exists, specific examples can be verified directly. In the Appendix, we show that (2.5) holds for the case of a disk cut out from a sphere.

Following D'Hoker–Phong,[6] we decompose the ϕ's and ψ's into two pieces,

$$\begin{aligned} \phi_i &= \tilde{\phi}_i + \chi_i, \\ \psi_i &= \tilde{\psi}_i + \zeta_i, \quad i = 1,2, \end{aligned}$$

where $\tilde{\phi}_i$ and $\tilde{\psi}_i$ have zero boundary conditions, and χ_i and ζ_i are harmonic. Note that, for example,

$$I(\phi_1) = I(\tilde{\phi}_1) + \frac{1}{8\pi} \int_{\partial D_\epsilon^1} dn^\mu \, \chi_1 \, \partial_\mu \chi_1,$$

where n^μ is the outward normal. Using the above, and by translation invariance of the measures, (2.5) becomes

$$\begin{aligned} Z_g = \int & \mathscr{D}\tilde{\phi}_1 \, \mathscr{D}\tilde{\phi}_2 \, \mathscr{D}\tilde{\psi}_1 \, \mathscr{D}\tilde{\psi}_2 \, e^{-I(\tilde{\phi}_1) - I(\tilde{\phi}_2) - I(\tilde{\psi}_1) - I(\tilde{\psi}_2)} \\ & \times \int \mathscr{D}\beta_1 \, \mathscr{D}\beta_2 \, \mathscr{D}\alpha^* \exp\left\{ -\frac{1}{8\pi} \int_{\partial R_\epsilon^1} dn^\mu \, \chi_1 \, \partial_\mu \chi_1 \right. \\ & - \frac{1}{8\pi} \int_{\partial R_\epsilon^2} dn^\mu \, \chi_2 \, \partial_\mu \chi_2 - \frac{1}{8\pi} \int_{\partial A_\epsilon^1} dn^\mu \, \zeta_1 \, \partial_\mu \zeta_1 \\ & \left. - \frac{1}{8\pi} \int_{\partial A_\epsilon^2} dn^\mu \, \zeta_2 \, \partial_\mu \zeta_2 \right\}. \end{aligned}$$

The integrals in the top line just give the regularized determinants of the Dirichlet problem for the regions R_ϵ^j and A_ϵ^j. We expand the boundary values in Fourier series,

$$\beta_j = \sum_{n \in \mathbb{Z}} b_n^j e^{in\theta},$$

and define $\beta_j^* = \beta_j - b_0^j$. Then the above expression becomes

$$Z_g = [\det{}_D \Delta_g(R^1_\epsilon)\det{}_D \Delta_g(R^2_\epsilon)\det{}_D \Delta_g(A^1_\epsilon)\det{}_D \Delta_g(A^2_\epsilon)]^{-1/2}\int \mathscr{D}\beta^*_1\,\mathscr{D}\beta^*_2\,db^1_0\,db^2_0\,\mathscr{D}\alpha^*$$
$$\times\exp\left\{-\frac{1}{8\pi}\int_{\partial R^1_\epsilon} dn^\mu\,\chi_1\,\partial_\mu\chi_1-\frac{1}{8\pi}\int_{\partial R^2_\epsilon} dn^\mu\,\chi_2\,\partial_\mu\chi_2-\frac{1}{8\pi}\int_{\partial A^1_\epsilon} dn^\mu\,\zeta_1\,\partial_\mu\zeta_1-\frac{1}{8\pi}\int_{\partial A^2_\epsilon} dn^\mu\,\zeta_2\,\partial_\mu\zeta_2\right\}. \tag{2.6}$$

Now consider the compact surfaces M_j with metrics g_j. A derivation similar to the one above shows

$$Z_{g_j} = [\det{}_D \Delta_{g_j}(R^j_\epsilon)\det{}_D \Delta_{g_j}(D^j_\epsilon)]^{-1/2}\int \mathscr{D}\beta^*_j \exp\left\{-\frac{1}{8\pi}\int_{\partial R^j_\epsilon} dn^\mu\,\chi_j\,\partial_\mu\chi_j-\frac{1}{8\pi}\int_{\partial D^j_\epsilon} dn^\mu\,\xi_j\,\partial_\mu\xi_j\right\},$$

where ξ_j is harmonic on D^j_ϵ and has boundary values β_j. Combining this with (2.6), we have

