LIPSCHITZ ANALYSIS OF GENERALIZED PHASE RETRIEVABLE MATRIX FRAMES*

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Abstract. The classical phase retrieval problem arises in contexts ranging from speech recognition to x-ray crystallography and quantum state tomography. The generalization to $U(r)$ phase retrieval of matrix frames is natural in the sense that it corresponds to quantum tomography of impure states. We provide computable global stability bounds for the quasi-linear analysis map β and a path forward for understanding related problems in terms of the differential geometry of key spaces. In particular, we manifest a Whitney stratification of the positive semidefinite matrices of low rank which allows us to "stratify" the computation of the global stability bound. We show that for the impure state case no such global stability bounds can be obtained for the non-linear analysis map α with respect to certain natural distance metrics. Finally, our computation of the global lower Lipschitz constant for the β analysis map provides novel conditions for a matrix frame to be generalized phase retrievable when $r > 1$.

Key words. Phase Retrieval, Generalized Phase Retrieval, Low Rank Matrix Analysis

AMS subject classifications. 42C15, 15B48, 30L05

1. Introduction.

Let $H = \mathbb{C}^{n\times r}$ with $n \geq r$ be the Hilbert space of tall matrices with complex entries, equipped with the real inner product $\langle z, w\rangle_{\mathbb{R}} = \Re\mathrm{tr}\{z^*w\}$, where z^* denotes the transpose complex conjugate of z (the hermitian conjugate). We denote by $\langle z, w\rangle_{\mathbb{C}} = \mathrm{tr}\{z^*w\}$ the complex inner product and by $\mathrm{Ran}(z) = \{zu | u \in \mathbb{C}^r\}$ the range of z as an operator $z : \mathbb{C}^r \to \mathbb{C}^n$. Let $\mathbb{C}^{n\times r}_*$ be the open subset of $\mathbb{C}^{n\times r}$ consisting of full rank tall matrices. For $p \geq 1$ we denote by $||z||_p$ the pth Schatten norm of z, that is to say the l_p norm of the singular values of z. The pseudo-inverse of z will be denoted $z^\dagger$. Let $U(r)$ be the Lie group of $r \times r$ matrices with entries in $\mathbb{C}$ satisfying $U^*U = \mathbb{I}$. We denote by $\mathbb{C}^{n\times r}/U(r)$ and $\mathbb{C}^{n\times r}_*/U(r)$ the set of equivalence classes in $\mathbb{C}^{n\times r}$ and $\mathbb{C}^{n\times r}_*$ respectively under the equivalence relation $z \sim w$ if and only if there exists $U \in U(r)$ such that $z = wU$. Let $S^{p,q}(\mathbb{C}^n)$ denote the set of symmetric operators (hermitian matrices) on $\mathbb{C}^n$ having at most p positive and q negative eigenvalues, and $\mathring{S}^{p,q}(\mathbb{C}^n)$ the set of symmetric operators (hermitian matrices) on $\mathbb{C}^n$ having exactly p positive and q negative eigenvalues. The set $\mathbb{C}^{n\times r}/U(r)$ may then be identified with $S^{r,0}(\mathbb{C}^n)$ and $\mathbb{C}^{n\times r}_*/U(r)$ with $\mathring{S}^{r,0}(\mathbb{C}^n)$ via Cholesky decomposition. Being a finite dimensional space, a *frame* for $\mathbb{C}^{n\times r}$ is a collection $\{f_j\}_{j=1}^m \subset \mathbb{C}^{n\times r}$ that spans $\mathbb{C}^{n\times r}$. In particular, $\{f_j\}_{j=1}^m$ is frame if and only if there exist $A, B > 0$ (called *frame bounds*) satisfying $A||z||_2^2 \leq \sum_{j=1}^m |\langle f_j, z\rangle_{\mathbb{R}}|^2 \leq B||z||_2^2$ for all $z \in \mathbb{C}^{n\times r}$. This condition may also be written $A||z||_2^2 \leq \sum_{j=1}^m \langle A_j, zz^*\rangle_{\mathbb{R}} \leq B||z||_2^2$ for all $z \in \mathbb{C}^{n\times r}$ where $A_j = f_jf_j^*$. Using this fact, we may extend the concept of a frame for $\mathbb{C}^{n\times r}$ to collections of symmetric matrices $\{A_j\}_{j=1}^m \subset \mathrm{Sym}(\mathbb{C}^n)$. Fix a frame for $\mathbb{C}^{n\times r}$, then that frame is called *generalized phase retrievable* if the following map is injective:

$$\begin{aligned} &\beta : \mathbb{C}^{n\times r}/U(r) \to \mathbb{R}^m \\ &\beta_j(z) = \langle A_j, zz^*\rangle_{\mathbb{R}}, \qquad j = 1, \ldots, m \end{aligned} \tag{1.1}$$

This definition is in agreement with the generalized phase retrieval problem laid out in [27] for the case $r = 1$. Note that if $A_j = f_jf_j^*$ then $\beta_j(z) = ||f_j^*z||_2^2$. A breadth of

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*Submitted to the editors 7/20/2021.

Funding: This work was supported in part by NSF under Grant DMS-1816608.

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literature exists on the classical phase retrieval problem where $r = 1$ and $H = \mathbb{C}^n$ or $H = \mathbb{R}^n$, see for example [4] for an explicit construction of Parseval phase retrievable frames and [1] for a proof of the stability of finite dimensional phase retrievability under perturbation of the frame vectors (in contrast to the finite dimensional case, it is shown in [10] that infinite dimensional phase retrieval is never stable). Probabilistic error bounds for the case of noisy phase retrieval may be found in [14] for frames sampled from a subgaussian distribution satisfying a so called "small ball" assumption. Efficient algorithms exist for doing classical phase retrieval (for example via Wirtinger flow as in [12]), as well for constructing frames with desirable properties (nearly tight with low coherence) as in [13]. See for example [25] for an analysis of the stability statistics for random frames and [21] for the interesting result that a large class of "non-peaky" vectors (so called μ-flat vectors) are recoverable even when frame vectors are chosen as Bernoulli random vectors, a case in which phase retrieval is well known to fail for arbitrary signals. Recently several advances have been made in understanding natural generalizations of the problem to arbitrary symmetric measurement matrices [27], unifying the problem of phase retrieval with that of fusion frame reconstruction. Lipschitz stability questions for the generalized phase retrieval are analyzed in [31]. The generalized phase retrieval problem in the case $r = 1$ has proven amenable to efficient implementations of gradient descent [22] and a probabilistic guarantee of global convergence of first order methods like gradient descent has been obtained in [23] for $O(n \log^3(n))$ frame vectors. In accordance with the classical phase retrieval we also define the α map as the entry-wise square root of the beta map (here we require that each $A_j \geq 0$):

$$\begin{aligned} &\alpha : \mathbb{C}^{n\times r}/U(r) \to \mathbb{R}^m \\ &\alpha_j(z) = \langle A_j, zz^* \rangle_{\mathbb{R}}^{\frac{1}{2}}, \qquad j = 1, \ldots, m \end{aligned} \tag{1.2}$$

Note that if we write $A_j = f_j f_j^*$ using Cholesky decomposition then $\alpha_j(z) = ||f_j^* z||_2$. In this paper we will study the global and local Lipschitz properties of these two maps in the case that the frame is generalized phase retrievable. In particular, we analyze the following (squared) global Lipschitz constants:

$$a_0 := \inf_{\substack{x,y \in \mathbb{C}^{n\times r}\\ x\neq y}} \frac{||\beta(x) - \beta(y)||_2^2}{||xx^* - yy^*||_2^2} \quad , \quad b_0 := \sup_{\substack{x,y \in \mathbb{C}^{n\times r}\\ x\neq y}} \frac{||\beta(x) - \beta(y)||_2^2}{||xx^* - yy^*||_2^2} \tag{1.3}$$

$$A_0 := \inf_{\substack{x,y \in \mathbb{C}^{n\times r}\\ x\neq y}} \frac{||\alpha(x) - \alpha(y)||_2^2}{||(xx^*)^{\frac{1}{2}} - (yy^*)^{\frac{1}{2}}||_2^2} \quad , \quad B_0 := \sup_{\substack{x,y \in \mathbb{C}^{n\times r}\\ x\neq y}} \frac{||\alpha(x) - \alpha(y)||_2^2}{||(xx^*)^{\frac{1}{2}} - (yy^*)^{\frac{1}{2}}||_2^2} \tag{1.4}$$

In doing so we will employ several distance metrics on $\mathbb{C}^{n\times r}/U(r)$ (equivalently on $S^{r,0}(\mathbb{C}^n)$), the relationships between which are contained in Theorem 3.7. The Lipschitz properties of α and β are intimately related to the geometry of $S^{r,0}(\mathbb{C}^n)$, which is the subject of Theorem 4.5. Theorem 4.5 continues the results in [8] on the geometry of the $n \times n$ positive definite matrices $\mathbb{P}(n)$. The main contributions of this work are thus:

- In Section 3 we introduce the novel distance

$$d(x, y) := \sqrt{(||x||_2^2 + ||y||_2^2)^2 - 4||x^*y||_1^2} \tag{1.5}$$

  on $\mathbb{C}^{n\times r}/U(r)$ and in Theorem 3.7 provide optimal Lipschitz constants with respect to natural embeddings of $(\mathbb{C}^{n\times r}/U(r), d)$ into the Euclidean space

$(\mathrm{Sym}(\mathbb{C}^n), ||\cdot||_2)$. This new distance metric allows us in 5.6 to compute local lower Lipschitz constants for the β map generalizing those in Theorem 2.5 of [6]. 3.7 also provides optimal Lipschitz constants with respect to natural embeddings of $(\mathbb{C}^{n\times r}/U(r), D)$ into $(\mathrm{Sym}(\mathbb{C}^n), ||\cdot||_2)$ for the Bures-Wasserstein distance $D(x,y) := \sqrt{||x||_2^2 + ||y||_2^2 - 2||x^*y||_1}$.

- In Section 4 Theorem 4.5 generalizes Theorem 5 in [8] by providing the geometry not just of manifold of positive definite matrices $\mathbb{P}(n)$ but of the algebraic semi-variety $S^{r,0}(\mathbb{C}^n)$. In particular we manifest a Whitney stratification of $S^{r,0}(\mathbb{C}^n)$, obtain the Riemannian metrics of the stratifying manifolds, and show that this family of metrics is compatible across the strata in the sense that geodesics of lower strata are limiting curves of geodesics in higher strata. In particular this proves that the geodesic in $S^{r,0}(\mathbb{C}^n)$ connecting two matrices of rank $k < r$ is completely contained in $\mathring{S}^{k,0}(\mathbb{C}^n)$. This stratification of the low rank positive-semidefinite matrices is crucial in simplifying the computation of the global lower Lipschitz bounds for β and α in Theorems 5.6 and 5.9 respectively.
- In Section 5 Theorem 5.6 provides an explicit formula for the global lower bound a_0 as the minimization over $U(n)$ of the $(2nr - r^2)$th eigenvalue of a family of matrices parametrized by $U(n)$. Theorem 5.6 also uses the distance d to provide a generalization of Theorem 2.5 in [6] to the case $r > 1$ and shows that the analog $\hat{Q}_z$ of $\mathcal{R}(\xi)$ can be used to control a_0 to within a factor of 2. We also show in Theorem 5.9 that the corresponding generalization of Theorem 2.2 in [6] to the case $r > 1$ is false, namely that $A_0 = 0$ when $r > 1$. Thus in the case $r > 1$ the more recently introduced β map (the entry-wise square of the α map) is a more natural and well behaved analysis map for generalized phase retrieval, owing primarily to the fact that it lifts to a linear map on the low rank positive semi-definite matrices. It should be noted that Theorem 5.9 does not rule out the possibility of a better distance metric with respect to which α is globally lower Lipschitz. Finally, in Theorem 5.14 we provide novel conditions for a frame $\{A_j\}_{j=1}^m$ for $\mathbb{C}^{n\times r}$ to be generalized phase retrievable.

A motivating example for the Lipschitz analysis of α and β is quantum tomography of impure states. A noisy quantum system is modeled as a statistical ensemble over pure quantum states. The standard example is unpolarized light. In such cases, all of the measurable information in the system is contained in a density matrix which, using bra-ket notation, has the form

$$\rho = \sum_{j\in\mathcal{I}} p_j |\psi_j\rangle\langle\psi_j| \tag{1.6}$$

where p_j is the ensemble probability that the system is in the pure quantum state $|\psi_j\rangle$ belonging to a Hilbert space H. If we assume the cardinality of $\mathcal{I}$ is finite and equal to r and that the state vectors themselves live in the Hilbert space $\mathbb{C}^n$ then $\rho \in S^{r,0}(\mathbb{C}^n) \cap \{x \in \mathrm{Sym}(\mathbb{C}^n) | \mathrm{tr}\{x\} = 1\}$. The expectation of a given observable A (a symmetric operator on $\mathbb{C}^n$) is therefore

$$\mathbb{E}_\rho[A] = \sum_{j\in\mathcal{I}} p_j \langle\psi_j|A|\psi_j\rangle = \sum_{j\in\mathcal{I}} p_j \mathrm{tr}\{|\psi_j\rangle\langle\psi_j|A\} = \mathrm{tr}\{\rho A\} = \Re\mathrm{tr}\{\rho A\} \tag{1.7}$$

By repeatedly measuring the observable A and then allowing the quantum system to relax one may estimate $\mathrm{tr}\{\rho A\}$ (and perhaps higher moments) but the aim is to infer ρ

itself. It was shown in [16] that sufficiently many randomly sampled Pauli observables can be used along with methods from compressed sensing (trace minimization, matrix Lasso) to reconstruct a low rank density matrix with high fidelity. In general, if a suite of observables is well-chosen (constitutes a generalized phase-retrievable frame) then the problem of inferring ρ from the expectation values of said observables is subordinate to the problem of phase retrieval on $\mathbb{C}^{n\times r}$. Asking if, for a collection of observables $\{A_j\}_{j=1}^m$, the density matrix ρ is recoverable is equivalent to asking if the map

$$\begin{aligned} &\tilde{\beta} : S^{r,0}(\mathbb{C}^n) \cap \{x \in \mathrm{Sym}(\mathbb{C}^n) | \mathrm{tr}\{x\} = 1\} \to \mathbb{R}^m \\ &\tilde{\beta}(\rho) = \begin{bmatrix} \langle \rho, A_1 \rangle_{\mathbb{R}} \\ \vdots \\ \langle \rho, A_m \rangle_{\mathbb{R}} \end{bmatrix} \end{aligned} \tag{1.8}$$

is injective. In fact, given that we can only approximate the expectations using finitely many measurements, we should hope that it is lower Lipschitz with respect to the Frobenius distance. Such stability questions for phase retrievable frames for $\mathbb{C}^n$ (the pure state case) are investigated in [1]. Given that ρ is positive semidefinite and rank at most r there exists a Cholesky factor $z \in \mathbb{C}^{n\times r}$ such that $\rho = zz^*$. Indeed we may take $z \in \mathbb{C}^{n\times r}/U(r)$ since ρ is invariant under $z \to zU$, in which case $\mathrm{tr}\{\rho\} = 1$ if and only if $||z||_2 = 1$. We may therefore concern ourselves with the Lipschitz properties of β restricted to $z \in \mathbb{C}^{n\times r}/U(r)$ with $||z||_2 = 1$, rather than $\tilde{\beta}$. For the time being we consider a Lipschitz analysis of $\beta : \mathbb{C}^{n\times r}/U(r) \to \mathbb{R}^m$, deferring discussion of a possible Lipschitz retract onto the unit sphere. Thus we seek information on the optimal global lower Lipschitz constant of the β map, namely $\sqrt{a_0}$. In the above example if $a_0 > 0$ this means that if we can measure each $E_\rho[A_j]$ to within error $\epsilon > 0$ then we can obtain an approximation $\hat{\rho}$ to ρ that satisfies

$$||\rho - \hat{\rho}||_2 \leq \frac{\epsilon\sqrt{m}}{\sqrt{a_0}} \tag{1.9}$$

In addition to quantum state tomography, Lipschitz analysis of spaces of low-rank matrices is central in a significant number of problems in science and engineering such as: the phase retrieval problem [4, 28], source separation and inverse problems [15], as well as the low-rank matrix completion problem [11].

We caution the reader that throughout the paper the scalar product $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ is a real inner product, however z^* denotes the conjugate with respect to the complex inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$. We also note that the norm $||z||_p$ for $p \geq 1$ is the pth Schatten norm of $z \in \mathbb{C}^{n\times r}$ seen as a $\mathbb{C}$-linear operator from $\mathbb{C}^r$ to $\mathbb{C}^n$. Hence the norm $||\cdot||_2$, while it refers to the Schatten 2 norm, is equivalently given as $||z||_2 = \sqrt{\langle z, z\rangle_{\mathbb{R}}} = \sqrt{\langle z, z\rangle_{\mathbb{C}}}$. If z were instead seen as an $\mathbb{R}$-linear operator from $\mathbb{C}^r$ to $\mathbb{C}^n$ then the resulting Schatten p norm would be amplified by a factor $2^{\frac{1}{p}}$ since the multiplicity of each singular value would double.