$$\frac{Z_{g_t}}{Z_{g_1}Z_{g_2}} = \left[\frac{\det{}_D \Delta_{g_1}(R^1_\epsilon)}{\det{}_D \Delta_{g_t}(R^1_\epsilon)}\frac{\det{}_D \Delta_{g_2}(R^2_\epsilon)}{\det{}_D \Delta_{g_t}(R^2_\epsilon)}\frac{\det{}_D \Delta_{g_1}(D^1_\epsilon)}{\det{}_D \Delta_{g_t}(A^1_\epsilon)}\frac{\det{}_D \Delta_{g_2}(D^2_\epsilon)}{\det{}_D \Delta_{g_t}(A^2_\epsilon)}\right]^{1/2}$$
$$\times\frac{\int\mathscr{D}\beta^*_1\,\mathscr{D}\beta^*_2\,db^1_0\,db^2_0\,\mathscr{D}\alpha^*\exp\{-1/8\pi\int_{\partial R^1_\epsilon}dn^\mu\,\chi_1\,\partial_\mu\chi_1-1/8\pi\int_{\partial R^2_\epsilon}dn^\mu\,\chi_2\,\partial_\mu\chi_2-1/8\pi\int_{\partial A^1_\epsilon}dn^\mu\,\zeta_1\,\partial_\mu\zeta_1-1/8\pi\int_{\partial A^2_\epsilon}dn^\mu\,\zeta_2\,\partial_\mu\zeta_2\}}{\int\mathscr{D}\beta^*_1\,\mathscr{D}\beta^*_2\exp\{-1/8\pi\int_{\partial R^1_\epsilon}dn^\mu\,\chi_1\,\partial_\mu\chi_1-1/8\pi\int_{\partial R^2_\epsilon}dn^\mu\,\chi_2\,\partial_\mu\chi_2-1/8\pi\int_{\partial D^1_\epsilon}dn^\mu\,\xi_1\,\partial_\mu\xi_1-1/8\pi\int_{\partial D^2_\epsilon}dn^\mu\,\xi_2\,\partial_\mu\xi_2\}}. \tag{2.7}$$

This expression is actually quite tractable, despite its appearance. The integrals (we shall call them *harmonic integrals*) may be evaluated explicitly. The determinants on the R^j_ϵ's can be related through the Liouville action. The remaining determinants are evaluated on disks or annuli, and these have closed expressions. We now proceed to discuss all these quantities in detail.

E. Evaluation of harmonic integrals

In this section, we compute

$$\int db^1_0\,db^2_0\,\mathscr{D}\alpha^*\exp\left\{-\frac{1}{8\pi}\int_{\partial A^1_\epsilon}dn^\mu\,\zeta_1\,\partial_\mu\zeta_1-\frac{1}{8\pi}\int_{\partial A^2_\epsilon}dn^\mu\,\zeta_2\,\partial_\mu\zeta_2\right\}. \tag{2.8}$$

Such integrals have been discussed in Ref. 13. We consider the general situation of an annulus A with inner radius r_1 and outer radius r_2 and with boundary conditions α and β, respectively. Let χ be the unique harmonic function on A with the given boundary conditions. If we expand α, β, and χ into Fourier coefficients,

$$\alpha=\sum_{n\in\mathbf{Z}}a_ne^{in\theta},\quad \beta=\sum_{n\in\mathbf{Z}}b_ne^{in\theta},\quad \chi=\sum_{n\in\mathbf{Z}}c_n(r)e^{in\theta},$$

we find the solutions,

$$c_0(r)=A_0+B_0\log r,$$
$$c_n(r)=A_nr^n+B_nr^{-n},\quad n\neq 0.$$

Imposing the boundary conditions, we have,

$$A_0=\frac{a_0\log r_2-b_0\log r_1}{\log r_2/r_1},\quad A_n=\frac{a_nr_1^n-b_nr_2^n}{r_1^{2n}-r_2^{2n}},$$
$$B_0=\frac{b_0-a_0}{\log r_2/r_1},\quad B_n=r_1^nr_2^n\left(\frac{b_nr_1^n-a_nr_2^n}{r_1^{2n}-r_2^{2n}}\right),\quad n\neq 0.$$

By computation,

$$\int_{\partial A}dn^\mu\,\chi\,\partial_\mu\chi=2\pi\frac{(b_0-a_0)^2}{\log r_2/r_1}+4\pi\sum_{n=1}^\infty nb_nb_{-n}\lambda_n+4\pi\sum_{n=1}^\infty na_na_{-n}\lambda_n-16\pi\sum_{n=1}^\infty\frac{na_nb_{-n}r_1^nr_2^n}{r_2^{2n}-r_1^{2n}}, \tag{2.9}$$

where, $\lambda_n=(r_2^{2n}+r_1^{2n})/(r_2^{2n}-r_1^{2n})$.