2. A review of quantitative phase retrievability. The question of phase retrievability criteria for frames for $\mathbb{R}^n$ was addressed in [4], in which it was shown that a frame $\mathcal{F}$ is phase retrievable if and only if it satisfies the "complementing property," that is if and only if for every subset $\mathcal{I} \subset \mathcal{F}$ either $\mathcal{I}$ or $\mathcal{F}\backslash\mathcal{I}$ spans $\mathbb{R}^n$. It was moreover shown in [4] that if $m < 2n-1$ then a frame for $\mathbb{R}^n$ of cardinality m will not be phase retrievable and also that a generic frame for $\mathbb{R}^n$ of size $m \geq 2n-1$ will be phase retrievable – that is to say the set $\{\mathcal{F} = \{f_1, \dots, f_m\} \subset \mathbb{R}^n | \mathcal{F} \text{ is phase retrievable}\}$ will be dense in the Zariski topology when $m \geq 2n - 1$. The question of phase retrievability *criteria* can be made quantitative by asking for which frames the analysis maps α and β are lower Lipschitz with respect to some natural distance metrics, and computing their lower Lipschitz constants. Intuitively, a frame is phase retrievable if and only if α (resp. β) is injective, thus it is natural to analyze (for a given frame) the lower Lipschitz constant of α (resp. β), which measures"how" injective α (resp. β) is. In answer to this refinement it was shown in [5] that for the α map and the distance $\rho(x, y) = \min\{||x - y||_2, ||x + y||_2\}$ we have:

THEOREM 2.1. *(See [5] Theorem 4.3.) For any index set $I \subset \{1, \dots, m\}$ let $\mathcal{F}[I] = \{f_k | k \in I\}$ and let $\sigma_1^2[I] = \lambda_{max}\Big(\sum_{k\in I} f_k f_k^*\Big)$ and $\sigma_n^2[I] = \lambda_{min}\Big(\sum_{k\in I} f_k f_k^*\Big)$. Then*

$$A_0 := \inf_{\substack{x,y\in\mathbb{R}^n \\ x\nsim y}} \frac{||\alpha(x) - \alpha(y)||_2^2}{\rho(x, y)^2} = \min_{I\subset\{1,\dots,m\}} \sigma_n^2[I] + \sigma_n^2[I^C] \tag{2.1}$$

This result implies in particular that for a phase retrievable frame for $\mathbb{R}^n$ the α map is globally lower Lipschitz. An analogous result was given in [5] for the β map and the distance $||xx^T - yy^T||_1$:

THEOREM 2.2. *(See [5] Theorem 2.1.) Let $\{f_j\}_{j=1}^m$ be a phase retrievable frame for $\mathbb{R}^n$ and let $R : \mathbb{R}^n \to Sym(\mathbb{R}^n)$ be given by $R(x) = \sum_{j=1}^m |\langle x, f_j\rangle|^2 f_j f_j^T$. Then*

$$a_0 := \inf_{\substack{x,y\in\mathbb{R}^n \\ x\nsim y}} \frac{||\beta(x) - \beta(y)||_2^2}{||xx^T - yy^T||_1^2} = \min_{\substack{x\in\mathbb{R}^n \\ ||x||_2=1}} \lambda_n(R(x)) > 0 \tag{2.2}$$

Regarding the complex case the following phase retrievability criterion was obtained in [7]:

THEOREM 2.3. *(See [7] Theorem 4.) Let $\{f_j\}_{j=1}^m$ be a frame for $\mathbb{C}^n$. For $u \in \mathbb{C}^n$ denote $S(u) = span_{\mathbb{R}}\{f_j f_j^* u\}_{j=1}^m$. Then the following are equivalent:*

(i) The frame $\{f_j\}_{j=1}^m \subset \mathbb{C}^n$ is phase retrievable.
(ii) $\dim_{\mathbb{R}} S(u) \geq 2n - 1$ for every $u \in \mathbb{C}^n \setminus \{0\}$.
(iii) $S(u) = span_{\mathbb{R}}\{iu\}^\perp$ for every $u \in \mathbb{C}^n \in \setminus\{0\}$.

In connection to this paper we note that the above result is extended to the case of generalized retrievability of frames for $\mathbb{C}^{n\times r}$ by Theorem 5.14. The quantitative lower Lipschitz variant of Theorem 2.3 was obtained for the β analysis map in [6], in which it was proved that for the beta map:

THEOREM 2.4. *(See [6] Theorem 2.3 and Theorem 2.5.) Let $\{f_j\}_{j=1}^m$ be a phase retrievable frame for $\mathbb{C}^n$. Define $\mathcal{R} : \mathbb{R}^{2n} \to Sym(\mathbb{R}^{2n})$ via $\mathcal{R}(\xi) = \sum_{j=1}^m \Phi_j \xi \xi^T \Phi_j$ where $\Phi_j = \phi_j \phi_j^T + J\phi_j \phi_j^T J^T$, $\phi_j = \begin{bmatrix} \Re f_j \\ \Im f_j \end{bmatrix}$ and J is the symplectic form $\begin{bmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix}$. Then*

$$a_0 := \inf_{\substack{x,y\in\mathbb{C}^n \\ x\nsim y}} \frac{||\beta(x) - \beta(y)||_2^2}{||xx^* - yy^*||_1^2} = \min_{\substack{\xi\in\mathbb{R}^{2n} \\ ||\xi||_2=1}} \lambda_{2n-1}(\mathcal{R}(\xi)) > 0 \tag{2.3}$$

The connection of the above to Theorem 2.3 is that the null space of $\mathcal{R}(\xi)$ includes the realification of $\text{span}_{\mathbb{R}}\{i\xi\}$ for every ξ. Theorem 2.4 is extended to the case of generalized phase retrievability of frames for $\mathbb{C}^{n\times r}$ by Theorem 5.6.

3. Relevant distances and Lipschitz embeddings.

DEFINITION 3.1. *We define the equivalence relation* $\sim$ *on* $\mathbb{C}^{n\times r}$ *via*

$$x \sim y \iff \exists U \in U(r) | x = yU \tag{3.1}$$

and denote by $[x]$ *the equivalence class of* $x \in \mathbb{C}^{n\times r}$*, and by* $\mathbb{C}^{n\times r}/U(r)$ *the collection of equivalence classes* $\{[x] | x \in \mathbb{C}^{n\times r}\}$.

The stability analysis that follows for β and α in Theorems 5.6 and 5.9 will rely heavily on the following natural metrics on $\mathbb{C}^{n\times r}/U(r)$.

DEFINITION 3.2. *We define* $D, d : \mathbb{C}^{n\times r} \times \mathbb{C}^{n\times r} \to \mathbb{R}$.

$$\begin{aligned} D(x,y) &= \min_{U\in U(r)} ||x - yU||_2 \\ &= \sqrt{||x||_2^2 + ||y||_2^2 - 2||x^*y||_1} \\ d(x,y) &= \min_{U\in U(r)} ||x - yU||_2 ||x + yU||_2 \\ &= \sqrt{(||x||_2^2 + ||y||_2^2)^2 - 4||x^*y||_1^2} \end{aligned} \tag{3.2}$$

We note that another distance on $\mathbb{C}^{n\times r}/U(r)$ given by

$$\begin{aligned} D'(x,y) &= \max_{U\in U(r)} ||x - yU||_2 \\ &= \sqrt{||x||_2^2 + ||y||_2^2 + 2||x^*y||_1} \end{aligned} \tag{3.3}$$

and is introduced and analyzed for the $r = 1$ case in [19]. We note merely that $d = D \cdot D'$. This does not imply d is a metric, however in fact we have the following proposition.

PROPOSITION 3.3. *Both* D *and* d *are metrics in the usual sense on* $\mathbb{C}^{n\times r}/U(r)$.

Proof. See A.1. ∎

The proof of Proposition 3.3 relies on Lemma A.1, an apparently simple result about the analytic geometry of parallelepipeds in $\mathbb{R}^3$ which may be of independent interest.

The minimizer U can be chosen to be the same for both d and D, and is characterized by the following:

PROPOSITION 3.4. *The unitary minimizer in both* d *and* D *is given by the polar factor in* $x^*yU = |x^*y|$. *The minimizer will be unique so long as* x^*y *is full rank. Otherwise, the minimizer will be of the form* $U = U_0 + U_1$ *where* $U_0 = V_0 W_0^*$ *with* $V_0, W_0 \in \mathbb{C}^{r\times rank(x^*y)}$ *the matrices whose columns are the right and left singular vectors respectively of the non-zero singular values of* x^*y *and* $U_1 \in \mathbb{C}^{r\times r}$ *any matrix such that* $U_1U_1^* = \mathbb{P}_{\ker(x^*y)}$ *and* $U_1^*U_1 = \mathbb{P}_{Ran(x^*y)^\perp}$.

Proof. See A.2 ∎

The metrics d and D can be compared to the usual Euclidean distance on $\mathrm{Sym}(\mathbb{C}^n)$ modulo certain embeddings.

DEFINITION 3.5. *We define* $\theta, \pi, \psi : \mathbb{C}^{n\times r} \to S^{r,0}(\mathbb{C}^n)$ *as*

$$\begin{aligned} \theta(x) &= (xx^*)^{\frac{1}{2}} \\ \pi(x) &= xx^* = \theta(x)^2 \\ \psi(x) &= ||x||_2 (xx^*)^{\frac{1}{2}} = ||\theta(x)||_2 \theta(x) \end{aligned} \tag{3.4}$$

PROPOSITION 3.6. *The embeddings π, θ, and ψ are rank-preserving, surjective, and injective modulo $\sim$, thus we write $\theta, \pi, \psi : \mathbb{C}^{n\times r}/U(r) \hookrightarrow Sym(\mathbb{C}^n)$.*

Proof. See A.3 ∎

THEOREM 3.7. *Let $x, y \in \mathbb{C}^{n\times r}/U(r)$. Then*

(i) $\theta : (\mathbb{C}^{n\times r}/U(r), D) \to (S^{r,0}(\mathbb{C}^n), ||\cdot||_2)$ *is a bi-Lipschitz map. In particular,*

$$C_n||\theta(x) - \theta(y)||_2 \le D(x, y) \le ||\theta(x) - \theta(y)||_2 \tag{3.5}$$

where $C_n = 1$ if $n = 1$ and $C_n = \frac{1}{\sqrt{2}}$ for $n > 1$. The constants C_n and 1 are optimal.

(ii) $\pi : (\mathbb{C}^{n\times r}/U(r), d) \to (S^{r,0}(\mathbb{C}^n), ||\cdot||_1)$ *is 1-Lipschitz and* $\psi^{-1} : (S^{r,0}(\mathbb{C}^n), ||\cdot||_2) \to (\mathbb{C}^{n\times r}/U(r), d)$ *is 2-Lipschitz for $r > 2$ and $\sqrt{2}$-Lipschitz for $r = 1$. In particular,*

$$||\pi(x) - \pi(y)||_2 \le ||\pi(x) - \pi(y)||_1 \le d(x, y) \le c_r||\psi(x) - \psi(y)||_2 \tag{3.6}$$

where $c_r = \sqrt{2}$ if $r = 1$ and $c_r = 2$ if $r > 1$. The constants 1 and c_r are optimal.

(iii) *For $r = 1$*

$$\psi(x) = \pi(x) \tag{3.7}$$

$$d(x, y) = ||\pi(x) - \pi(y)||_1 \tag{3.8}$$

The identity (3.8) was noticed and used in [6], its proof is included here for the benefit of the reader.

(iv) *For $r > 1$, there is no constant C satisfying $C||\pi(x) - \pi(y)||_2 \ge d(x, y)$ for each $x, y \in \mathbb{C}^{n\times r}$ (hence the use of the alternate embedding ψ).*

Proof. See A.4 ∎

Remark 3.8. While d and D are evidently not Lipschitz equivalent (they scale differently), they do generate the same topology on $\mathbb{C}^{n\times r}/U(r)$ since $d(x, y) \le D(x, y)^2$ and given sufficiently small $\epsilon > 0$ we have $d(x, y) < ||x||\sqrt{\epsilon} \implies D(x, y) < \epsilon$.

4. Geometry of the matrix phase retrieval. It will be essential in the analysis and computation of (1.3) to understand the geometry of the spaces $S^{r,0}(\mathbb{C}^n)$. In order to do so, we will demonstrate that $S^{r,0}(\mathbb{C}^n)$ has a Whitney stratification over the smooth Riemannian manifolds $\mathring{S}^{i,0}(\mathbb{C}^n)$ for $i = 0, \ldots, r$ of real dimension $2ni - i^2$. We recall the following definitions, due to John Mather and sourced from [20]:

DEFINITION 4.1. *Let V_i, V_j be disjoint real manifolds embedded in $\mathbb{R}^d$ such that $\dim V_j > \dim V_i$ and $V_i \cap \overline{V_j}$ non-empty. Let $x \in V_i \cap \overline{V_j}$. Then a triple (V_j, V_i, x) is called a$-$ (resp. b$-$) regular if*

(a) *If a sequence $(y_n)_{n\ge1} \subset V_j$ converges to x in $\mathbb{R}^d$ and $T_{y_n}(V_j)$ converges in the Grassmannian $Gr_{\dim V_j}(\mathbb{R}^d)$ to a subspace τ_x of $\mathbb{R}^d$ then $T_x(V_i) \subset \tau_x$.*

(b) *If sequences $(y_n)_{n\ge1} \subset V_j$ and $(x_n)_{n\ge1} \subset V_i$ converge to x in $\mathbb{R}^d$, the unit vector $(x_n - y_n)/||x_n - y_n||_2$ converges to a vector $v \in \mathbb{R}^d$, and $T_{y_n}(V_j)$ converges in the Grassmannian $Gr_{\dim V_j}(\mathbb{R}^d)$ to a subspace τ_x of $\mathbb{R}^d$ then $v \in \tau_x$.*

DEFINITION 4.2. *Let V be a real semi-algebraic variety. A disjoint decomposition*

$$V = \bigsqcup_{i\in I} V_i, \qquad V_i \cap V_j = \emptyset \text{ for } i \ne j \tag{4.1}$$

into smooth manifolds $\{V_i\}_{i\in I}$, termed strata, is a Whitney stratification if

(a) Each point has a neighborhood intersecting only finitely many strata
(b) The boundary sets $\overline{V_j} \setminus V_j$ *of each stratum* V_j *are unions of other strata.*
(c) Every triple (V_j, V_i, x) *such that* $x \in V_i \subset \overline{V_j}$ *is a-regular and b-regular as in Definition 4.1.*

A simple example of a semi-algebraic variety that is not a manifold but admits a Whitney stratification is the cone $\mathcal{C} = \{(x,y)|xy \geq 0\} \subset \mathbb{R}^2$ consisting off the first and third quadrant of the coordinate plane. A possible Whitney stratification of this set is given by $V_0 = \{0\}$, $V_1 = \{(x,0)|x \neq 0\}$, $V_2 = \{(0,y)|y \neq 0\}$, and $V_3 = \{(x,y)|x \neq 0, y \neq 0\}$. In this case note that condition (a) is trivially satisfied since there are only finitely many strata, and moreover that (b) is satisfied since $\overline{V_3} \setminus V_3 = V_0 \cup V_1 \cup V_2$, $\overline{V_2} \setminus V_2 = V_0$, $\overline{V_1} \setminus V_1 = V_0$, and that $\overline{V_0} \setminus V_0 = \phi$ (an empty union of the other strata). That this stratification is both (a) and (b) regular may be readily observed. For example the tangent space at any point of V_3 is simply $\mathbb{R}^2$, and thus the Grassmanian limit of a convergent sequence of such tangent spaces is also $\mathbb{R}^2$ and certainly contains the one dimensional tangent space at any point of V_2 (identified with the y axis), the one dimensional tangent space at any point of V_1 (identified with the x axis), and the zero dimensional tangent space associated with V_0 (identified with the origin).

We will also need the following:

Definition 4.3. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be smooth manifolds and let* $\pi : \mathcal{M} \to \mathcal{N}$ *be a smooth map. For each* $x \in \mathcal{M}$ *let*

$$T_x(\mathcal{M}) := \{\gamma'(0)|\gamma : [-1,1] \to \mathcal{M} \text{ is a smooth curve with } \gamma(0) = x\} \tag{4.2}$$

be the tangent space of $\mathcal{M}$ *at* x*. Similarly for* $T_{\pi(x)}(\mathcal{N})$*. Let* $D\pi(x) : T_x(\mathcal{M}) \to T_{\pi(x)}(\mathcal{N})$ *be the differential of* π *at* x*, that is to say* $D\pi(x)(v) := \alpha'(0)$ *where* $\alpha = \pi \circ \gamma$, $\gamma(0) = x$*, and* $\gamma'(0) = v$ *(that* $D\pi(x)$ *does not depend on the exact choice of curve* γ *is an elementary result of differential geometry). Then*
(a) For each $x \in \mathcal{M}$ *define the vertical space at* x *as:*

$$V_{\pi,x}(\mathcal{M}) \subset T_x(\mathcal{M}) := \ker D\pi(x) = \{w \in T_x(\mathcal{M})|D\pi(x)(w) = 0\} \tag{4.3}$$

(b) If $\mathcal{M}$ *is equipped with a Riemannian metric* $g : \mathcal{M} \times T_x(\mathcal{M}) \times T_x(\mathcal{M}) \to \mathbb{R}$ *then we may define the horizontal space at each* x *via the canonical orthogonal complement of the vertical space:*

$$H_{\pi,x}(\mathcal{M}) \subset T_x(\mathcal{M}) := V_{\pi,x}(\mathcal{M})^{\perp} = \{v \in T_x(\mathcal{M})|g(x,v,w) = 0 \forall w \in V_{\pi,x}(\mathbb{C}_*^{n\times r})\} \tag{4.4}$$

The following proposition will be essential both in proving the geometric results in Theorem 4.5 and in the analysis of the Lipschitz constants for β and α set out in Theorems 5.6, 5.9, and 5.13:

Proposition 4.4. *Let* $\pi : \mathbb{C}_*^{n\times r} \to \mathring{S}^{r,0}(\mathbb{C}^n)$ *be as in Definition 3.5 and let* $V_{\pi,x}(\mathbb{C}_*^{n\times r})$ *and* $H_{\pi,x}(\mathbb{C}_*^{n\times r})$ *denote the vertical and horizontal spaces as in Definition 4.3 of the manifold* $\mathbb{C}_*^{n\times r}$ *at* x *with respect to the embedding* π*. Here the Riemmanian metric on* $\mathbb{C}_*^{n\times r}$ *is of course* $g : \mathbb{C}_*^{n\times r} \times \mathbb{C}^{n\times r} \times \mathbb{C}^{n\times r} \to \mathbb{R}$ *given by* $g(x,v,w) = \Re tr\{z^* w\}$*. Let* $T_{\pi(x)}(\mathring{S}^{r,0}(\mathbb{C}^n))$ *denote the tangent space of* $\mathring{S}^{r,0}(\mathbb{C}^n)$ *at*

$\pi(x)$. *Then*

$$V_{\pi,x}(\mathbb{C}_*^{n\times r}) = \{xK | K \in \mathbb{C}^{r\times r}, K^* = -K\} \tag{4.5}$$

$$H_{\pi,x}(\mathbb{C}_*^{n\times r}) = \{Hx + X | H \in \mathbb{C}^{n\times n}, H^* = H = \mathbb{P}_{Ran(x)}H, X \in \mathbb{C}^{n\times r}, \mathbb{P}_{Ran(x)}X = 0\} \tag{4.6}$$

$$T_{\pi(x)}(\mathring{S}^{r,0}(\mathbb{C}^n)) = \{W \in Sym(\mathbb{C}^n) | \mathbb{P}_{Ran(x)^\perp} W \mathbb{P}_{Ran(x)^\perp} = 0\} = D\pi(x)(H_{\pi,x}(\mathbb{C}_*^{n\times r})) \tag{4.7}$$

Proof. See B.1 ∎

Employing similar techniques to [8], but generalizing from the manifold of positive definite matrices to the semi-algebraic variety $S^{r,0}(\mathbb{C}^n)$ semidefinite matrices, we prove:

THEOREM 4.5. *Let π be as in Definition 3.5 and the distance D be as in (3.2). Then*

(i) $\mathring{S}^{p,q}(\mathbb{C}^n)$ *is a real analytic manifold for each $p, q > 0$ of real dimension $2n(p+q) - (p+q)^2$.*

(ii) $\pi : \mathbb{C}_*^{n\times r} \to \mathring{S}^{r,0}(\mathbb{C}^n)$ *can be made into a Riemannian submersion by choosing the following unique Riemannian metric on $\mathring{S}^{r,0}(\mathbb{C}^n)$:*

$$h(Z_1, Z_2) = tr\{Z_2^{\|} \int_0^\infty e^{-uxx^*} Z_1^{\|} e^{-uxx^*} du\} + \Re tr\{Z_1^{\perp *} Z_2^{\perp} (xx^*)^\dagger\} \tag{4.8}$$

Where $Z_1, Z_2 \in T_{\pi(x)}(\mathring{S}^{r,0}(\mathbb{C}^n))$, $(xx^*)^\dagger$ denotes the pseudo-inverse of xx^*, and

$$Z_i^{\|} = \mathbb{P}_{Ran(x)} Z_i \mathbb{P}_{Ran(x)} \qquad Z_i^{\perp} = \mathbb{P}_{Ran(x)^\perp} Z_i \mathbb{P}_{Ran(x)} \tag{4.9}$$

(iii) $\mathring{S}^{r,0}(\mathbb{C}^n)$ *equipped with the metric h is a Riemannian manifold with D as its geodesic distance.*

(iv) The semi-algebraic variety $S^{r,0}(\mathbb{C}^n)$ admits as an explicit Whitney stratification $(\mathring{S}^{i,0})_{i=0}^r$.

(v) The geometry associated to h is compatible with the Whitney stratification in the following sense: If $(A_i)_{i\geq 1}, (B_i)_{i\geq 1} \subset \mathring{S}^{p,0}$ have limits A and B respectively in $\mathring{S}^{q,0}$ for $q < p$ and if $\gamma_i : [0,1] \to \mathring{S}^{p,0}$ are geodesics in $\mathring{S}^{p,0}$ connecting A_i to B_i chosen in such a way that the limiting curve $\delta : [0,1] \to \overline{\mathring{S}^{p,0}}$ given by

$$\delta(t) = \lim_{i\to\infty} \gamma_i(t) \tag{4.10}$$

exists, then the image of δ lies in $\mathring{S}^{q,0}$ and is a geodesic curve in $\mathring{S}^{q,0}$ connecting A to B.

Proof. See B.2 ∎

5. Computation of Lipschitz bounds. We are primarily interested in computing a_0 and A_0, the squared global lower Lipschitz constants for the β and α analysis maps respectively. Owing to the linearity of the β analysis map when interpreted as in (1.8), we will be able to show in Theorem 5.6 that the optimal global lower Lipschitz bound a_0 can be obtained via local considerations. For the α analysis map we will be able to show in Theorem 5.9 that the optimal global lower Lipschitz bound A_0 is actually zero for $r > 1$. Since the global lower Lipschitz bound for the α analysis map is trivial we emphasize the analysis of the local lower Lipschitz bounds. Recall that

$$a_0 = \inf_{\substack{x,y\in\mathbb{C}^{n\times r}\\ [x]\neq[y]}} \frac{||\beta(x)-\beta(y)||_2^2}{||\pi(x)-\pi(y)||_2^2} = \inf_{\substack{x,y\in\mathbb{C}^{n\times r}\\ [x]\neq[y]}} \frac{\sum_{j=1}^m (\langle xx^*, A_j\rangle_{\mathbb{R}} - \langle yy^*, A_j\rangle_{\mathbb{R}})^2}{||xx^*-yy^*||_2^2} \tag{5.1}$$

From purely topological considerations, we may obtain

Proposition 5.1. *The constant a_0 is strictly positive whenever the map β is injective, equivalently whenever $\{A_j\}_{j=1}^m$ is a generalized phase retrievable frame of symmetric matrices.*

Proof. See C.1 ∎

Definition 5.2. *Let $z \in \mathbb{C}^{n\times r}$ have rank k. We will analyze the following four types of local lower Lipschitz bounds for β, the first two with respect to the norm induced metric and the second two with respect to the metric d:*

$$\begin{aligned}
a_1(z) &= \lim_{R\to 0} \inf_{\substack{x\in\mathbb{C}^{n\times r}\\ ||\pi(x)-\pi(z)||_2<R}} \frac{||\beta(x)-\beta(z)||_2^2}{||\pi(x)-\pi(z)||_2^2}\\
a_2(z) &= \lim_{R\to 0} \inf_{\substack{x,y\in\mathbb{C}^{n\times r}\\ ||\pi(x)-\pi(z)||_2<R\\ ||\pi(y)-\pi(z)||_2<R}} \frac{(||\beta(x)-\beta(y)||_2^2}{||\pi(x)-\pi(y)||_2^2}\\
\hat{a}_1(z) &= \lim_{R\to 0} \inf_{\substack{x\in\mathbb{C}^{n\times r}\\ d(x,z)<R\\ rank(x)\leq k}} \frac{||\beta(x)-\beta(z)||_2^2}{d(x,z)^2}\\
\hat{a}_2(z) &= \lim_{R\to 0} \inf_{\substack{x,y\in\mathbb{C}^{n\times r}\\ d(x,z)<R\\ d(y,z)<R\\ rank(x)\leq k\\ rank(y)\leq k}} \frac{||\beta(x)-\beta(y)||_2^2}{d(x,y)^2}
\end{aligned} \tag{5.2}$$

Note that in the definition of $\hat{a}_1(z)$ and $\hat{a}_2(z)$ we do not allow the ranks of x and y to exceed that of z. As we shall prove, without the rank constraints these local lower bounds would be zero.

The following two "geometric" local lower bounds will prove helpful in our analysis.

Definition 5.3. *Let $z \in \mathbb{C}^{n\times r}$ have rank k and let $\hat{z} \in \mathbb{C}_*^{n\times k}$ be such that there exists $U \in U(r)$ with $[\hat{z}|0]U = z$. Let $T_{\pi(\hat{z})}(\mathring{S}^{k,0}(\mathbb{C}^n))$ and $H_{\pi,\hat{z}}(\mathbb{C}_*^{n\times k})$ be as 4.7 and 4.6. We define:*

$$a(z) := \min_{\substack{W\in T_{\pi(\hat{z})}(\mathring{S}^{k,0}(\mathbb{C}^n))\\ ||W||_2=1}} \sum_{j=1}^m |\langle W, A_j\rangle_{\mathbb{R}}|^2 \tag{5.3}$$

$$\hat{a}(z) := \min_{\substack{w\in H_{\pi,\hat{z}}(\mathbb{C}_*^{n\times k})\\ ||w||_2=1}} \sum_{j=1}^m |\langle D\pi(\hat{z})(w), A_j\rangle_{\mathbb{R}}|^2 \tag{5.4}$$

The following two families of matrices, Q_z and $\hat{Q}_z$, indexed by $\mathbb{C}^{n\times r}$, will allow us to write the local lower Lipschitz bounds with respect to $||xx^* - yy^*||_2$ and $d(x,y)$ as eigenvalue problems.

DEFINITION 5.4. *Given $z \in \mathbb{C}^{n\times r}$ having rank $k > 0$ we define a matrix $Q_z \in \mathbb{R}^{(2nk-k^2)\times(2nk-k^2)}$ in the following way. Let $U_1 \in \mathbb{C}^{n\times k}$ be a matrix whose columns are left singular vectors of z corresponding to non-zero singular values of z, so that $U_1U_1^* = \mathbb{P}_{Ran(z)}$. Let $U_2 \in \mathbb{C}^{n\times(n-k)}$ be a matrix whose columns are left singular vectors of z corresponding to the zero singular values of z, so that $U_2U_2^* = \mathbb{P}_{Ranz^\perp}$. Then*

$$Q_z := \sum_{j=1}^{m} \begin{bmatrix} \tau(U_1^*A_jU_1) \\ \mu(U_2^*A_jU_1) \end{bmatrix} \begin{bmatrix} \tau(U_1^*A_jU_1) \\ \mu(U_2^*A_jU_1) \end{bmatrix}^T \tag{5.5}$$

where the isometric isomorphisms τ and μ are given by

$$\begin{aligned} &\tau : Sym(\mathbb{C}^k) \to \mathbb{R}^{k^2} && \mu : \mathbb{C}^{p\times q} \to \mathbb{R}^{2pq} \\ &\tau(X) = \begin{bmatrix} D(X) \\ \sqrt{2}T(\Re X) \\ \sqrt{2}T(\Im X) \end{bmatrix} && \mu(X) = vec(\begin{bmatrix} \Re X \\ \Im X \end{bmatrix}) \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} &D : Sym(\mathbb{C}^k) \to \mathbb{R}^k && T : Sym(\mathbb{R}^k) \to \mathbb{R}^{\frac{1}{2}k(k-1)} \\ &D(W) = \begin{bmatrix} X_{11} \\ \vdots \\ X_{kk} \end{bmatrix} && T(X) = \begin{bmatrix} X_{12} \\ X_{13} \\ X_{23} \\ \vdots \\ X_{k-1k} \end{bmatrix} \end{aligned} \tag{5.7}$$

and

$$vec : \mathbb{R}^{p\times q} \to \mathbb{R}^{pq} \qquad vec(X) = vec([X_1|\cdots|X_q]) = \begin{bmatrix} X_1 \\ \vdots \\ X_q \end{bmatrix} \tag{5.8}$$

We note that Q_z depends only on $\mathrm{Ran}(z)$, in particular it is invariant under $(U_1, U_2) \to (U_1P, U_2Q)$ for $P \in U(k), Q \in U(n-k)$. We will also refer to Q_z as $Q_{[U_1|U_2]}$ where $[U_1|U_2] \in U(n)$.

DEFINITION 5.5. *Given $z \in \mathbb{C}^{n\times r}$ having rank $k > 0$ we define a matrix $\hat{Q}_z \in \mathbb{R}^{2nk\times 2nk}$ in the following way. Let $F_j = \mathbb{I}_{k\times k} \otimes j(A_j) \in \mathbb{R}^{2nk\times 2nk}$ where*

$$\begin{aligned} &j : \mathbb{C}^{m\times n} \to \mathbb{R}^{2m\times 2n} \\ &j(X) = \begin{bmatrix} \Re X & -\Im X \\ \Im X & \Re X \end{bmatrix} \end{aligned} \tag{5.9}$$

is an injective homomorphism. Then

$$\hat{Q}_z := 4\sum_{j=1}^{m} F_j\mu(\hat{z})\mu(\hat{z})^T F_j \tag{5.10}$$

With these definitions in mind, we will prove the following:

THEOREM 5.6. *Let $z \in \mathbb{C}^{n\times r}$ have rank $k > 0$. Then* *(i) The global lower bound a_0 is given as*

$$a_0 = \inf_{z \in \mathbb{C}^{n\times r}\setminus\{0\}} a(z) \tag{5.11}$$

(ii) The local lower bounds $a_1(z)$ and $a_2(z)$ are squeezed between a_0 and $a(z)$

$$a_0 \leq a_2(z) \leq a_1(z) \leq a(z) \tag{5.12}$$

So that in particular

$$a_0 = \inf_{z \in \mathbb{C}^{n\times r}\setminus\{0\}} a_i(z) \tag{5.13}$$

(iii) The infimization problem in $a(z)$ may be reformulated as an eigenvalue problem. Let Q_z be the $2nk - k^2 \times 2nk - k^2$ matrix given in Definition 5.4. Then

$$a(z) = \lambda_{2nk-k^2}(Q_z) \tag{5.14}$$

(iv) For $r = 1$, $\hat{a}(z)$ differs from $a(z)$ by a constant factor, hence for $r = 1$ the infimum $\inf_{z\in\mathbb{C}^{n\times r}\setminus\{0\}} \hat{a}(z)$ is non-zero. For $r > 1$ this infimum is zero and hence there is no non-trivial global lower bound $\hat{a}_0$ analogous to a_0 for the alternate metric d.

(v) The local lower bounds with respect to the alternate metric d satisfy

$$\hat{a}_1(z) = \hat{a}_2(z) = \frac{1}{4||z||_2^2}\hat{a}(z) \tag{5.15}$$

(vi) The infimization problem in $\hat{a}(z)$ may be reformulated as an eigenvalue problem. Let $\hat{Q}_z$ be the $2nk \times 2nk$ matrix given in Definition 5.5. Then $\hat{a}(z)$ is directly computable as

$$\hat{a}(z) = \lambda_{2nk-k^2}(\hat{Q}_z) \tag{5.16}$$

(vii) We have the following local inequality relating $a(z)$ and $\hat{a}(z)$.

$$\frac{1}{4||z||_2^2}\hat{a}(z) \leq a(z) \leq \frac{1}{2\sigma_k(z)^2}\hat{a}(z) \tag{5.17}$$

(viii) Computation of the global lower bound a_0 may be reformulated as the minimization of a continuous quantity over the compact Lie group $U(n)$.

$$a_0 = \min_{\substack{U\in U(n)\\ U=[U_1|U_2]\\ U_1\in\mathbb{C}^{n\times r}\\ U_2\in\mathbb{C}^{n\times(n-r)}}} \lambda_{2nr-r^2}(Q_{[U_1|U_2]}) \tag{5.18}$$

(ix) While (iv) makes clear that a_0 cannot be upper bounded by $\inf_{z\in\mathbb{C}^{n\times r}\setminus\{0\}} \hat{a}(z)$, we can achieve a similar end by constraining z to have orthonormal columns. Namely

$$\frac{1}{4}\inf_{\substack{z\in\mathbb{C}_*^{n\times r}\\ z^*z=\mathbb{I}_{r\times r}}} \hat{a}(z) \leq a_0 \leq \frac{1}{2}\inf_{\substack{z\in\mathbb{C}_*^{n\times r}\\ z^*z=\mathbb{I}_{r\times r}}} \hat{a}(z) \tag{5.19}$$

Proof. See C.2 ∎

We now move on to analyzing the local lower Lipschitz bounds for the α map $x \mapsto \langle xx^*, A_j\rangle_{\mathbb{R}}^{\frac{1}{2}}$. This was done for the case $r = 1$ in [6]. Recall that $\theta(x) = (xx^*)^{\frac{1}{2}}$ and that

$$A_0 = \inf_{\substack{x,y\in\mathbb{C}^{n\times r}\\ [x]\neq[y]}} \frac{||\alpha(x)-\alpha(y)||_2^2}{||\theta(x)-\theta(y)||_2^2} = \inf_{\substack{x,y\in\mathbb{C}^{n\times r}\\ [x]\neq[y]}} \frac{\sum_{j=1}^m (\langle xx^*, A_j\rangle_{\mathbb{R}}^{\frac{1}{2}} - \langle yy^*, A_j\rangle_{\mathbb{R}}^{\frac{1}{2}})^2}{||(xx^*)^{\frac{1}{2}} - (yy^*)^{\frac{1}{2}}||_2^2} \tag{5.20}$$

In analogy with Definition 5.2, we consider the local lower Lipschitz bounds for the α map.

Definition 5.7. *Let $z \in \mathbb{C}^{n\times r}$ have rank k. We define*

$$\begin{aligned}
A_1(z) &= \lim_{R\to 0} \inf_{\substack{x\in\mathbb{C}^{n\times r}\\ ||\theta(x)-\theta(z)||_2\le R\\ rank(x)\le k}} \frac{||\alpha(x)-\alpha(z)||_2^2}{||\theta(x)-\theta(z)||_2^2}\\
A_2(z) &= \lim_{R\to 0} \inf_{\substack{x,y\in\mathbb{C}^{n\times r}\\ ||\theta(x)-\theta(z)||_2\le R\\ ||\theta(y)-\theta(z)||_2\le R\\ rank(x)\le k\\ rank(y)\le k}} \frac{||\alpha(x)-\alpha(y)||_2^2}{||\theta(x)-\theta(y)||_2^2}\\
\hat{A}_1(z) &= \lim_{R\to 0} \inf_{\substack{x\in\mathbb{C}^{n\times r}\\ D(x,z)\le R\\ rank(x)\le k}} \frac{||\alpha(x)-\alpha(z)||_2^2}{D(x,z)^2}\\
\hat{A}_2(z) &= \lim_{R\to 0} \inf_{\substack{x,y\in\mathbb{C}^{n\times r}\\ D(x,z)\le R\\ D(y,z)\le R\\ rank(x)\le k\\ rank(y)\le k}} \frac{||\alpha(x)-\alpha(y)||_2^2}{D(x,y)^2}
\end{aligned} \tag{5.21}$$

Definition 5.8. *Given $z \in \mathbb{C}^{n\times r}$ having rank $k > 0$ we define two matrices $\hat{T}_z, \hat{R}_z \in \mathbb{R}^{2nk\times 2nk}$. Let $I_0(z) \subset \{1,\dots,m\}$ be the indices such that $\alpha_j(z) = 0$ (or equivalently such that α_j is not differentiable) for $j \in I_0(z)$, and let $I(z) = \{1,\dots,m\}\setminus I_0(z)$. Once again let $F_j = \mathbb{I}_{k\times k}\otimes j(A_j) \in \mathbb{R}^{2nk\times 2nk}$, then define $\hat{T}_z$ and $\hat{R}_z$ via*

$$\hat{T}_z = \sum_{j\in I(z)} \frac{1}{\mu(\hat{z})^T F_j \mu(\hat{z})} F_j \mu(\hat{z})\mu(\hat{z})^T F_j \tag{5.22}$$

$$\hat{R}_z = \sum_{j\in I_0(z)} F_j \tag{5.23}$$

With these definitions in mind we prove:

Theorem 5.9. *Let $z \in \mathbb{C}^{n\times r}$ have rank $k > 0$. Then*

(i) For $r > 1$ it is the case that $\inf_{z\in\mathbb{C}^{n\times r}\setminus\{0\}} A_i(z) = 0$ for $i = 1,2$, as such $A_0 = 0$.