On the space of functions: $\alpha=\Sigma_{n\in\mathbf{Z}}a_ne^{in\theta}$, $\bar a_n=a_{-n}$, we have the inner product:

$$\langle\alpha,\beta\rangle=(1/2\pi)a_0b_0+(\alpha^*,\bar\beta^*)+(\beta^*,\bar\alpha^*),$$

where, $(\alpha^*,\bar\beta^*)=\Sigma_{n=1}^\infty na_n\bar b_n$. The path integrals are normalized such that $1=\int\mathscr{D}\alpha\,e^{-(1/2)\langle\alpha,\alpha\rangle}$. Computing as in Sec. II C, we find

$$\int\mathscr{D}\alpha^*\,e^{-(\alpha^*,\bar\alpha^*)}=(2\pi)^{-1}. \tag{2.10}$$

Applying the result (2.9) to the integral (2.8), we have,

$$\int db^1_0\,db^2_0\,\mathscr{D}\alpha^*\exp\left\{-\frac{1}{8\pi}\int_{\partial A^1_\epsilon}dn^\mu\,\zeta_1\,\partial_\mu\zeta_1-\frac{1}{8\pi}\int_{\partial A^2_\epsilon}dn^\mu\,\zeta_2\,\partial_\mu\zeta_2\right\}$$
$$=\int db^1_0\exp-\frac{(b^1_0)^2}{4\log r_2/r_1}\int db^2_0\exp-\frac{(b^2_0)^2}{4\log r_2/r_1}\int\mathscr{D}\alpha^*\,e^{-(Q\alpha^*,\bar\alpha^*)-(T\alpha^*,\bar\beta^*)-(1/2)(Q\beta^*_1,\bar\beta^*_1)-(1/2)(Q\beta^*_2,\bar\beta^*_2)}.$$

The operators $Q-1$ and T are trace class, depending on r_1 and r_2, and,

$$\left.\begin{matrix} Q-1\to 0 \\ T\to 0 \end{matrix}\right\} \quad \text{as } r_1\to 0.$$

We may therefore take the limit. Using (2.10),

$$\int \mathscr{D}\alpha^* \, e^{-(Q\alpha^*,\bar{\alpha}^*)-(T\alpha^*,\bar{\beta}^*)-(1/2)(Q\beta_1^*,\bar{\beta}_1^*)-(1/2)(Q\beta_2^*,\bar{\beta}_2^*)}$$

$$\to (2\pi)^{-1}\exp\{-\tfrac{1}{2}(\beta_1^*,\bar{\beta}_1^*)-\tfrac{1}{2}(\beta_2^*,\bar{\beta}_2^*)\},$$

as $r_1\to 0$.

A similar computation is easily carried out for harmonic integrals over the disk of radius r_2 with boundary values β^*. We find,

$$\int_{\partial D} dn^\mu \, \xi \, \partial_\mu \xi = 4\pi(\beta^*\bar{\beta}^*).$$

Putting these results together, we have for the quotient of harmonic integrals in (2.7),

$$\frac{\int \mathscr{D}\beta_1^* \, \mathscr{D}\beta_2^* \, db_0^1 \, db_0^2 \, \mathscr{D}\alpha^* \cdots}{\int \mathscr{D}\beta_1^* \, \mathscr{D}\beta_2^* \cdots} = 2\log\frac{\epsilon}{|t|^{1/2}} H_I(t,\epsilon). \tag{2.11}$$

H_I is continuous, and $\lim_{t\to 0} H_I(t,\epsilon)=1$ for fixed ϵ.

F. Case of a nonseparating node

In (2.11) the appearance of the $\log(\epsilon/|t|^{1/2})$ was due to the integration over both b_0^1 and b_0^2. This corresponds roughly to the creation of an extra zero mode as the surface M is pulled apart. We shall see that the situation is quite different for the case where M degenerates to a surface with a nonseparating node.

Recall the construction of Sec. II B. Again, we choose ϵ small and define three curves:

$$C_\epsilon^a=\{|z_a|=\epsilon\},$$
$$C_\epsilon^b=\{|z_b|=\epsilon\},$$
$$C_t=\{|z_1|=|z_2|=|t|^{1/2}\}.$$

We define, as in Sec. II D, the disks $D_\epsilon^a, D_\epsilon^b$, and the annuli $A_\epsilon^a, A_\epsilon^b$. Let $R_\epsilon=M-D_\epsilon^a-D_\epsilon^b$. We factorize the functional integral as in (2.5), except that now we allow constant values on C_t and restrict to nonconstants on C_ϵ^b. Let the boundary values be α on C_t, μ_a, on C_ϵ^a, and μ_b^* on C_ϵ^b. Then we have

$$Z_g=\int \mathscr{D}\phi \, \mathscr{D}\psi_a \, \mathscr{D}\psi_b \, \mathscr{D}\mu_a \, \mathscr{D}\mu_b^* \, \mathscr{D}\alpha \times e^{-I(\phi)-I(\psi_a)-I(\psi_b)},$$

where ϕ,ψ_a, and ψ_b are defined by analogy with Sec. II D. We split the functions into harmonic pieces and functions with zero boundary values,

$$\phi=\tilde{\phi}+\chi, \quad \psi_a=\tilde{\psi}_a+\zeta_a, \quad \psi_b=\tilde{\psi}_b+\zeta_b,$$

and simplify,

$$Z_g=[\det\nolimits_D \Delta_g(R_\epsilon)\det\nolimits_D \Delta_g(A_\epsilon^a)\det\nolimits_D \Delta_g(A_\epsilon^b)]^{-1/2}$$
$$\times\int \mathscr{D}\mu_a \, \mathscr{D}\mu_b^* \, \mathscr{D}\alpha \exp -\frac{1}{8\pi}\left\{\int_{\partial R_\epsilon} dn^j \chi \, \partial_j \chi + \int_{\partial A_\epsilon^a} dn^j \zeta_a \, \partial_j \zeta_a + \int_{\partial A_\epsilon^b} dn^j \zeta_b \, \partial_j \zeta_b\right\}.$$

If we denote by $\hat{g}$ the original metric on M, then,

$$\frac{Z_{g_t}}{Z_{\hat{g}}}=\left[\frac{\det_D \Delta_{\hat{g}}(R_\epsilon)}{\det_D \Delta_{g_t}(R_\epsilon)}\frac{\det_D \Delta_{\hat{g}}(D_\epsilon^a)}{\det_D \Delta_{g_t}(A_\epsilon^a)}\frac{\det_D \Delta_{\hat{g}}(D_\epsilon^b)}{\det_D \Delta_{g_t}(A_\epsilon^b)}\right]^{1/2}$$
$$\times\frac{\int \mathscr{D}\mu_a \, \mathscr{D}\mu_b^* \, \mathscr{D}\alpha \exp-(1/8\pi)\{\int_{\partial R_\epsilon} dn^j \chi \, \partial_j\chi+\int_{\partial A_\epsilon^a} dn^j \zeta_a \, \partial_j\zeta_a+\int_{\partial A_\epsilon^b} dn^j \zeta_b \, \partial_j\zeta_b\}}{\int \mathscr{D}\mu_a \, \mathscr{D}\mu_b^* \exp-(1/8\pi)\{\int_{\partial R_\epsilon} dn^j \chi \, \partial_j\chi+\int_{\partial D_\epsilon^a} dn^j \xi_a \, \partial_j\xi_a+\int_{\partial D_\epsilon^b} dn^j \xi_b \, \partial_j\xi_b\}}, \tag{2.12}$$

where ξ_a is harmonic on D_ϵ^a and has boundary values μ_a, etc. Referring to the result (2.9), and expanding μ by its Fourier coefficients

$$\mu_a=\sum_{k\in\mathbf{Z}} m_k e^{ik\theta},$$

we have,

$$\int_{\partial A_\epsilon^a} dn^j \zeta_a \, \partial_j \zeta_a = 2\pi\frac{(m_0-a_0)^2}{\log(\epsilon/|t|^{1/2})}+4\pi(Q\mu_a^*,\bar{\mu}_a^*) + 4\pi(Q\alpha^*,\bar{\alpha}^*)+8\pi(T\alpha^*,\bar{\mu}_a^*),$$

$$\int_{\partial A_\epsilon^b} dn^j \zeta_b \, \partial_j \zeta_b = 2\pi\frac{a_0^2}{\log(\epsilon/|t|^{1/2})}+4\pi(Q\mu_b^*,\bar{\mu}_b^*) + 4\pi(Q\alpha^*,\bar{\alpha}^*)+8\pi(T\alpha^*,\bar{\mu}_b^*).$$