(ii) Let $\hat{T}_z$ and $\hat{R}_z$ be as in Definition 5.8. Then $\hat{A}_1(z)$ and $\hat{A}_2(z)$ are directly computable as

$$\hat{A}_1(z) = \lambda_{2nk-k^2}(\hat{T}_z + \hat{R}_z) \tag{5.24}$$

$$\hat{A}_2(z) = \lambda_{2nk-k^2}(\hat{T}_z) \tag{5.25}$$

(iii) We have the following inequality between $A_i(z)$ and $\hat{A}_i(z)$ for $i = 1, 2$, which justifies not treating them separately.

$$\hat{A}_i(z) \leq A_i(z) \leq \sqrt{2}\hat{A}_i(z) \tag{5.26}$$

Proof. See C.3 ∎

For the sake of completeness we also include the following theorem on the global upper Lipschitz bounds for the α and β analysis maps.

Definition 5.10. *We define the following (squared) upper Lipschitz constants for β and α respectively:*

$$b_0 := \sup_{\substack{x,y\in\mathbb{C}^{n\times r}\\ [x]\neq[y]}} \frac{||\beta(x)-\beta(y)||_2^2}{||xx^*-yy^*||_2^2} \tag{5.27}$$

$$B_0 := \sup_{\substack{x,y\in\mathbb{C}^{n\times r}\\ [x]\neq[y]}} \frac{||\alpha(x)-\alpha(y)||_2^2}{||(xx^*)^{\frac{1}{2}}-(yy^*)^{\frac{1}{2}}||_2^2} \tag{5.28}$$

A somewhat simplifying alternate upper Lipschitz constant for β is

$$b_{0,1} := \sup_{\substack{x,y\in\mathbb{C}^{n\times r}\\ [x]\neq[y]}} \frac{||\beta(x)-\beta(y)||_2^2}{||xx^*-yy^*||_1^2} \tag{5.29}$$

Definition 5.11. *The β map is the pullback of a linear operator acting on symmetric matrices which we refer to as $\mathcal{A}$. Specifically,*

$$\begin{aligned}\mathcal{A} &: Sym(\mathbb{C}^n) \to \mathbb{R}^m\\ \mathcal{A}_j(X) &= \langle X, A_j\rangle_{\mathbb{R}}\end{aligned} \tag{5.30}$$

Definition 5.12. *When $A_j \geq 0$ for each j, we define the operator T_r.*

$$\begin{aligned}T_r &: \mathbb{C}^{n\times r} \to (\mathbb{C}^{n\times r})^m\\ T_r(x) &= (A_j^{\frac{1}{2}}x)_{j=1}^m\end{aligned} \tag{5.31}$$

In a slight abuse of notation we write for $r = 1$

$$\begin{aligned}T_1 &: \mathbb{C}^n \to \mathbb{C}^{n\times m}\\ T_1(x) &= [A_1^{\frac{1}{2}}x|\cdots|A_m^{\frac{1}{2}}x]\end{aligned} \tag{5.32}$$

We compute explicitly b_0, $b_{0,1}$, and B_0 via different norms of the operators $\mathcal{A}$ and T_r, as well as providing formulas for b_0 and B_0 analogous to (5.18) and (5.25). Specifically, we prove:

Theorem 5.13. *Let b_0, $b_{0,1}$, B_0, $\mathcal{A}$, and T_r be as above. Then*

(i) The global upper bound b_0 is given by

$$b_0 = \max_{\substack{U\in U(n)\\ U=[U_1|U_2]\\ U_1\in\mathbb{C}^{n\times r}, U_2\in\mathbb{C}^{n\times n-r}}} \lambda_1(Q_{[U_1|U_2]}) \tag{5.33}$$

Where Q_U is as in Definition 5.4.

(ii) The global upper bound $b_{0,1}$ *is given by*

$$b_{0,1} = ||\mathcal{A}||_{1\to 2}^2 \tag{5.34}$$

Additionally if $A_j \geq 0$ *for all* j *then*

$$b_{0,1} = ||T_r||_{2\to(2,4)}^4 = ||T_1||_{2\to(2,4)}^4 \tag{5.35}$$

Where the $||\cdot||_{2,4}$ *norm of a matrix is the* l^4 *norm of the vector of* l^2 *norms of its columns.*

(iii) The global upper bound B_0 *is given by*

$$B_0 = \sup_{\substack{z\in\mathbb{C}^{n\times r}\\ z\neq 0}} \lambda_1(\hat{T}_z) = B \tag{5.36}$$

Where $\hat{T}_z$ *is as in Definition 5.8 and* B *is the optimal upper frame bound for* $\{A_j\}_{j=1}^m$.

Proof. See C.4. ∎

It turns out that Theorem 5.6 allows us to find novel algebraic conditions for a frame for $\mathbb{C}^{n\times r}$ to be generalized phase retrievable, generalizing Theorem 4 in [7]. The benefit of condition (vi) over the definition of phase retrievability is that they involve checking a quantity over all $n\times r$ matrices with orthonormal columns, that is to say over the Stiefel manifold of dimension $2nr - r^2$, as opposed to over all pairs of $n\times r$ matrices.

THEOREM 5.14. *Let* $\{A_j\}_{j=1}^m$ *be a frame for* $\mathbb{C}^{n\times r}$*. Then the following are equivalent:*

(i) $\{A_j\}_{j=1}^m$ *is generalized phase retrievable.*

(ii) For all $U_1 \in \mathbb{C}^{n\times r}$*,* $U_2 \in \mathbb{C}^{n\times(n-r)}$ *such that* $[U_1|U_2] \in U(n)$ *the* $2nr - r^2 \times 2nr - r^2$ *matrix*

$$Q_{[U_1|U_2]} = \sum_{j=1}^m \begin{bmatrix}\tau(U_1^*A_jU_1)\\ \mu(U_2^*A_jU_1)\end{bmatrix}\begin{bmatrix}\tau(U_1^*A_jU_1)\\ \mu(U_2^*A_jU_1)\end{bmatrix}^T \tag{5.37}$$

is invertible.

(iii) For all $z \in \mathbb{C}^{n\times r}$ *such that* z *has orthonormal columns, the* $2nr \times 2nr$ *matrix*

$$\hat{Q}_z = 4\sum_{j=1}^m (\mathbb{I}_{k\times k}\otimes j(A_j))\mu(z)\mu(z)^T(\mathbb{I}_{k\times k}\otimes j(A_j)) \tag{5.38}$$

has as its null space precisely the r^2 *dimensional* $\mathcal{V}_z = \{\mu(u)|u \in V_{\pi,z}(\mathbb{C}_*^{n\times r})\}$.

(iv) For all $U_1 \in \mathbb{C}^{n\times r}$*,* $U_2 \in \mathbb{C}^{n\times(n-r)}$ *such that* $[U_1|U_2] \in U(n)$*,* $H \in Sym(\mathbb{C}^r)$*,* $B \in \mathbb{C}^{(n-r)\times r}$ *there exist* $c_1,\dots c_m \in \mathbb{R}$ *such that*

$$U_1^*(\sum_{j=1}^m c_jA_j)U_1 = H \tag{5.39a}$$

$$U_2^*(\sum_{j=1}^m c_jA_j)U_1 = B \tag{5.39b}$$

(v) For all $U_1 \in \mathbb{C}^{n\times r}$ with orthonormal columns

$$span_{\mathbb{R}}\{A_jU_1\}_{j=1}^m = \{U_1K | K \in \mathbb{C}^{r\times r}, K^* = -K\}^{\perp} \tag{5.40}$$

(vi) For all $U_1 \in \mathbb{C}^{n\times r}$ with orthonormal columns

$$dim_{\mathbb{R}}\{A_jU_1\}_{j=1}^m \geq 2nr - r^2 \tag{5.41}$$

Proof. See C.5 □

6. Numerical experiments. The main benefit of lower Lipschitz results like Theorem 5.1 is that they provide quantitative control over reconstruction error in the generalized phase retrieval problem, as opposed to the topological result in Proposition 5.1 that the error is bounded whenever the matrix frame is generalized phase retrievable (i.e. that $a_0 > 0$). This is only true, however, if for a given frame one can make headway in computing the lower Lipschitz constant a_0. Unfortunately (5.18) yields a_0 as a non-convex optimization problem, so for the time being we content ourselves with examining the statistics of the local lower Lipschitz constants $\hat{a}_2(z)$ and $a(z)$. We also verify numerically the result in Theorem 5.9 that α is not globally lower Lipschitz (i.e. that $A_0 = 0$) by examining the statistics of the local lower Lipschitz constant $\hat{A}_2(z)$.

For each experiment we use a fixed frame set of cardinality $m = 4nk - 4k^2$, noting that Theorem 2.1 in [30] implies that a generic frame for $\mathbb{C}^{n\times k}$ with cardinality $m \geq 4nk - 4k^2$ will be generalized phase retrievable when $2k \leq n$. The experiment shown in Figure 1 supports the result in Theorem 5.9 that $\inf_{z\in\mathbb{C}^{n\times r}\setminus\{0\}} \hat{A}_2(z) = 0$ for $r > 1$, thus that the α analysis map is not globally lower Lipschitz with respect to either $D(x, y)$ or $||(xx^*)^{\frac{1}{2}} - (yy^*)^{\frac{1}{2}}||_2$ when $r > 1$. This experiment also supports the earlier result in [6] that when $r = 1$ $\inf_{z\in\mathbb{C}^{n\times r}\setminus\{0\}} \hat{A}_2(z) > 0$. The experiment shown in Figure 2 supports the result noted in the proof of Theorem 5.6 that $\inf_{z\in\mathbb{C}^{n\times r}\setminus\{0\}} \hat{a}_2(z) = 0$ for $r > 1$, thus that the β analysis map is not globally lower Lipschitz with respect to $d(x, y)$ when $r > 1$. That this quantity is non-zero when $r = 1$ follows from the fact that for $r = 1$ we have $d(x, y) = ||xx^* - yy^*||_1$ (see Theorem 3.7). Finally, the experiment shown in Figure 3 supports the result in Theorem 5.6 that $a_0 = \inf_{z\in\mathbb{C}^{n\times r}\setminus\{0\}} a(z) > 0$ even when $r > 1$, thus that the β analysis map is globally lower Lipschitz with respect to $||xx^* - yy^*||_2$ whenever the frame $(A_j)_{j\geq 1}$ is generalized phase retrievable. Code for all numerical experiments can be found at github.com/cbartondock/LipschtizAnalysisofGenPR.

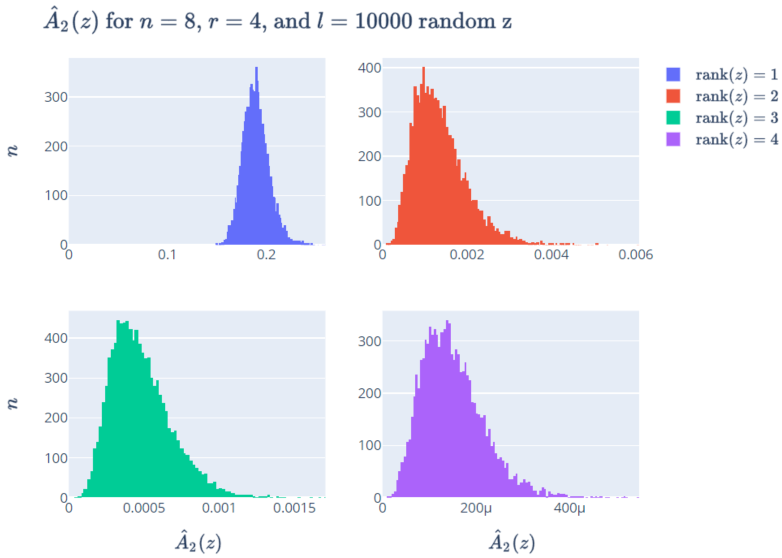

FIG. 1. *In all experiments $\hat{A}_2(z)$ is computed for a fixed frame of $4nk - 4k^2$ matrices in $\mathbb{C}^{n\times k}$ for $l = 10^4$ samples of z having rank k. The entries of both z and the frame matrices are sampled from a complex Gaussian with unit variance and zero mean. As can clearly be seen only the $k = 1$ case has a clear separation from zero.*

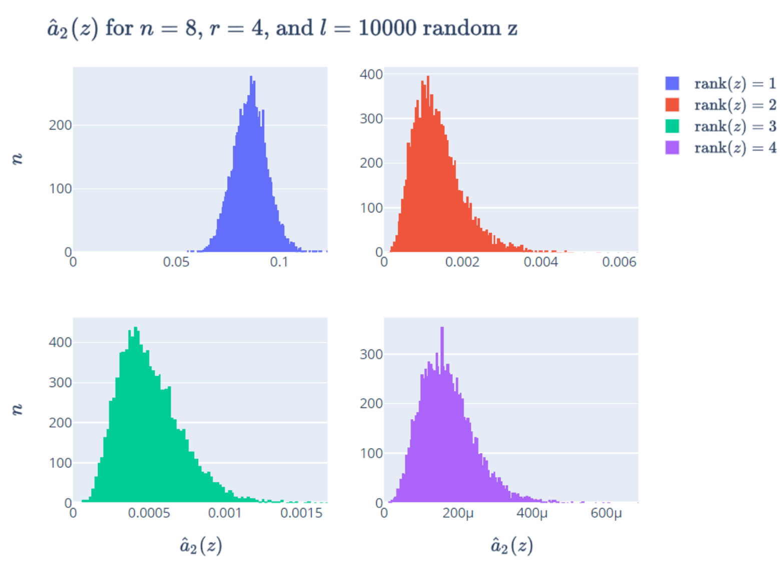

FIG. 2. *In all experiments $\hat{a}_2(z)$ is computed for a fixed frame of $4nk - 4k^2$ matrices in $\mathbb{C}^{n\times k}$ for $l = 10^4$ samples of z having rank k. The entries of both z and the frame matrices are sampled from a complex Gaussian with unit variance and zero mean. As can clearly be seen only the $k = 1$ case has a clear separation from zero.*

FIG. 3. ’ *In all experiments* $a(z) = \lambda_{2nk-k^2}(Q_{[U_1|U_2]})$ *is computed for a fixed frame of* $4nk - 4k^2$ *matrices in* $\mathbb{C}^{n\times k}$ *for* $l = 10^4$ *samples of* $U \in U(n)$ *distributed according to the uniform Haar distribution on* $U(n)$*.* $U_1 \in \mathbb{C}^{n\times k}$ *is composed of the first* k *columns of* U *so that* $Q_{[U_1|U_2]} \in \mathbb{C}^{2nk-k^2\times 2nk-k^2}$*. The entries of the frame matrices are sampled from a complex Gaussian with unit variance and zero mean. In this case an overlapping log-plot is also included, in which clear separation from zero can be seen for* $k = 1, \ldots, 4$*.*

7. Conclusion. This paper extends known results about the stability of generalized phase retrieval to the “impure state” case where the phase no longer comes from $U(1)$ but instead the non-abelian groups $U(r)$ where $r > 1$. We showed that the situation changes drastically in this case, both because $U(r)$ is non-abelian and because for $r > 1$ a sequence in $\mathbb{C}_*^{n\times r}/U(r)$ with $||x_n||_2 = 1$ can come arbitrarily close to dropping in rank. In particular, we showed that while the β analysis map remains lower Lipschitz with respect to the norm induced distance on $\mathrm{Sym}(\mathbb{C}^n)$ (Theorem 5.6), the α analysis map does not (Theorem 5.9). Our analysis relies on several Lipschitz embeddings of $\mathbb{C}^{n\times r}/U(r)$ into the Euclidean space $\mathrm{Sym}(\mathbb{C}^n)$ (Theorem 3.7) and a Whitney stratification of the positive semidefinite matrices into positive semidefinite matrices of fixed rank (Theorem 4.5). This investigation of the geometry of positive semidefinite matrices incidentally provided the interesting and (to the best of our knowledge) previously unknown result that the Riemannian geometry of the stratifying manifolds given by the Bures-Wasserstein metric is compatible with the stratification. In particular geodesics of positive semi-definite matrices with respect to the Bures-Wasserstein metric are rank preserving and may be approximated by geodesics of higher rank. We note that the fact that $a_0 > 0$ and can be explicitly computed as in (5.18) suggests that known convergent algorithms for generalized phase retrieval may be extended to the case $r > 1$. Finally, the explicit computation of the lower Lipschitz bound for the β map allowed for a novel characterization of generalized phase retrievable frames in the impure state case $r > 1$ (Theorem 5.14).

Appendix A. Proofs for Section 3.

A.1. Proof of Proposition 3.3.

Proof. Both $d(x, y)$ and $D(x, y)$ are obviously positive and symmetry follows from the fact that that $U(r)$ is a group. Moreover, owing to the compactness of $U(r)$, both $D(x, y)$ and $d(x, y)$ are zero if and only if there exists U_0 such that $x = yU_0$, that is if and only if $[x] = [y]$. It remains to prove the triangle inequality. For $D(x, y)$ the computation is straightforward and follows from the unitary invariance of the Frobenius norm. If U_1 and U_2 are unitary minimizers for $D(x, z)$ and $D(z, y)$ respectively then

$$\begin{aligned} D(x,z) + D(y,z) &= ||x - zU_1||_2 + ||z - yU_2||_2 \\ &= ||x - zU_1||_2 + ||zU_1 - yU_2U_1||_2 \\ &\geq ||x - yU_2U_1||_2 \geq D(x,y) \end{aligned} \tag{A.1}$$

We note that the above argument also holds for any unitarily invariant norm $|||\cdot|||$ so that each $D_{|||\cdot|||}(x, y) := \min_{U\in U(r)} |||x - yU|||$ is a metric on $\mathbb{C}^{n\times r}/U(r)$. A similar trick can be employed regarding $d(x, y)$, but it requires the following lemma which does not readily generalize to arbitrary unitarily invariant norms or even $p \neq 2$:

LEMMA A.1. *The following triangle inequality holds for all $x, y, z \in \mathbb{C}^{n\times r}$*

$$||x - y||_2||x + y||_2 \leq ||x - z||_2||x + z||_2 + ||z - y||_2||z + y||_2 \tag{A.2}$$

Proof. This is essentially a statement about the geometry of parallelepipeds in $\mathbb{R}^3$, namely that the sum of the product of face diagonals from any two sides sharing a vertex will always exceed the product of the two on the remaining side sharing the vertex. The lemma follows from the observation that for $x, y \in \mathbb{R}^n$

$$
\begin{aligned}
||x-y||_2||x+y||_2 &= \sqrt{(||x||_2^2+||y||_2^2)^2-4|\langle x,y\rangle_{\mathbb{R}}|^2} \\
&= \frac{1}{2}\left(||x||_2^2-||y||_2^2+\sqrt{(||x||_2^2+||y||_2^2)^2-4|\langle x,y\rangle_{\mathbb{R}}|^2}\right) \\
&\quad -\frac{1}{2}\left(||x||_2^2-||y||_2^2-\sqrt{(||x||_2^2+||y||_2^2)^2-4|\langle x,y\rangle_{\mathbb{R}}|^2}\right) \\
&= \lambda_+(xx^T-yy^T)-\lambda_-(xx^T-yy^T) \\
&= ||xx^T-yy^T||_1
\end{aligned} \tag{A.3}
$$

See the proof of Theorem 3.7 for a direct computation of the eigenvalues of $xx^T - yy^T$ (the theorem deals with the complex case but the real case is identical). This identity proves the lemma immediately since the latter obeys the triangle inequality and

$$
\begin{aligned}
||x-y||_2||x+y||_2 &= ||\mu(x)-\mu(y)||_2||\mu(x)+\mu(y)||_2 \\
&= ||\mu(x)\mu(x)^T-\mu(y)\mu(y)^T||_1 \\
&\le ||\mu(x)\mu(x)^T-\mu(z)\mu(z)^T||_1+||\mu(z)\mu(z)^T-\mu(y)\mu(y)^T||_1 \\
&= ||x-z||_2||x+z||_2+||z-y||_2||z+y||_2
\end{aligned} \tag{A.4}
$$

Where $\mu : \mathbb{C}^{n\times r} \to \mathbb{R}^{2nr}$ is complex matrix vectorization. ∎

The proposition then follows via a similar argument to (A.1), namely if U_1, U_2 are the minimizers in $d(x,z)$ and $d(z,y)$ respectively then

(A.5) ∎

$$
\begin{aligned}
d(x,z)+d(z,y) &= ||x-zU_1||_2||x+zU_1||_2+||z-yU_2||_2||z+yU_2||_2 \\
&= ||x-zU_1||_2||x+zU_1||_2+||zU_1-yU_2U_1||_2||zU_1+yU_2U_1||_2 \\
&\ge ||x-yU_2U_1||_2||x+yU_2U_1||_2 \ge d(x,y)
\end{aligned}
$$

A.2. Proof of Proposition 3.4.

Proof. Both the trace $\mathrm{tr}\{x^*yU\}$ in that appears in D and its square as it appears in d will be maximized when x^*yU is positive semidefinite, thus we may take the minimizer to be the polar factor for x^*y, the polar factor of course being the unique unitary for which x^*yU is non-negative only when x^*y is full rank. The non-uniqueness of the minimizer arises precisely from the non-uniqueness in choice of polar factor when x^*y does not have full rank. Note that even if y is full rank, x^*y will have rank less than r whenever $\mathrm{Ran}(y) \cap \mathrm{Ran}(x)^\perp \neq 0$. ∎

A.3. Proof of Proposition 3.6.

Proof. Note that the non-zero eigenvalues of $\pi(x)$ are precisely the squares of the singular values of x, the non-zero eigenvalues of $\theta(x)$ agree with the non-zero singular values of x, and the non-zero eigenvalues values of $\psi(x)$ differ from the non-zero singular values of x only by a factor of $||x||_2$. This proves that the embeddings preserve rank. It is readily checked that the embeddings are surjective and injective modulo $\sim$. In particular for $A \in S^{r,0}(\mathbb{C}^n)$, we have

$$\pi^{-1}(A) = [\mathrm{Cholesky}(A)] \tag{A.6}$$

$$\theta^{-1}(A) = [\mathrm{Cholesky}(A^2)] \tag{A.7}$$

$$\psi^{-1}(A) = [\mathrm{Cholesky}(A^2/||A||_2)] \tag{A.8}$$

where Cholesky(A) is a Cholesky decomposition of A in $\mathbb{C}^{n\times r}$ (note that the Cholesky decomposition is unique up to equivalence class). ∎

A.4. Proof of Theorem 3.7.

Proof. To prove (3.5) we analyze the following quantity:

$$Q(x,y) = \frac{D(x,y)^2}{||\theta(x)-\theta(y)||_2^2} = \frac{||x||_2^2+||y||_2^2-2||x^*y||_1}{||x||_2^2+||y||_2^2-2\text{tr}\{(xx^*)^{\frac{1}{2}}(yy^*)^{\frac{1}{2}}\}} \tag{A.9}$$

We first note that $||x^*y||_1 = ||(xx^*)^{\frac{1}{2}}(yy^*)^{\frac{1}{2}}||_1$ since $(xx^*)^{\frac{1}{2}}(yy^*)^{\frac{1}{2}}$ and x^*y have the same non-zero singular values. Hence if we define $A=\theta(x)=(xx^*)^{\frac{1}{2}}$ and $B=\theta(y)=(yy^*)^{\frac{1}{2}}$ we can abuse notation slightly and write

$$Q(A,B) = \frac{||A||_2^2+||B||_2^2-2||AB||_1}{||A||_2^2+||B||_2^2-2\text{tr}\{AB\}} \tag{A.10}$$

Now $\text{tr}\{AB\}\le ||AB||_1$, so we conclude that $Q(x,y)\le 1$. On the other hand this bound is achievable by any x and y for having the same left singular vectors, since in this case A and B commute hence $AB\ge 0$ and $||AB||_1=\text{tr}\{AB\}$. We conclude that the upper Lipschitz constant is 1, and in particular

$$\sup_{\substack{x,y\in\mathbb{C}^{n\times r}/U(r)\\ x\neq y}} Q(x,y) = \max_{\substack{x,y\in\mathbb{C}^{n\times r}/U(r)\\ x\neq y}} Q(x,y) = 1 \tag{A.11}$$

We now turn our attention to the lower bound. It is shown in [9] that for any unitarily invariant norm $|||\cdot|||$ and positive semidefinite matrices A and B the following generalization of the arithmetic-geometric mean inequality holds:

$$4|||AB|||^2 \le |||(A+B)^2||| \tag{A.12}$$

We apply this inequality to the nuclear norm and conclude that

$$\begin{aligned} 4||AB||_1 &\le ||(A+B)^2||_1 \\ &= \text{tr}\{(A+B)^2\} \\ &= ||A||_2^2+||B||_2^2+2\text{tr}\{AB\} \end{aligned} \tag{A.13}$$

We employ this fact in the analysis of $Q(x,y)$:

$$\begin{aligned} Q(A,B) &= \frac{1}{2}\cdot\frac{2||A||_2^2+2||B||_2^2-4||AB||_1}{||A||_2^2+||B||_2^2-2\text{tr}\{AB\}} \\ &\ge \frac{1}{2}\cdot\frac{2||A||_2^2+2||B||_2^2-(||A||_2^2+||B||_2^2+2\text{tr}\{AB\})}{||A||_2^2+||B||_2^2-2\text{tr}\{AB\}} = \frac{1}{2} \end{aligned} \tag{A.14}$$

This implies a lower Lipschitz constant of at least $\frac{1}{\sqrt{2}}$. For the trivial case $n=r=1$ the ratio is 1. To prove the constant of $\frac{1}{\sqrt{2}}$ is optimal for $n>1$, let e_1 and e_2 be any two orthogonal unit vectors in $\mathbb{C}^n$ and let $x=e_1$ and $(y_j)_{j\ge1}$ be given by $y_j=\sqrt{1-\frac{1}{j^2}}e_1+\frac{1}{j}e_2$. Define $A=\theta(x)$ and $B_j=\theta(y_j)$, then both A and each B_j have unit norm and are rank 1 hence are idempotent, so that

$$\begin{aligned} AB_j &= (xx)^{\frac{1}{2}}(y_jy_j^*)^{\frac{1}{2}} = xx^*y_jy_j^* \\ &= \langle x,y_j\rangle_{\mathbb{R}}xy_j^* \\ &= (1-\frac{1}{j^2})e_1e_1^* + \frac{\sqrt{1-\frac{1}{j^2}}}{j}e_1e_2^* \end{aligned} \tag{A.15}$$

Thus $\mathrm{tr}\{AB_j\} = 1 - \frac{1}{j^2}$. On the other hand, $||AB_j||_1 = ||x^*y_j||_1 = |\langle x, y_j\rangle_{\mathbb{R}}| = \sqrt{1 - \frac{1}{j^2}}$. We find

$$\begin{aligned} \lim_{j\to\infty} Q(A, B_j) &= \lim_{j\to\infty} \frac{1 - ||AB_j||_1}{1 - \mathrm{tr}\{AB_j\}} \\ &= \lim_{j\to\infty} j^2(1 - \sqrt{1 - \frac{1}{j^2}}) = \frac{1}{2} \end{aligned} \tag{A.16}$$

Thus we conclude

$$\inf_{\substack{x,y\in\mathbb{C}^{n\times r}\\ x\neq y}} Q(x, y) = \frac{1}{2} \tag{A.17}$$

We now concern ourselves with proving (3.6). To prove the lower bound, let U_0 be the minimizer in $d(x, y)$. Then

$$\begin{aligned} ||\pi(x) - \pi(y)||_1 &= ||xx^* - yy^*||_1 \\ &= ||\frac{1}{2}(x - yU_0)(x + yU_0)^* + \frac{1}{2}(x + yU_0)(x - yU_0)^*||_2 \\ &\leq \frac{1}{2}||(x - yU_0)(x + yU_0)^*||_1 + \frac{1}{2}||(x - yU_0)(x + yU_0)^*||_1 \\ &\leq ||x - yU_0||_2||x + yU_0||_2 = d(x, y) \end{aligned} \tag{A.18}$$

This implies a lower Lipschitz constant of at least 1, but in fact this constant is optimal since the two are equal for $r = 1$. Turning our attention to the upper bound, we will in fact prove the following stronger inequality:

$$||\psi(x) - \psi(y)||_2 \geq \frac{1}{4}d(x, y)^2 + \frac{1}{4}D(x, y)^4 + (||x||_2 - ||y||_2)^2\left(||x^*y||_1 + \frac{1}{2}(||x||_2 + ||y||_2)^2\right) \tag{A.19}$$

We prove (A.19) by direct computation:

$$\begin{aligned} &||\psi(x) - \psi(y)||_2^2 - \frac{1}{4}d(x, y)^2 \\ &= ||x||_2^4 + ||y||_2^4 - 2||x||_2||y||_2\mathrm{tr}\{(xx^*)^{\frac{1}{2}}(yy^*)^{\frac{1}{2}}\} - \frac{1}{4}\left((||x||_2^2 + ||y||_2^2)^2 - 4||x^*y||_1^2\right) \\ &= \frac{3}{4}||x||_2^4 + \frac{3}{4}||y||_2^4 + ||x^*y||_1^2 - \frac{1}{2}||x||_2^2||y||_2^2 - 2||x||_2||y||_2\mathrm{tr}\{(xx^*)^{\frac{1}{2}}(yy^*)^{\frac{1}{2}}\} \\ &\geq \frac{3}{4}||x||_2^4 + \frac{3}{4}||y||_2^4 + ||x^*y||_1^2 - \frac{1}{2}||x||_2^2||y||_2^2 - 2||x||_2||y||_2||(xx^*)^{\frac{1}{2}}(yy^*)^{\frac{1}{2}}||_1 \\ &= \frac{1}{4}(||x||_2^2 - ||y||_2^2)^2 + \frac{1}{2}||x||_2^4 + \frac{1}{2}||y||_2^4 + ||x^*y||_1^2 - 2||x||_2||y||_2||x^*y||_1 \end{aligned} \tag{A.20}$$

We then note that

$$\begin{aligned} \frac{1}{4}D(x, y)^4 &= \frac{1}{4}(||x||^2 + ||y||^2 - 2||x^*y||_1)^2 \\ &= \frac{1}{4}||x||_2^4 + \frac{1}{4}||y||_2^4 + \frac{1}{2}||x||_2^2||y||_2^2 + ||x^*y||_1^2 - (||x||_2^2 + ||y||_2^2)||x^*y||_1 \end{aligned} \tag{A.21}$$

So that if we add and subtract $\frac{1}{4}D(x,y)^4$ from (A.20) we obtain the result

(A.22)

$$\begin{aligned} ||\psi(x) - \psi(y)||_2^2 - \frac{1}{4}d(x,y)^2 & \\ &\geq \frac{1}{2}(||x||_2^2 - ||y||_2^2)^2 + \frac{1}{4}D(x,y)^4 + (||x||_2 - ||y||_2)^2||x^*y||_1 \\ &= \frac{1}{4}D(x,y)^4 + (||x||_2 - ||y||_2)^2\Big((||x^*y||_1 + \frac{1}{2}(||x||_2 + ||y||_2)^2\Big) \end{aligned}$$

This immediately proves that $2||\psi(x) - \psi(y)||_2 \geq d(x,y)$ and hence that the upper Lipschitz constant in (3.6) is at most 2. For $r = 1$, we will prove shortly claim (iii), implying that $d(x,y) = ||\pi(x) - \pi(y)||_1 = ||\psi(x) - \psi(y)||_1$, hence in this case the optimal constant is $\sqrt{2}$, owing to the fact that $\psi(x) - \psi(y)$ will have rank at most 2 and in that case $d(x,y) = ||\psi(x) - \psi(y)||_1 \leq \sqrt{2}||\psi(x) - \psi(y)||_2$. For $r > 1$, however, we show that the upper Lipschitz constant of 2 is optimal by considering a sequence of matrices in $\mathbb{C}^{n\times 2}$. As before let e_1 and e_2 be any unit orthonormal vectors in $\mathbb{C}^n$. Let $x = [e_1|0]$, $(y_j)_{j\geq 1}$ be given by $y_j = [\sqrt{1 - \frac{1}{j^2}}e_1|\frac{1}{j}e_2]$. As before let $A = \theta(x)$, $B_n = \theta(y_j)$. We first note that A and each B_j commute and are positive semidefinite, so that AB_j is also positive semidefinite and we have $\mathrm{tr}\{AB_j\} = ||AB_j||_1$ and the inequality in (A.20) is actually an equality. This makes clear the impediment to a rank 1 sequence achieving the upper Lipschitz constant of 2: A and B_j could not be made to commute without x and y_j lying in the same equivalence class. Finally, we observe that $||x||_2 = ||y_j||_2 = 1$ so the remainder term in (A.19) disappears and we obtain

$$||\psi(x) - \psi(y_j)||_2^2 = \frac{1}{4}d(x,y)^2 + \frac{1}{4}D(x,y)^4 \quad \text{(A.23)}$$

We note moreover that $d(x,y)^2 = D(x,y)^2(||x||_2^2 + ||y||_2^2 + 2||x^*y||_1)$ so that

$$\begin{aligned} \frac{||\psi(x) - \psi(y_j)||_2^2}{d(x,y_j)^2} &= \frac{1}{4}\left(1 + \frac{D(x,y_j)^4}{d(x,y_j)^2}\right) \\ &= \frac{1}{4}\left(1 + \frac{1 - ||x^*y_j||_1}{1 + ||x^*y_j||_1}\right) \end{aligned} \quad \text{(A.24)}$$

Now $||x^*y_j||_1 = || \left[\frac{e_1^*}{0}\right] \begin{bmatrix} \sqrt{1 - \frac{1}{j^2}} & 0 \\ 0 & \frac{1}{j} \end{bmatrix} \left[e_1|e_2\right] ||_1 = \sqrt{1 - \frac{1}{j^2}}$ so that

$$\lim_{j\to\infty} \frac{||\psi(x) - \psi(y_j)||_2^2}{d(x,y_j)^2} = \lim_{j\to\infty} \frac{1}{4}\left(1 + \frac{1 - \sqrt{1 - \frac{1}{j^2}}}{1 + \sqrt{1 + \frac{1}{j^2}}}\right) = \frac{1}{4} \quad \text{(A.25)}$$