Thus we may evaluate the harmonic integral

$$\int \mathscr{D}\alpha \exp -\frac{1}{8\pi}\left\{\int_{\partial A_\epsilon^a} dn^j \zeta_a \, \partial_j\zeta_a + \int_{\partial A_\epsilon^b} dn^j \zeta_b \, \partial_j\zeta_b\right\}$$
$$=\int da_0 \exp -\frac{a_0^2+(a_0-m_0)^2}{4\log(\epsilon/|t|^{1/2})}$$
$$\times\int \mathscr{D}\alpha^* \exp[-(Q\alpha^*,\bar{\alpha}^*)-(T\alpha^*,\bar{\mu}_a^*)$$
$$-(T\alpha^*,\bar{\mu}_b^*)-1/2(Q\mu_a^*,\bar{\mu}_a^*)-1/2(Q\mu_b^*,\bar{\mu}_b^*)].$$

As before, $Q\to 1$ and $T\to 0$ as $t\to 0$, so by (2.10), the α^* integral $\to(2\pi)^{-1}$. The a_0 integral is easily evaluated. The ratio of the harmonic integrals in (2.12) may therefore be written

$$\frac{\int \mathscr{D}\mu_a \, \mathscr{D}\mu_b^* \, \mathscr{D}\alpha \cdots}{\int \mathscr{D}\mu_a \, \mathscr{D}\mu_b^* \cdots}=(2\pi)^{-1}\left[\log\frac{\epsilon}{|t|^{1/2}}\right]^{1/2} H_{II}(t,\epsilon), \tag{2.13}$$

with $\lim_{t\to 0} H_{II}(t,\epsilon) = 1$ for fixed ϵ. The fact that $\log|t|$ is raised to the power 1/2 rather than 1 will mean that the singularity for the logarithm of the determinant carries and extra $\log(-\log|t|)$ term [see Eq. (2.17)].

G. The Liouville action

We now recall the relationship between determinants on surfaces with boundary for conformally related metrics.[14] Suppose that on a surface R with boundary we have metrics g and $\hat{g}$ related by a conformal factor, $g_{z\bar{z}} = e^{2\sigma}\hat{g}_{z\bar{z}}$. Then for the Dirichlet problem on R, we have,

$$\det{}_D \Delta_g(R) = \det{}_D \Delta_{\hat{g}}(R) e^{S_L(\sigma,R)},$$

where

$$S_L(\sigma,R) = -\frac{1}{6\pi}\int_R d^2\xi\,\sqrt{\hat{g}}\hat{K}\sigma - \frac{1}{6\pi}\int_{\partial R} d\hat{s}\,k_{\hat{g}}\sigma - \frac{1}{12\pi}\int_R d^2\xi\,\sqrt{\hat{g}}\hat{g}^{ab}\partial_a\sigma\,\partial_b\sigma - \frac{1}{4\pi}\int_{\partial R} d\hat{s}\,n\cdot\partial\sigma, \tag{2.14}$$

where $\hat{K}$ is the scalar curvature and $k_{\hat{g}}$ is the geodesic curvature, both with respect to the metric $\hat{g}$.

Using this expression and (2.11), we can rewrite (2.7) as,

$$\frac{Z_{g_t}}{Z_{g_1}Z_{g_2}} = \left(\frac{\det_D \Delta_0(D_\epsilon)}{\det_D \Delta_0(A_\epsilon)}\right) 2\log\frac{\epsilon}{|t|^{1/2}} H_I(t,\epsilon) \times\exp\tfrac{1}{2}\{-S_L(\sigma_t,R_\epsilon^1) - S_L(\sigma_t,R_\epsilon^2) + S_L(\sigma_1,D_\epsilon^1) + S_L(\sigma_2,D_\epsilon^1) - S_L(\sigma_t,A_\epsilon^1) - S_L(\sigma_t,A_\epsilon^2)\}. \tag{2.15}$$

Here, σ_t is the conformal factor relating g_t to g_j on R_ϵ^j, and that relating g_t to the Euclidean metric in the annuli A_ϵ^j. σ_j is the conformal factor relating g_j to the Euclidean metric on D_ϵ^j. Δ_0 is the Laplace operator in the Euclidean metric.

The form of (2.15) makes it easy to compute the asymptotics of the determinant when one has sufficient estimates for the behavior of the metric. The Euclidean determinants, as mentioned, may be evaluated explicitly—we recall these results for reference (see Ref. 14):

$$\det{}_D \Delta_0(D_\epsilon) = 2^{-1/6}\pi^{-1/2}\epsilon^{-1/3} \times\exp[-2\zeta'(-1) - 5/12],$$
$$\det{}_D \Delta_0(A_\epsilon) = \pi^{-1}|t|^{1/6}\epsilon^{-1/3} \times\log(\epsilon/|t|^{1/2})[f(|t|/\epsilon^2)]^{-2}, \tag{2.16}$$

where $\zeta(s)$ is the Riemann zeta function, and f is the partition function

$$f(x) = \prod_{n=1}^{\infty}(1-x^n)^{-1}.$$

For the case of a nonseparating node, we have, from (2.12) and (2.13),

$$\frac{Z_{g_t}}{Z_{\hat{g}}} = \left(\frac{\det_D \Delta_0(D_\epsilon)}{\det_D \Delta_0(A_\epsilon)}\right)(2\pi)^{-1/2}\left[\log\frac{\epsilon}{|t|^{1/2}}\right]^{1/2} H_{II}(t,\epsilon) \times\exp\frac{1}{2}\{-S_L(\sigma_t,R_\epsilon) + S_L(\hat{\sigma},D_\epsilon^a) + S_L(\hat{\sigma},D_\epsilon^b) - S_L(\sigma_t,A_\epsilon^a) - S_L(\sigma_t,A_\epsilon^b)\}, \tag{2.17}$$

with the obvious definitions.

At this point, we can see the difference in the behavior of the determinant depending upon whether the degeneration is to a separating or nonseparating node. Roughly speaking, one expects that for "reasonable" families of metrics g_t, the Liouville actions will contribute singularities of the order $\sim\log|t|$. Comparing (2.15), (2.16), and (2.17), we see that asymptotically $\log\det'\Delta$ has an additional term $\log(-\log|t|)$ if and only if the node is nonseparating.

III. EXAMPLE—THE ARAKELOV METRIC

We use the results of Sec. II to obtain the asymptotics of the determinant in the Arakelov metric. Comparing this with our previous results on the asymptotics of the Faltings invariant, we are able to obtain the exact additive constant relating the two.

A. Estimating the Liouville action

We now evaluate the terms in the exponential of (2.15) for surfaces degenerating with the Arakelov metric.[7] We shall treat only the case of degeneration to a separating node—recall the construction of Sec. II A. The asymptotics of the metric are[5]

$$\log g_{z\bar{z}}^t = 2(h_2/h)^2\log|t| + \log g_{z\bar{z}}^{(1)} - 4(h_2/h)\log G^{(1)}(z,p) + o(1),$$

for $z\in M_1$ and $G^{(1)}(z,w)$ the Arakelov–Green's function for M_1. A similar expression holds for $z\in M_2$.

Recall that in (2.15), the conformal factor near p relates the above metric to the Euclidean metric, and away from p the relation is with respect to $g^{(1)}$. To simplify notation, we write g for $g^{(1)}$ and $\hat{g}$ for the Euclidean metric.

For z near the node p, we use $G^{(1)}(z,p)\sim|z|$, in local coordinates.

$$\sigma_t(z) = -(h_2/h)\log|z|^2 + \cdots, \tag{3.1}$$
$$\partial_z\sigma_t(z) = -(h_2/h)(1/z) + \cdots. \tag{3.2}$$

Away from p, we have,

$$\sigma_t(z) = (h_2/h)^2\log|t| - 2(h_2/h)\log G^{(1)}(z,p) + \cdots, \tag{3.3}$$

$$\partial_z\partial_{\bar{z}}\sigma_t(z) = -2\pi i(h_2/h)\mu_{z\bar{z}}^{(1)} + \cdots, \tag{3.4}$$

where $\mu^{(1)}$ is the canonical metric on the surface M_1.

Near p we compute, using (3.1) and (3.2), to order ϵ,

$$-\frac{1}{12\pi}\int_{A_\epsilon^1} d^2\xi\,\sqrt{\hat{g}}\hat{g}^{ab}\partial_a\sigma\,\partial_b\sigma = -\frac{1}{3\pi}\int_0^{2\pi}d\theta\int_{|t|^{1/2}}^{\epsilon} r\,dr\left(\frac{h_2}{h}\right)^2\frac{1}{r^2}$$

$$=\frac{1}{3}\left(\frac{h_2}{h}\right)^2\log|t|-\frac{2}{3}\left(\frac{h_2}{h}\right)^2\log\epsilon,$$

$$-\frac{1}{6\pi}\int_{\partial A_\epsilon^1}d\hat{s}\,k_{\hat{g}}\sigma=-\frac{1}{6\pi}\int_0^{2\pi}d\theta\{\sigma(\epsilon)-\sigma(|t|^{1/2})\}$$