Thus we have proven claims (i) and (ii). To prove the first claim of (iii) note that for $r = 1$, $(xx^*)^{\frac{1}{2}} = \frac{xx^*}{||x||_2}$. The second part of (iii) follows from direct computation of $||xx^* - yy^*||_1$ via the method of moments. Clearly $xx^* - yy^*$ will have one positive and one negative eigenvalue, which we denote λ_+ and λ_-. In this case

$$\begin{aligned} \lambda_+ + \lambda_- &= \mathrm{tr}\{xx^* - yy^*\} \\ &= ||x||_2^2 - ||y||_2^2 \\ \lambda_+\lambda_- &= \frac{1}{2}\left(\mathrm{tr}\{xx^* - yy^*\}^2 - \mathrm{tr}\{(xx^* - yy^*)^2\}\right) \\ &= ||x||^2||y||^2 - |\langle x, y\rangle_{\mathbb{R}}|^2 \end{aligned} \tag{A.26}$$

A little bit of algebra then yields

$$\lambda_\pm = \frac{1}{2}\left(||x||_2^2 - ||y||_2^2 \pm \sqrt{(||x||^2 + ||y||^2)^2 - 4|\langle x, y\rangle_{\mathbb{R}}|^2}\right) \tag{A.27}$$

Thus we find $||xx^* - yy^*||_1 = \lambda_+ - \lambda_- = \sqrt{(||x||^2 + ||y||^2)^2 - 4|\langle x, y\rangle_{\mathbb{R}}|^2} = d(x, y)$. It strikes the authors that this is a minor miracle. Finally, to prove claim (iv) consider x and y having a common basis of singular vectors with singular values $(\sigma_i)_{i=1}^r$ and $(\mu_i)_{i=1}^r$ respectively. Then

$$||\pi(x) - \pi(y)||_2^2 = \sum_{i=1}^r (\sigma_i^2 - \mu_i^2)^2 \tag{A.28}$$

$$d(x, y)^2 = \sum_{i,j=1}^r (\sigma_i + \mu_i)^2(\sigma_j - \mu_j)^2 \tag{A.29}$$

The latter is obviously larger, consistent with (3.6). If it were additionally the case that $d(x, y) \leq C||\pi(x) - \pi(y)||_2$ we would have

$$\sum_{i\neq j} (\sigma_i + \mu_i)^2(\sigma_j - \mu_j)^2 \leq (C - 1)\sum_{i=1}^r (\sigma_i^2 - \mu_i^2)^2 \tag{A.30}$$

In the case $r = 1$ the left hand side is zero and so we may take $C = 1$. For $r > 1$, in contradiction of the above take $\sigma_1 = \mu_1 = \delta$, $\sigma_2 \neq \mu_2$ and all other singular values zero. We then would obtain

$$4\delta^2(\sigma_2 - \mu_2)^2 \leq (C - 1)(\sigma_2^2 - \mu_2^2)^2 \tag{A.31}$$

There is evidently no such C since δ may be chosen arbitrarily large. Thus claim (v) is proved, justifying the use of the alternate embedding ψ in (3.6). This concludes the proof of Theorem 3.7. ∎

Appendix B. Proofs for Section 4.

B.1. Proof of Proposition 4.4.

Proof. The proof of (4.5) is by direct computation. Namely

$$V_{\pi,x}(\mathbb{C}_*^{n\times r}) = \ker D\pi(x) = \{w \in \mathbb{C}^{n\times r} | xw^* + wx^* = 0\} \tag{B.1}$$

We would like to obtain a direct parametrization, however, and note that

$$\begin{aligned} w \in V_{\pi,x}(\mathbb{C}_*^{n\times r}) &\iff wx^* = \tilde{K} && \tilde{K} \in \mathbb{C}^{n\times n}, \tilde{K}^* = -\tilde{K}, \mathbb{P}_{\mathrm{Ran}(x)}\tilde{K} = \tilde{K} \\ &\iff wx^* = xKx^* && K \in \mathbb{C}^{r\times r}, K^* = -K \\ &\iff w = xK && K \in \mathbb{C}^{r\times r}, K^* = -K \end{aligned} \tag{B.2}$$

In the first line note that w is recoverable from such a $\tilde{K}$ via $w = \tilde{K}x(x^*x)^{-1}$. In the second note that $K = (xx^*)^\dagger x^* \tilde{K} x (xx^*)^\dagger$. The third "if and only if" is obtained by right multiplying $x(x^*x)^{-1}$. The horizontal space is then computable as $V_{\pi,x}(\mathbb{C}_*^{n\times r})^\perp$:

$$
\begin{aligned}
w \in H_{\pi,x}(\mathbb{C}_*^{n\times r}) &\iff \Re\mathrm{tr}\{w^* x K\} = 0 \quad \forall K \in \mathbb{C}^{n\times n}, K^* = -K \\
&\iff x^* w = \tilde{H} \quad \tilde{H} \in \mathbb{C}^{r\times r}, \tilde{H}^* = \tilde{H} \\
&\iff x^* w = x^* H x \quad H \in \mathbb{C}^{n\times n}, H^* = H, \mathbb{P}_{\mathrm{Ran}(x)} H = H \\
&\iff \mathbb{P}_{\mathrm{Ran}(x)} w = Hx \quad H \in \mathbb{C}^{n\times n}, H^* = H, \mathbb{P}_{\mathrm{Ran}(x)} H = H \\
&\iff w = Hx + X \quad H \in \mathbb{C}^{n\times n}, H^* = H = \mathbb{P}_{\mathrm{Ran}(x)} H, X \in \mathbb{C}^{n\times r}, \mathbb{P}_{\mathrm{Ran}(x)} X = 0
\end{aligned}
\tag{B.3}
$$

The second line follows from the fact that $\mathbb{C}^{n\times n}$ decomposes orthogonally into Hermitian and skew-Hermitian matrices. In the second note that $H = (x^*x)^{-1} x \tilde{H} x^* (x^*x)^{-1}$. The third follows from left multiplying by $(xx^*)^\dagger x$. Finally, the tangent space can be parametrized via the horizontal space as its image through $D\pi(x)$ as

$$
\begin{aligned}
T_{\pi(x)}(\mathring{S}^{r,0}(\mathbb{C}^n)) &= D\pi(x)(H_{\pi,x}(\mathbb{C}_*^{n\times r})) \\
&= \{Hxx^* + xx^*H + xX^* + Xx^* | H \in \mathbb{C}^{n\times n}, H^* = H, \mathbb{P}_{\mathrm{Ran}(x)} H = H, \mathbb{P}_{\mathrm{Ran}(x)} X = 0\}
\end{aligned}
\tag{B.4}
$$

This provides a direct parametrization, but for our purposes the simpler indirect description given by (4.7) will be more useful. It is clear from (B.4) that $T_{\pi(x)}(\mathring{S}^{r,0}(\mathbb{C}^n)) \subset \{W \in \mathrm{Sym}(\mathbb{C}^n) | \mathbb{P}_{\mathrm{Ran}(x)^\perp} W \mathbb{P}_{\mathrm{Ran}(x)^\perp} = 0\}$. To prove the reverse, note that if $W \in \mathrm{Sym}(\mathbb{C}^n)$ and $\mathbb{P}_{\mathrm{Ran}(x)^\perp} W \mathbb{P}_{\mathrm{Ran}(x)^\perp}$ then $W = W_1 + W_2 + W_2^*$ where $\mathbb{P}_{\mathrm{Ran}(x)} W_1 \mathbb{P}_{\mathrm{Ran}(x)} = W_1$ and $\mathbb{P}_{\mathrm{Ran}(x)} W_2 \mathbb{P}_{\mathrm{Ran}(x)^\perp} = W_2$. Any such W_2 is representable as xX^* where X is as in the description of the horizontal space. Indeed, take $X = W_2^* x (x^*x)^{-1}$. Finally, the Sylvester equation $xx^*H + Hxx^* = W_1$ has the unique solution

$$
H = \int_0^\infty e^{-txx^*} W_1 e^{-txx^*} dt \tag{B.5}
$$

□

B.2. Proof of Theorem 4.5.

Proof. To prove (i) in relatively short order we employ the following theorem:

THEOREM B.1 (see [26] and [18] Appendix B). *Let $\phi : G \times M \to M$ be a smooth action of a Lie group G on a smooth manifold M. If the action is semi-algebraic, then orbits of ϕ are smooth submanifolds of M.*

We apply this theorem in the case of $\mathring{S}^{p,q}(\mathbb{C}^n)$. Sylvester's Inertia Theorem says that $A \in \mathring{S}^{p,q}(\mathbb{C}^n)$ if and only if $A = K I_{p,q} K^*$ for some $K \in \mathrm{GL}(\mathbb{C}^n)$ where $I_{p,q} = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1, 0, \ldots, 0)$ is the matrix of inertia indices. Thus $\mathring{S}^{p,q}(\mathbb{C}^n)$ is precisely the orbit of $I_{p,q}$ under the smooth Lie group action:

$$
\begin{aligned}
\psi : \mathrm{GL}(\mathbb{C}^n) \times \mathbb{C}^{n\times n} &\to \mathbb{C}^{n\times n} \\
\psi(K, L) &= KLK^*
\end{aligned}
\tag{B.6}
$$

Noting that $\psi(KJ,L)=\psi(K,\psi(J,L))$ for $K,J\in \mathrm{GL}(\mathbb{C}^n)$. We need to check that the action is semi-algebraic. For a fixed $L\in\mathbb{C}^{n\times n}$ the action has as its graph

$$\begin{aligned}&\left\{(K,Y)\middle| K\in\mathrm{GL}(\mathbb{C}^n), Y=KLK^*\right\}\\ =&\left\{(k_{ij},y_{ij})\middle| i,j\in 1,\dots,n, \mathrm{Det}(k_{ij})\neq 0, y_{ij}-Q_{ij}(k_{ij})=0\right\}\end{aligned}\tag{B.7}$$

where each Q_{ij} is a quadratic polynomial in $(k_{ij})_{i,j=1}^n$ determined by L. This set is manifestly semi-algebraic, so by Theorem B.1 each $\mathring{S}^{p,q}(\mathbb{C}^n)$ is a smooth submanifold of $\mathbb{C}^{n\times n}$. To prove that the dimension of $\mathring{S}^{p,q}(\mathbb{C}^n)$ is given by $2n(p+q)-(p+q)^2$ note that the $\dim \mathring{S}^{p,q}(\mathbb{C}^n)=\dim \mathring{S}^{p+q,0}$ since matrix absolute value

$$\begin{aligned}&|\cdot|:\mathring{S}^{p,q}(\mathbb{C}^n)\to\mathring{S}^{p+q,0}\\ &|A|=(AA^*)^{\frac{1}{2}}\end{aligned}\tag{B.8}$$

is surjective and injective of up to permutation of eigenvalues. The dimension of $\mathring{S}^{p+q,0}$ can be computed from $T_{\pi(x)}(\mathring{S}^{r,0}(\mathbb{C}^n))$ as found in Lemma 4.4. Taking $r=p+q$ then

$$\dim T_{\pi(x)}(\mathring{S}^{r,0}(\mathbb{C}^n))=n^2-(n-r)^2=2nr-r^2=2n(p+q)-(p+q)^2\tag{B.9}$$

It remains to prove analyticity of $\mathring{S}^{r,0}(\mathbb{C}^n)$. It is proved in Lemma 3.11 of [3] that $\mathring{S}^{1,0}(\mathbb{C}^n)$ is real analytic. The proof in the general case is analagous. First note that owing to Sylvester's inertia theorem $\mathrm{GL}(\mathbb{C}^n)$ acts transitively on $\mathring{S}^{p,q}(\mathbb{C}^n)$ via conjugation, since if $X,Y\in\mathring{S}^{p,q}(\mathbb{C}^n)$ then we may obtain $G_1,G_2\in\mathrm{GL}(\mathbb{C}^n)$ so that $G_1XG_1^*=I_{p,q}=G_2YG_2^*$, hence $(G_2^{-1}G_1)X(G_2^{-1}G_1)^*=Y$. It remains to obtain that the stabilizer group is closed in $\mathrm{GL}(\mathbb{C}^n)$ so that we can invoke the homogeneous space construction theorem. If $Z\in\mathring{S}^{p,q}(\mathbb{C}^n)$ then $Z=zI_{p,q}z^*$ for some $z=U_z\left[\frac{\Lambda_z}{0}\right]V_z^*\in\mathbb{C}_*^{n\times r}$. The stabilizer group at Z is given by $T\in\mathrm{GL}(\mathbb{C}^n)$ such that $Tz\in\{zU|U\in U(p,q)\}$. In a basis $e_1,\dots e_n$ for $\mathbb{C}^n$ where $e_1,\dots e_r$ span $\mathrm{Ran}(z)$ and $e_{r+1},\dots,e_n$ span $\mathrm{Ran}(z)^\perp$ the stabilizer is therefore given by

$$\mathbb{H}_Z^{r,0}=\left\{\left[\begin{array}{c|c}\Lambda_zU\Lambda_z^{-1}&M_1\\\hline 0&M_2\end{array}\right]\middle| \quad U\in U(p,q), M_1\in\mathbb{C}^{r\times n-r}, M_2\in\mathbb{C}^{r\times r},\det(M_2)\neq 0\right\}\tag{B.10}$$

It is easy to see that $\mathbb{H}_Z^{r,0}$ is a (relatively) closed subset of $\mathrm{GL}(\mathbb{C}^n)$, hence by the homogeneous space construction theorem $\mathring{S}^{r,0}(\mathbb{C}^n)$ is diffeomorphic to the analytic manifold $\mathrm{GL}(\mathbb{C}^n)/\mathbb{H}_Z^{r,0}$. This concludes the proof of (i). Claims (ii) and (iii) represent slight generalizations over the analogous results in [8] for positive definite matrices, but the same key theorems apply. Namely, we employ the following:

Theorem B.2 (see [17] Proposition 2.28). *Let (M,g) be a Riemannian manifold and let G be a compact Lie group of isometries acting freely on M. Then let $N=M/G$ and $\pi:M\to N$ be the quotient map. Then there exists a unique Riemannian metric h on N so that $\pi:(M,g)\to(N,h)$ is a Riemannian submersion; and in particular that $D\pi(z):H_{\pi,z}\to T_{\pi(z)}(N)$ is isometric for each $z\in M$.*

Theorem B.3 (see [17] Proposition 2.109). *If $\pi:(M,g)\to(N,h)$ is a Riemannian submersion and γ is a geodesic in (M,g) such that $\dot\gamma(0)$ is horizontal (i.e. $\dot\gamma(0)\in H_{\pi,\gamma(0)}$) then*

(i) $\dot{\gamma}(t)$ *is horizontal for all* t
(ii) $\pi \circ \gamma$ *is a geodesic in* (N, h) *of the same length as* γ

In our case we are interested in the geometry of $\mathbb{C}_*^{n\times r}/U(r)$, where $\mathbb{C}_*^{n\times r}$ is an open subset of $\mathbb{C}^{n\times r}$ and is therefore a smooth Riemannian manifold of constant metric when equipped with the standard real inner product on $\mathbb{C}^{n\times r}$

$$\langle A, B\rangle_{\mathbb{R}} = \Re\mathrm{tr}\{A^*B\} \tag{B.11}$$

The relevant compact Lie group of isometries will be $U(r)$, acting by matrix multiplication on the right. We note that while $U(r)$ does not act freely on $\mathbb{C}^{n\times r}$, it does act freely on $\mathbb{C}_*^{n\times r}$ since for $x \in \mathbb{C}_*^{n\times r}$ and $W \in U(r)$

$$x = xW \iff x^*x = x^*xW \iff (x^*x)^{-1}(x^*x) = W \iff \mathbb{I}_{r\times r} = W \tag{B.12}$$

Therefore by Theorem B.2 there exists a metric h on $\mathbb{C}_*^{n\times r}/U(r)$ such that the differential of π at x

$$\begin{aligned} &D\pi(x) : (H_{\pi,x}(\mathbb{C}_*^{n\times r}), \langle\cdot,\cdot\rangle_{\mathbb{R}}) \to (T_{\pi(x)}(S^{r,0}(\mathbb{C}^n)), h) \\ &D\pi(x)(w) = xw^* + wx^* \end{aligned} \tag{B.13}$$

is an isometric isomorphism. Indeed

$$h(Z_1, Z_2) = \langle D\pi(x)^\dagger Z_1, D\pi(x)^\dagger Z_2\rangle_{\mathbb{R}} \tag{B.14}$$

Where $D\pi(x)^\dagger$ is the pseudo-inverse of the linear operator $D\pi(x)$. In this case, for $w_1, w_2 \in H_{\pi,x}(\mathbb{C}_*^{n\times r})$

$$h(D\pi(w_1), D\pi(w_2)) = \langle D\pi(x)^\dagger D\pi(w_1), D\pi(x)^\dagger D\pi(w_2)\rangle_{\mathbb{R}} = \langle w_1, w_2\rangle_{\mathbb{R}} \tag{B.15}$$

We now determine h explicitly. Namely, if $Z_1, Z_2 \in T_{\pi(x)}(\mathring{S}^{r,0}(\mathbb{C}^n)) = D\pi(H_{\pi,x}(\mathbb{C}_*^{n\times r}))$ then $Z_i = D\pi(x)(H_i x + X_i)$ where H_i, X_i are as in (4.6). We must have

$$\begin{aligned} h(Z_1, Z_2) &= \Re\mathrm{tr}[(H_1x + X_1)^*(H_2x + X_2)] \\ &= \Re\mathrm{tr}[x^*H_1H_2x] + \Re\mathrm{tr}[X_1^*X_2] \end{aligned} \tag{B.16}$$

We define $Z_i^{\|} := \mathbb{P}_{\mathrm{Ran}(x)} Z_i \mathbb{P}_{\mathrm{Ran}(x)} = xx^*H_i + H_ixx^*$ and $Z_i^{\perp} := \mathbb{P}_{\mathrm{Ran}(x)^\perp} Z_i \mathbb{P}_{\mathrm{Ran}(x)} = X_ix^*$. Then

$$\begin{aligned} H_i &= \int_0^\infty e^{-txx^*} Z_i^{\|} e^{-txx^*} dt \\ X_i &= Z_i^{\perp} x(x^*x)^{-1} \end{aligned} \tag{B.17}$$