$$=\frac{2}{3}\frac{h_2}{h}\log\epsilon-\frac{1}{3}\frac{h_2}{h}\log|t|.$$

The other terms in the Liouville action are order ϵ. Away from p, we use (3.3) and the expression for the curvature of the Arakelov metric $\partial_z\partial_{\bar{z}}\log g^{z\bar{z}}=4\pi i(h-1)\mu_{z\bar{z}}$:

$$-\frac{1}{6\pi}\int_{R_\epsilon^1}dv\,K\sigma=\frac{2}{3}(h_1-1)\int_{R_\epsilon^1}\mu_1\left\{\left(\frac{h_2}{h}\right)^2\log|t|-2\frac{h_2}{h}\log G^{(1)}(z,p)\right\}$$

$$\to\frac{2}{3}\frac{h_1h_2^2}{h^2}\log|t|-\frac{2}{3}\left(\frac{h_2}{h}\right)^2\log|t|,$$

as $\epsilon\to0$,

by the normalization of the Green's function:

$$\int_M\mu(z)\log G(z,w)=0.$$

We also have,

$$-\frac{1}{6\pi}\int_{\partial R_\epsilon^1}ds\,k_g\sigma=\frac{1}{3}\left\{\left(\frac{h_2}{h}\right)^2\log|t|-2\frac{h_2}{h}\log\epsilon\right\}\quad\text{as }\epsilon\to0.$$

And finally, using (3.4),

$$-\frac{1}{12\pi}\int_{R_\epsilon^1}d^2\xi\sqrt{g}g^{ab}\partial_a\sigma\,\partial_b\sigma=-\frac{1}{12\pi}4\left(\frac{h_2}{h}\right)^2\int_{R_\epsilon^1}dv(z)g^{ab}\partial_a\log G^{(1)}(z,p)\partial_b\log G^{(1)}(z,p)$$

$$=\frac{1}{3\pi}\left(\frac{h_2}{h}\right)^2\int_{R_\epsilon^1}dv(z)\log G^{(1)}(z,p)\Delta_g\log G^{(1)}(z,p)$$

$$-\frac{1}{3\pi}\left(\frac{h_2}{h}\right)^2\int_{\partial R_\epsilon^1}dn^\mu\log G^{(1)}(z,p)\partial_\mu\log G^{(1)}(z,p).$$

The first term vanishes as $\epsilon\to0$ by $\partial_z\partial_{\bar{z}}\log G(z,w)=\pi i\mu_{z\bar{z}}$, $z\neq w$, and the above normalization. The second term is

$$\sim\frac{1}{3\pi}\left(\frac{h_2}{h}\right)^2\int_0^{2\pi}\epsilon\,d\theta\frac{1}{\epsilon}\log G^{(1)}(z,p)\sim\frac{2}{3}\left(\frac{h_2}{h}\right)^2\log\epsilon.$$

The computations for R_ϵ^2 are identical. Adding these together, we have

$$S_L(\sigma_t,R_\epsilon^1)+S_L(\sigma_t,R_\epsilon^2)+S_L(\sigma_t,A_\epsilon^1)+S_L(\sigma_t,A_\epsilon^2)$$

$$=\frac{2}{3}\frac{h_1h_2}{h}\log|t|-\frac{1}{3}\log|t|+o(\epsilon,t).\tag{3.5}$$

Furthermore, it is easy to see that $S_L(\sigma_1,D_\epsilon^1)$, and $S_L(\sigma_2,D_\epsilon^2)$ are also $o(\epsilon)$.

B. Asymptotics of the determinant

Combining the results of (2.15), (2.16), and (3.5), it follows that for g_t the Arakelov metric,

$$\lim_{t\to0}\log\frac{\det'\Delta_{g_t}}{\text{Area}(M_t,g_t)}-\frac{2}{3}\frac{h_1h_2}{h}\log|t|$$

$$=\log\frac{\det'\Delta_{g_1}}{\text{Area}(M_1,g_1)}+\log\frac{\det'\Delta_{g_2}}{\text{Area}(M_2,g_2)}-\frac{c_0}{6},\tag{3.6}$$

where $c_0=-24\zeta'(-1)-6\log2\pi-2\log2-5$.