Plugging these expressions into (B.16) yields the expression

(B.18)
$$\begin{aligned} h(Z_1, Z_2) &= \Re\mathrm{tr}\{xx^* \int_0^\infty e^{-txx^*} Z_1^{\|} e^{-txx^*} dt \int_0^\infty e^{-sxx^*} Z_2^{\|} e^{-sxx^*} ds\} + \Re\mathrm{tr}\{Z_1^{\perp *} Z_2^{\perp} (xx^*)^\dagger\} \\ &:= h_0(Z_1, Z_2) + h_1(Z_1, Z_2) \end{aligned}$$

The first term in (B.18) $h_0(Z_1, Z_2)$ can be simplified via the change of coordinates $u = t + s$ and $v = t - s$ as

$$\begin{aligned} h_0(Z_1,Z_2) &= \int_0^\infty \int_0^\infty \Re\mathrm{tr}\{e^{-xx^*(t+s)} Z_1^{\|} e^{-xx^*(t+s)} xx^* Z_2^{\|}\} ds dt \\ &= \frac{1}{2}\int_0^\infty \int_{-u}^{u} \Re\mathrm{tr}\{e^{-uxx^*} Z_1^{\|} e^{-uxx^*} xx^* Z_2^{\|}\} dv du \\ &= \int_0^\infty u \Re\mathrm{tr}\{e^{-uxx^*} Z_1^{\|} e^{-uxx^*} xx^* Z_2^{\|}\} du \\ &= \int_0^\infty u\mathrm{tr}\{e^{-uxx^*} Z_1^{\|} e^{-uxx^*} xx^* Z_2^{\|} + Z_2^{\|} xx^* e^{-uxx^*} Z_1^{\|} e^{-uxx^*}\} du \\ &= -\mathrm{tr}\{Z_2^{\|} \int_0^\infty u \frac{\partial}{\partial u} e^{-uxx^*} Z_1^{\|} e^{-uxx^*} du\} \\ &= \mathrm{tr}\{Z_2^{\|} \int_0^\infty e^{-uxx^*} Z_1^{\|} e^{-uxx^*} du\} \\ &= \langle H_1, Z_2\rangle_{\mathbb{R}} = \langle Z_1, H_2\rangle_{\mathbb{R}} \end{aligned} \tag{B.19}$$

Where the last equality follows from cycling under the trace immediately and then repeating the same calculation. With this metric in hand we have shown (ii), namely that the map

$$\pi : (\mathbb{C}_*^{n\times r}, \langle\cdot,\cdot\rangle_{\mathbb{R}}) \to (\mathring{S}^{r,0}(\mathbb{C}^n), h) \tag{B.20}$$

is a Riemannian submersion. To prove (iii), let $A, B \in \mathring{S}^{r,0}(\mathbb{C}^n)$ and let xx^* and yy^* be their respective Cholesky decompositions, so that $x, y \in \mathbb{C}_*^{n\times r}$. Consider the following straight line curve in $\mathbb{C}^{n\times r}$:

$$\begin{gathered} \sigma_{x,y} : [0,1] \to \mathbb{C}^{n\times r} \\ \sigma_{x,y}(t) = (1-t)x + tyU \end{gathered} \tag{B.21}$$

Where U is a polar factor such that $x^*yU = |x^*y|$ (equivalently U is a minimizer of the distance D, as in Proposition 3.4). The claim is that we will be able to apply Theorem B.3 to the pushforward of $\sigma_{x,y}$, proving that it is a geodesic connecting $A = \pi(x)$ to $B = \pi(yU)$. Specifically, we would like to prove

$$\sigma_{x,y}(t) \in \mathbb{C}_*^{n\times r} \qquad \forall t \in [0,1] \tag{B.22}$$

$$\dot\sigma_{x,y}(0) \in H_{\pi,x}(\mathbb{C}_*^{n\times r}) \tag{B.23}$$

We first prove (B.22), namely that $\sigma_{x,y}(t)$ does not drop rank as t varies from 0 to 1 even though $\mathbb{C}_*^{n\times r}$ is not convex. The endpoints $\sigma_{x,y}(0) = x$ and $\sigma_{x,y}(1) = yU$ are of course full rank, so it is enough to prove it for $t \in (0,1)$. Consider $x^*\sigma_{x,y}(t)$:

$$x^*\sigma_{x,y}(t) = (1-t)\underbrace{x^*x}_{\in\,\mathbb{P}(r)} \quad + \quad t\underbrace{x^*yU}_{|x^*y|\,\in\,PSD(r)} \in \mathbb{P}(r) \text{ for } t \in (0,1) \tag{B.24}$$

This implies that $\sigma_{x,y}(t) \in \mathbb{C}_*^{n\times r}$ for $t \in (0,1)$, so (B.22) is proved. Let $v = \dot\sigma_{x,y}(0) =$

$yU - x$. Then

$$\begin{aligned} x^*v &= -x^*x + x^*yU = -x^*x + (x^*yy^*x)^{\frac{1}{2}} \\ \mathbb{P}_{\mathrm{Ran}(x)}v &= -(xx^*)^\dagger xx^*x + (xx^*)^\dagger x(x^*yy^*x)^{\frac{1}{2}} \\ \mathbb{P}_{\mathrm{Ran}(x)}v &= \underbrace{\left(-\mathbb{P}_{\mathrm{Ran}(x)} + (xx^*)^\dagger x(x^*yy^*x)^{\frac{1}{2}}x^*(xx^*)^\dagger\right)}_{H} x \\ v &= Hx + X, \quad \mathbb{P}_{\mathrm{Ran}(x)}X = 0, \quad H^* = \mathbb{P}_{\mathrm{Ran}(x)}H = H \end{aligned} \tag{B.25}$$

Hence (B.23) is proved and so by Theorem B.3 we have that $\gamma_{A,B} := \pi \circ \sigma_{x,y}$ is a geodesic on $(\mathring{S}^{r,0}(\mathbb{C}^n), h)$ connecting A and B. We find specifically that this geodesic is given by

$$\begin{aligned} \gamma_{A,B}(t) &= \pi((1-t)x + tyU) \\ &= ((1-t)x + tyU)((1-t)x + tyU)^* \\ &= (1-t)^2 xx^* + t^2 yy^* + t(1-t)(xU^*y^* + yUx^*) \end{aligned} \tag{B.26}$$

Clearly $A = xx^*$ and $B = yy^*$, but what about xU^*y^* and yUx^*? Fortunately, a minor miracle occurs. Namely,

$$\begin{aligned} (yUx^*)^2 &= yUx^*yUx^* = yU|x^*y|x^* = y(|x^*y|U^*)^*x^* = y(x^*y)^*x^* = yy^*xx^* \\ (xU^*y^*)^2 &= xU^*y^*xU^*y^* = x(x^*yU)^*U^*y^* = x|x^*y|U^*y^* = xx^*yy^* \end{aligned} \tag{B.27}$$

Thus in fact xU^*y^* and yUx^* are matrix square roots (not necessarily symmetric, but having positive non-zero eigenvalues) for BA and AB respectively. We obtain the following expression for the family of geodesics on $\mathring{S}^{r,0}(\mathbb{C}^n)$ connecting A and B

$$\gamma_{A,B}(t) = (1-t)^2 xx^* + t^2 yy^* + t(1-t)(xU_0^*y^* + yU_0x^*) + t(1-t)(xU_1^*y^* + yU_1x^*) \tag{B.28}$$

Where U_0 and U_1 are as in Proposition 3.4. The fact that the form of this expression is independent of r is somewhat surprising, and motivates claims (iv) and (v). In order to prove (iv) we must first check that the collection of smooth manifolds $(\mathring{S}^{i,0}(\mathbb{C}^n))_{i=0}^r$ provide a stratification of the cone $S^{r,0}(\mathbb{C}^n)$ (conditions (a) and (b) of Definition 4.2). Condition (a) is satisfied trivially and for (b) we note that

$$\overline{\mathring{S}^{i,0}(\mathbb{C}^n)} \setminus \mathring{S}^{i,0}(\mathbb{C}^n) = \{0\} \cup \mathring{S}^{1,0} \cup \cdots \cup S^{i-1,0} \tag{B.29}$$

It remains to check that whenever $p > q$ the triple $(\mathring{S}^{p,0}(\mathbb{C}^n), \mathring{S}^{q,0}(\mathbb{C}^n), A)$ is a-regular and b-regular for $A \in \mathring{S}^{q,0} \subset \overline{\mathring{S}^{p,0}}$. It was noted by John Mather in Proposition 2.4 of [24] that b-regularity implies a-regularity, but we will use a-regularity in our proof of b-regularity so we need to prove a-regularity first. Specifically, a-regularity in this case states that if $(A_i)_{i\geq 1} \subset \mathring{S}^{p,0}(\mathbb{C}^n)$ converges to $A \in \mathring{S}^{q,0}(\mathbb{C}^n)$ and if $T_{A_i}(\mathring{S}^{p,0}(\mathbb{C}^n))$ converges in Grassmannian sense to the vector space τ_A then $T_A(\mathring{S}^{q,0}(\mathbb{C}^n)) \subset \tau_A$. Upon examining the form of the tangent space as given by (4.7) it becomes clear that convergence of the tangent spaces $T_{A_i}(\mathring{S}^{p,0}(\mathbb{C}^n))$ is equivalent to convergence of $\mathrm{Ran}A_i$ to a space we denote L, so that the Grassmannian limit of the tangent spaces is given by

$$\tau_A = \{W \in \mathrm{Sym}(\mathbb{C}^n) | \mathbb{P}_{L^\perp} W \mathbb{P}_{L^\perp} = 0\} \tag{B.30}$$

It is evident that L should contain as a subspace Ran$A$, and that this would prove that the stratification given is a-regular. Indeed, if $A_i = U_i\Lambda_i U_i^*$ is the low rank diagonalization of A_i so that $\Lambda_i = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ is the diagonal matrix of non-zero eigenvalues of A_i and $U_iU_i^* = \mathbb{P}_{\mathrm{Ran}A_i}$, $U_i^*U_i = \mathbb{I}_{p\times p}$ then by compactness we can obtain a subsequence of $(U_i)_{i\geq 1}$ that converges to a matrix U such that the columns of U are precisely an orthonormal basis for L. In this case, we may write $A = U\Lambda U^*$ since $A = \lim_{i\to\infty} U_i\Lambda_i U_i^*$ and the sequences of eigenvalues converge (some to zero), so that if $U = [u_1|\cdots|u_p]$ then

$$\mathrm{Ran}A = \mathrm{span}\{u_i|\Lambda_{ii} \neq 0\} \subset \mathrm{span}\{u_i\}_{i=1}^p = L \tag{B.31}$$

Thus, owing to (B.30) and the description of the tangent space in (4.7) we conclude that $\mathbb{T}_A(\mathring{S}^{q,0}(\mathbb{C}^n)) \subset \tau_A$ and our stratification is a-regular. As for b-regularity, let $(A_i)_{i\geq 1} \subset \mathring{S}^{p,0}(\mathbb{C}^n)$, $A \in \mathring{S}^{q,0}(\mathbb{C}^n)$, and τ_A be as before (specifically we assume the Grassmannian limit defining τ_A converges) and let $(B_i)_{i\geq 1} \subset \mathring{S}^{q,0}(\mathbb{C}^n)$ be convergent also to A such that the following limit exists

$$Q = \lim_{i\to\infty} Q_i := \lim_{i\to\infty} \frac{A_i - B_i}{||A_i - B_i||_2} \tag{B.32}$$

We claim that $Q \in \tau_A$. Specifically, let $\Theta_i = A_i - \mathbb{P}_{\mathrm{Ran}(A_i)}B_i\mathbb{P}_{\mathrm{Ran}(A_i)}$ and $\Psi_i = \mathbb{P}_{\mathrm{Ran}(A_i)}B_i\mathbb{P}_{\mathrm{Ran}(A_i)} - B_i$. Then either $\Psi_i = 0$, in which case $Q_i = \Theta_i/||\Theta_i||_2$, or $\Psi_i \neq 0$, so that

$$Q_i = \frac{||\Theta_i||_2}{||A_i - B_i||_2}\frac{\Theta_i}{||\Theta_i||_2} + \frac{||\Psi_i||_2}{||A_i - B_i||_2}\frac{\Psi_i}{||\Psi_i||_2} \tag{B.33}$$

We will obtain convergent subsequences for the sequences of unit norm matrices $\Theta_i/||\Theta_i||_2$ and $\Psi_i/||\Psi_i||_2$, but first note that

$$\frac{||\Theta_i||_2}{||A_i - B_i||_2} = \frac{||\mathbb{P}_{\mathrm{Ran}(A_i)}(A_i - B_i)\mathbb{P}_{\mathrm{Ran}(A_i)}||_2}{||A_i - B_i||_2} \leq 1 \tag{B.34}$$

Hence $||\Psi_i||_2/||A_i - B_i||_2$ is also a bounded sequence (if it were not Q_i would fail to converge). Next note that for i sufficiently large $\Psi_i = \mathbb{P}_{\mathrm{Ran}(A_i)}B_i\mathbb{P}_{\mathrm{Ran}(A_i)} - B_i$ is the difference of two matrices in $\mathring{S}^{q,0}(\mathbb{C}^n)$, both converging to A. Therefore, owing to the fact that $\mathring{S}^{q,0}(\mathbb{C}^n)$ is an analytic manifold, any convergent subsequence of $\Psi_i/||\Psi_i||_2$ will have its limit lying in $T_A(\mathring{S}^{q,0}(\mathbb{C}^n))$ (see for example Lemma 4.12 in [29]). Owing to the already proved a-regularity we conclude that the limit of any convergent subsequence of $\Psi_i/||\Psi_i||_2$ lies in τ_A. Similarly, $\Theta_i = \mathbb{P}_{\mathrm{Ran}(A_i)}(A_i - B_i)\mathbb{P}_{\mathrm{Ran}(A_i)}$ hence any convergent subsequence of $\Theta_i/||\Theta_i||_2$ must lie in τ_A. Thus we may obtain a subsequence such that the sequences of real numbers $||\Theta_{i_j}||_2/||A_{i_j} - B_{i_j}||_2$ and $||\Psi_{i_j}||_2/||A_{i_j} - B_{i_j}||_2$ converge to some $\alpha, \beta \in \mathbb{R}$ and the sequences of unit norm matrices $\Theta_{i_j}/||\Theta_{i_j}||_2$ and $\Psi_{i_j}/||\Psi_{i_j}||_2$ converge to some $\hat{\Theta}, \hat{\Psi} \in \tau_A$. Since $(Q_i)_{i\geq 1}$ converges, we find that

$$Q = \alpha\hat{\Theta} + \beta\hat{\Psi} \in \tau_A \tag{B.35}$$

Thus the stratification $(\mathring{S}^{i,0}(\mathbb{C}^n))_{i=0}^r$ is b-regular and in particular is a Whitney stratification of $S^{r,0}(\mathbb{C}^n)$.

In order to prove (v), let $A_i = x_i x_i^*$ and $B_i = y_i y_i^*$ be Cholesky decompositions of A_i and B_i such that $x_i, y_i \in \mathbb{C}^{n\times p}$ and note that we are told the following limit exists at each t

$$\delta(t) = \lim_{i\to\infty} (1-t)^2 x_i x_i^* + t^2 y_i y_i^* + t(1-t)(x_i U_i^* y_i^* + y_i U_i x_i^*) \tag{B.36}$$

Where $U_i \in U(p)$ is such that $x_i^* y_i U_i \geq 0$. We note that since $(A_i)_{i\geq 1}$ and $(B_i)_{i\geq 1}$ converge we may obtain convergent subsequences for their Cholesky factors x_i and y_i ($||x_i||_2$ and $||y_i||_2$ must both be bounded or else A_i and B_i would not converge). We may also obtain a convergent subsequence for $(U_i)_{i\geq 1}$ owing to the compactness of $U(p)$. Denote these subsequential limits by x, y, and U respectively and consider a combined subsequential indexing such that each occurs. Let V_x and V_y be the matrices of right singular vectors for x and y so that $x = [\hat{x}|0]V_x$ and $y = [\hat{y}|0]V_y$ for some $\hat{x}, \hat{y} \in \mathbb{C}_*^{n\times q}$. Then clearly

$$\delta(t) = (1-t)^2 \hat{x}\hat{x}^* + t^2 \hat{y}\hat{y}^* + t(1-t)(\hat{x}\hat{U}^*\hat{y}^* + \hat{y}\hat{U}\hat{x}^*) \tag{B.37}$$

Where $\hat{U}$ is the upper left $q \times q$ block of $V_y U V_x^*$. We will prove that in fact

$$V_y U V_x^* = \left[\begin{array}{c|c} \hat{U} & 0 \\ \hline 0 & \tilde{U} \end{array}\right] \tag{B.38}$$

In particular, this will imply that $\hat{U} \in U(q)$ since $V_y U V_x^* \in U(p)$ hence the upper left $q \times q$ blocks of $(V_y U V_x^*)(V_y U V_x^*)^*$ and $(V_y U V_x^*)^*(V_y U V_x^*)$ must both be equal to the $q \times q$ identity matrix. In order to prove (B.38), note that $U = VW^*$ where

$$x^* y = W \left[\begin{array}{c|c} \Sigma & 0 \\ \hline 0 & 0 \end{array}\right] V^* \tag{B.39}$$

is a singular value decomposition of $x^* y$. On the other hand if

$$\hat{x}^* \hat{y} = P \left[\begin{array}{c|c} \Lambda & 0 \\ \hline 0 & 0 \end{array}\right] Q^* \tag{B.40}$$

is a singular value decomposition for $\hat{x}^*\hat{y}$ then

$$x^* y = \underbrace{V_x^* \left[\begin{array}{c|c} P & 0 \\ \hline 0 & \tilde{P} \end{array}\right]}_{W} \left[\begin{array}{c|c} \begin{array}{c|c} \Lambda & 0 \\ \hline 0 & 0 \end{array} & 0 \\ \hline 0 & 0 \end{array}\right] \underbrace{\left[\begin{array}{c|c} Q & 0 \\ \hline 0 & \tilde{Q} \end{array}\right] V_y}_{V^*} \tag{B.41}$$

Where $\tilde{P}, \tilde{Q} \in U(p-q)$ are in general arbitrary, but may of course be chosen in accordance with W and V. Thus

$$V_y U V_x^* = V_y V W^* V_x = \begin{bmatrix} PQ & 0 \\ 0 & \tilde{P}\tilde{Q} \end{bmatrix} \tag{B.42}$$

is as in (B.38). The question remains whether $\hat{x}^*\hat{y}\hat{U} \geq 0$, but we note that

$$
\begin{aligned}
x^*yU &= V_x^* \begin{bmatrix} \hat{x}^*\hat{y} & 0 \\ 0 & 0 \end{bmatrix} V_y U \\
&= V_x^* \begin{bmatrix} \hat{x}^*\hat{y} & 0 \\ 0 & 0 \end{bmatrix} V_y U V_x^* V_x \\
&= V_x^* \begin{bmatrix} \hat{x}^*\hat{y} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{U} & 0 \\ 0 & \tilde{U} \end{bmatrix} V_x \\
&= V_x^* \begin{bmatrix} \hat{x}^*\hat{y}\hat{U} & 0 \\ 0 & 0 \end{bmatrix} V_x
\end{aligned} \tag{B.43}
$$

Thus x^*yU will be positive semidefinite only if $\hat{x}^*\hat{y}\hat{U}$ is positive semidefinite, and since $x^*yU = \lim_{i\to\infty} x_i^* y_i U_i = \lim_{i\to\infty} |x_i^* y_i| \geq 0$ we conclude that $\hat{x}^*\hat{y}\hat{U} \geq 0$. A nearly identical proof shows that $Ux^*y \geq 0$. We conclude that δ is a geodesic in $\mathring{S}^{q,0}(\mathbb{C}^n)$ connecting A and B. ∎

Appendix C. Proofs for Section 5.