The Faltings invariant is related to the zeta regularized determinant with respect to the Arakelov metric by[2,9]

$$\delta(M)=c_h-6\log[\det'\Delta_g/\text{Area}(M,g)],\tag{3.7}$$

where c_h is constant depending only on the genus. Our previous results on the degeneration of $\delta(M)$ show that, under the degeneration above,[5]

$$\lim_{t\to0}\delta(M_t)+4(h_1h_2/h)\log|t|=\delta(M_1)+\delta(M_2).$$

Comparing this with (3.6) allows us to evaluate c_h,

$$c_h=hc_1+(1-h)c_0,\tag{3.8}$$

where c_0 is as above, and c_1 is the constant in the elliptic case. This may be evaluated explicitly (see Ref. 15 for determinants on the torus and Ref. 8 for the Arakelov metric): $c_1=-8\log2\pi$.

The constant c_h is related to the absolute constants appearing in bosonization formulas.[9-11] For example, the spin-1 formula reads

$$\|\vartheta\|(p_1+\cdots+p_{h-1}+x-y-\Delta)$$

$$=e^{-c_h/8}\left(\frac{\det'\Delta_g}{\det\text{Im}\,\Omega\,\text{Area}(M,g)}\right)^{3/4}\|\det\omega_j(p_k)\|\times\frac{\Pi_{j=1}^hG(p_j,y)}{\Pi_{j<k}G(p_j,p_k)}.$$

APPENDIX

Here we give an example of the "sewing" property (2.5) for functional integrals on Riemann surfaces. The glueing of the two ends of a cylinder to obtain a torus was shown in Ref. 6 and other examples in the plane can be found in Ref. 13.

We consider the example of glueing two disks to obtain a sphere.

Let M be the sphere of radius 1 (regarded as $\mathbb{C}\cup\{\infty\}$) with constant curvature metric g given by

$$d^2s = 4[|dz|^2/(1+|z|^2)^2].$$

The conformal factor relating g to the Euclidean metric $\hat{g}$, defined by $g = e^{2\sigma}\hat{g}$, is

$$\sigma = \log 2 - \log(1+|z|^2).$$

From (2.3) we have

$$(2\pi \det{}' \Delta_g)^{-1/2} = \int \mathscr{D}\phi^* \, e^{-I(\phi)}. \tag{A1}$$

In the local coordinate z, we cut out a disk $B(R) = \{|z| < R\}$. We must evaluate $\det_D \Delta_g(B(R))$ where D denotes the Dirichlet problem. From Sec. II G, this reduces to evaluating the Euclidean determinant and the Liouville action (2.14). This is easily done—the result is

$$S_L(\sigma,R) = -\frac{1}{3}\log 2 + \frac{R^2}{1+R^2} - \frac{1}{3(1+R^2)} + \frac{1}{3}.$$

Substituting R for ϵ in (2.16), we have

$$\det{}_D \Delta_g(B(R)) = (2\pi)^{-1/2} R^{-1/3} \exp\left\{-2\zeta'(-1) + \frac{R^2}{1+R^2} - \frac{1}{3(1+R^2)} - \frac{1}{12}\right\}. \tag{A2}$$

For the disk $M - B(R)$, we use the coordinate $1/z$ to obtain

$$\det{}_D \Delta_g(B(1/R)) = (2\pi)^{-1/2} R^{1/3} \exp\left\{-2\zeta'(-1) + \frac{1}{1+R^2} - \frac{R^2}{3(1+R^2)} - \frac{1}{12}\right\}. \tag{A3}$$

We now decompose the path integral as in Sec. III D:

$$\begin{aligned}\int \mathscr{D}\phi^* \, e^{-I(\phi)} &= \int \mathscr{D}\phi_1 \, \mathscr{D}\phi_2 \, \mathscr{D}\alpha^* \, e^{-I(\phi_1) - I(\phi_2)} \\ &= [\det{}_D \Delta_g(B(R)) \det{}_D \Delta_g(B(1/R))]^{-1/2} \\ &\quad \times \int \mathscr{D}\alpha^* \exp\left\{-\frac{1}{8\pi}\int_{\partial B(R)} dn^\mu \chi_1 \partial_\mu \chi_1 \right. \\ &\quad \left. - \frac{1}{8\pi}\int_{\partial B(1/R)} dn^\mu \chi_2 \partial_\mu \chi_2 \right\}.\end{aligned}$$

The harmonic integral is evaluated as in Sec. II E. The answer is simply $(2\pi)^{-1}$. Putting this together with (A1), (A2), and (A3), we have for the determinant on the sphere,

$$\det{}' \Delta_g = \exp(-4\zeta'(-1) + 1/2),$$

which is the result obtained directly by evaluating the zeta function for the eigenvalues of the sphere (see Ref. 16).

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