C.1. Proof of Proposition 5.1.

Proof. We may first note that $\langle xx^*, A_j\rangle_{\mathbb{R}} - \langle yy^*, A_j\rangle_{\mathbb{R}} = \langle xx^* - yy^*, A_j\rangle_{\mathbb{R}}$. The expression (1.3) then becomes

$$
a_0 = \inf_{\substack{L\in S^{r,r}(\mathbb{C}^n)\\ ||L||_2=1}} \sum_{j=1}^m \langle L, A_j\rangle^2 \tag{C.1}
$$

The claim follows by contradiction if $S^{r,r}$ is closed. Explicitly, if $S^{r,r}$ is closed then $S^{r,r} \cap \{x \in \mathbb{C}^{n\times n} : ||x||_2 = 1\}$ is compact. Assume $a_0 = 0$, then there exists $L_0 \in S^{r,r} \cap \{x \in \mathbb{C}^{n\times n} : ||x||_2 = 1\}$ so that

$$
0 = \sum_{j=1}^m \langle L_0, A_j\rangle^2 \tag{C.2}
$$

This implies that the map β is not injective since, in particular, if $xx^* = (L_0)_+$ and $yy^* = (L_0)_-$ then $xx^* \neq yy^*$ since $||L_0||_2 = 1$ but $\beta(x) = \beta(y)$. It remains to show that the spaces $S^{p,q}$ and in particular $S^{r,r}$ are closed. Consider the map $\eta : \mathbb{C}^{n\times n} \to \{0,\dots,n\}^2$ with $\eta(A) = (\operatorname{rank}(A_+), \operatorname{rank}(A_-))$ taking A to its Sylvester indices (p, q). Then η is continuous with respect to the usual topology on $\mathbb{C}^{n\times n}$ and with respect to the "upper box" topology τ_{ub} on $\{0,\dots,n\}^2$ generated by the base

$$
\mathcal{B}_{\text{ub}} = \{\{x,\dots,n\} \times \{y,\dots,n\} | (x,y) \in \{0,\dots,n+1\}\} \tag{C.3}
$$

The maps $A \to A_\pm$ are continuous and it is well known that $\operatorname{rank}(A + B) \geq \operatorname{rank}(A)$ whenever $||B||_{2\to 2} < \sigma_{p+q}(A)$, hence η is continuous. Moreover $\{0,\dots,p\} \times \{0,\dots,q\}$ is closed in τ_{ub} hence $S^{p,q}$, its pullback through the continuous map η, is closed in $\mathbb{C}^{n\times n}$. ∎

C.2. Proof of Theorem 5.6.

Proof. We first prove that $a_0 = \inf_{z\in\mathbb{C}^{n\times r}} a(z)$. We note that

$$
a_0 = \inf_{\substack{x,y\in\mathbb{C}^{n\times r}\\ xx^*\neq yy^*}} \frac{1}{||xx^* - yy^*||_2^2} \sum_{j=1}^m |\langle xx^* - yy^*, A_j\rangle_{\mathbb{R}}|^2 \tag{C.4}
$$

We may change coordinates to $z = \frac{1}{2}(x+y)$ and $w = x - y$ so that

$$a_0 = \inf_{\substack{z,w\in\mathbb{C}^{n\times r}\\ zw^*+wz^*\neq 0}} \frac{1}{||zw^*+wz^*||_2^2}\sum_{j=1}^m |\langle zw^*+wz^*, A_j\rangle_{\mathbb{R}}|^2 \tag{C.5}$$

Recall that z has rank k, and therefore we may take $z = [\hat{z}|0]U$ for $\hat{z} \in \mathbb{C}_*^{n\times k}$ and $U \in U(r)$. We then define $\hat{w} \in \mathbb{C}^{n\times k}$ via the first k columns of wU^* then $zw^* + wz^* = \hat{z}\hat{w}^* + \hat{w}\hat{z}^* = D\pi(\hat{z})(\hat{w})$, so that in fact we may take $\hat{w} \in H_{\pi,\hat{z}}(\mathbb{C}_*^{n\times k}) \setminus \{0\}$. We obtain

$$\begin{aligned} a_0 &= \inf_{z\in\mathbb{C}^{n\times r}\setminus\{0\}} \inf_{\hat{w}\in H_{\pi,\hat{z}}(\mathbb{C}_*^{n\times k})\setminus\{0\}} \frac{1}{||D\pi(\hat{z})(\hat{w})||_2^2}\sum_{j=1}^m |\langle D\pi(\hat{z})(\hat{w}), A_j\rangle_{\mathbb{R}}|^2 \\ &= \inf_{z\in\mathbb{C}^{n\times r}\setminus\{0\}} \min_{\substack{W\in T_{\pi(\hat{z})}(\mathring{S}^{k,0}(\mathbb{C}^n))\\ ||W||_2=1}} \sum_{j=1}^m |\langle W, A_j\rangle_{\mathbb{R}}|^2 \\ &= \inf_{\substack{z\in\mathbb{C}^{n\times r}\\ ||z||_2=1}} \min_{\substack{W\in T_{\pi(\hat{z})}(\mathring{S}^{k,0}(\mathbb{C}^n))\\ ||W||_2=1}} \sum_{j=1}^m |\langle W, A_j\rangle_{\mathbb{R}}|^2 \\ &= \inf_{\substack{z\in\mathbb{C}^{n\times r}\\ ||z||_2=1}} a(z) \end{aligned} \tag{C.6}$$

This proves (5.11). The first two inequalities of (5.12) are clear from the definitions of the quantities involved, namely $a_0 \leq a_2(z) \leq a_1(z)$. It remains to prove that $a_1(z) \leq a(z)$. We will need the following families of real-linear subspaces of $\mathbb{C}^{n\times r}$ indexed by $z \in \mathbb{C}^{n\times r}$.

(C.7)
$$H_z = \{Hz + X | H \in \mathbb{C}^{n\times n}, H^* = H = \mathbb{P}_{\mathrm{Ran}(z)}H, X \in \mathbb{C}^{n\times r}, \mathbb{P}_{\mathrm{Ran}(z)}X = 0, X\mathbb{P}_{\ker(z)} = 0\}$$

(C.8)
$$\Delta_z = \{w \in \mathbb{C}^{n\times r}| \quad \exists\rho > 0 \quad \forall|\epsilon| < \rho \quad z^*(z+\epsilon w) \geq 0\}$$

(C.9)
$$\Gamma_z = \{y \in \mathbb{C}^{n\times r}|\mathbb{P}_{\mathrm{Ran}(z)}y = 0, \quad y\mathbb{P}_{\ker(z)} = y\}$$ $\blacksquare$

LEMMA C.1. *The space Δ_z is alternately characterized as*

$$\Delta_z = \{w \in \mathbb{C}^{n\times r}|z^*w = w^*z\} \tag{C.10}$$

And is thus manifestly a real-linear subspace. Moreover, Δ_z decomposes orthogonally into

$$\Delta_z = H_z \oplus \Gamma_z \tag{C.11}$$

Finally, if $z = [\hat{z}|0]U$ for $\hat{z} \in \mathbb{C}_*^{n\times k}$ then

$$H_z = \left[H_{\pi,\hat{z}}(\mathbb{C}_*^{n\times k})\middle|0\right]U \tag{C.12}$$

Proof. Clearly a necessary and sufficient condition for $w \in \Delta_z$ is that $z^*w = w^*z$, for in this case take $|\epsilon| < \sigma_k(z)/||w||_2$. We can use this condition to obtain a parametrization for Δ_z:

$$\begin{aligned}
w \in \Delta_z &\iff z^*w = w^*z \\
&\iff z^*w = \tilde{H} \qquad \tilde{H} \in \mathbb{C}^{r\times r}, \tilde{H}^* = \tilde{H} = \mathbb{P}_{\ker(z)^\perp}\tilde{H} \\
&\iff z^*w = z^*Hz \quad H \in \mathbb{C}^{n\times n}, H^* = H = \mathbb{P}_{\mathrm{Ran}(z)}H \\
&\iff w = Hz + X \quad H \in \mathbb{C}^{n\times n}, H^* = H = \mathbb{P}_{\mathrm{Ran}(z)}H, X \in \mathbb{C}^{n\times r}, \mathbb{P}_{\mathrm{Ran}(z)}X = 0
\end{aligned} \tag{C.13}$$

This proves (C.11), with orthogonality easily verified. To prove (C.12) note that if $z = [\hat{z}|0]U$ for $\hat{z} \in \mathbb{C}_*^{n\times k}$, $U \in U(r)$, and $w = Hz + X \in H_z$ then the condition $X\mathbb{P}_{\ker(z)} = 0$ implies $X = [\tilde{X}|0]U$ for $\tilde{X} \in \mathbb{C}^{n\times k}$ and $\mathbb{P}_{\mathrm{Ran}(z)}X = 0$ if and only if $\mathbb{P}_{\mathrm{Ran}(z)}\tilde{X} = 0$. Thus

$$\begin{aligned}
H_z &= \{H[\hat{z}|0]U + [\tilde{X}|0]U | H \in \mathbb{C}^{n\times n}, H^* = H = \mathbb{P}_{\mathrm{Ran}(z)}H, \tilde{X} \in \mathbb{C}^{n\times k}, \mathbb{P}_{\mathrm{Ran}(z)}\tilde{X} = 0\} \\
&= \{[H\hat{z} + \tilde{X}|0]U | H \in \mathbb{C}^{n\times n}, H^* = H = \mathbb{P}_{\mathrm{Ran}(\hat{z})}, \tilde{X} \in \mathbb{C}^{n\times k}, \mathbb{P}_{\mathrm{Ran}(\hat{z})}\tilde{X} = 0\} \\
&= [H_{\pi,\hat{z}}(\mathbb{C}_*^{n\times k})|0]U
\end{aligned} \tag{C.14}$$

∎

With this lemma in mind, we may transform $a_1(z)$ into a linear minimization problem over Δ_z. Namely

$$\begin{aligned}
a_1(z) &= \lim_{R\to 0} \inf_{\substack{x\in\mathbb{C}^{n\times r} \\ ||xx^*-zz^*||_2<R}} \frac{\sum_{j=1}^m |\langle xx^* - zz^*, A_j\rangle_\mathbb{R}|^2}{||xx^* - zz^*||_2^2} \\
&= \lim_{R\to 0} \inf_{\substack{x\in\mathbb{C}^{n\times r} \\ ||xx^*-zz^*||_2<R \\ z^*x\geq 0}} \frac{\sum_{j=1}^m |\langle xx^* - zz^*, A_j\rangle_\mathbb{R}|^2}{||xx^* - zz^*||_2^2}
\end{aligned} \tag{C.15}$$

We can add the $z^*x \geq 0$ constraint without altering the infimimum since doing so amounts to a choice of representative for x, but x only appears as $\pi(x) = xx^*$. We now show the following lemma, implying that we may instead minimize over $||x-z||_2 < R$.

Lemma C.2. *For all $z \in \mathbb{C}^{n\times r}$ and $\epsilon > 0$ there exists $\delta > 0$ such that if $z^*x \geq 0$ and $||zz^* - xx^*||_2 < \delta$ then $||z - x||_2 < \epsilon$.*

Proof. We begin with the fact that the operation

$$\begin{aligned}
\zeta &: PSD(n) \to PSD(n) \\
\zeta(A) &= \sqrt{\mathrm{tr}A}\sqrt{A}
\end{aligned} \tag{C.16}$$

is continuous with respect to the topology induced by the Frobenius norm. Note that $\zeta(xx^*) = ||x||_2(xx^*)^{\frac{1}{2}} = \psi(x)$ (the embedding ψ as given in Definition 3.5). Therefore, given any $z \in \mathbb{C}^{n\times r}$ and ϵ_1 there exists δ such that

$$||xx^* - zz^*||_2 < \delta \implies ||||x||_2(xx^*)^{\frac{1}{2}} - ||z||_2(zz^*)^{\frac{1}{2}}||_2 < \epsilon_1 \tag{C.17}$$

The latter expression here is of course $||\psi(x) - \psi(z)||_2$, which satisfies $||\psi(x) - \psi(z)||_2 \geq \frac{1}{2}D(x,z)^2$ by (A.19). If $z^*x \geq 0$ then $D(x,z) = ||x - z||_2$, so if we take $\epsilon_1 = \frac{\epsilon^2}{2}$ then the above δ satisfies the lemma. ∎

With this lemma in hand we may freely replace $||xx^* - zz^*||_2$ by $||x-z||_2$ in the infimization constraint for $a_1(z)$ (note that the converse of the lemma is immediate since π is continuous with respect to the topology induced by the Frobenius norm). After doing so, we change variables from x to $w = x - z$ so that

$$\begin{aligned}
a_1(z) &= \lim_{R\to 0} \inf_{\substack{x\in\mathbb{C}^{n\times r}\\ ||x-z||_2<R \\ z^*x\geq 0}} \frac{\sum_{j=1}^m |\langle xx^*-zz^*, A_j\rangle_{\mathbb{R}}|^2}{||xx^*-zz^*||_2^2} \\
&= \lim_{R\to 0} \inf_{\substack{w\in\mathbb{C}^{n\times r}\\ ||w||_2<R \\ z^*(z+w)\geq 0}} \frac{\sum_{j=1}^m |\langle zw^*+wz^*+ww^*, A_j\rangle_{\mathbb{R}}|^2}{||zw^*+wz^*+ww^*||_2^2} \\
&= \lim_{R\to 0} \inf_{\substack{w\in\Delta_z\\ ||w||_2<R}} \frac{\sum_{j=1}^m |\langle zw^*+wz^*+ww^*, A_j\rangle_{\mathbb{R}}|^2}{||zw^*+wz^*+ww^*||_2^2} \\
&\leq \lim_{R\to 0} \inf_{\substack{w\in H_z\\ ||w||_2<R}} \frac{\sum_{j=1}^m |\langle zw^*+wz^*+ww^*, A_j\rangle_{\mathbb{R}}|^2}{||zw^*+wz^*+ww^*||_2^2} \\
&= \lim_{R\to 0} \inf_{\substack{w\in H_z\\ ||w||_2<R}} \frac{\sum_{j=1}^m |\langle zw^*+wz^*+ww^*, A_j\rangle_{\mathbb{R}}|^2}{||zw^*+wz^*||_2^2 + ||ww^*||_2^2 + 4\Re\mathrm{tr}\{zw^*ww^*\}} \\
&\leq \lim_{R\to 0} \inf_{\substack{w\in H_z\\ ||w||_2<R}} \frac{\sum_{j=1}^m |\langle zw^*+wz^*+ww^*, A_j\rangle_{\mathbb{R}}|^2}{||zw^*+wz^*||_2^2(1+4\frac{\Re\mathrm{tr}\{zw^*ww^*\}}{||zw^*+wz^*||_2^2})}
\end{aligned} \tag{C.18}$$

We need to show that the ratio

$$R(w) = 4\frac{|\Re\mathrm{tr}\{zw^*ww^*\}|}{||zw^*+wz^*||_2^2} \tag{C.19}$$

is $O(||w||)$ when $w \in H_z$. We employ the parametrization of H_z given in (C.7) and note that for $w = Hz + X$

$$||zw^*+wz^*||_2^2 = 2(||z^*Hz||_2^2 + ||zz^*H||_2^2 + ||zX^*||_2^2) \tag{C.20}$$

$$\Re\mathrm{tr}\{zw^*ww^*\} = \Re\mathrm{tr}\{z^*H^2zz^*Hz\} + \Re\mathrm{tr}\{X^*Xz^*Hz\} \tag{C.21}$$

Thus we find

$$\begin{aligned}
R(w) &\leq \frac{2|\Re\mathrm{tr}\{z^*H^2zz^*Hz\}| + 2|\Re\mathrm{tr}\{X^*Xz^*Hz\}|}{||z^*Hz||_2^2 + ||zz^*H||_2^2 + ||zX^*||_2^2} \\
&\leq 2\frac{|\Re\mathrm{tr}\{z^*H^2zz^*Hz\}|}{||z^*Hz||_2^2} + 2\frac{|\Re\mathrm{tr}\{X^*Xz^*Hz\}|}{||zX^*||_2^2 + ||z^*Hz||_2^2} \\
&\leq 2\frac{||z^*H^2z||_2}{||z^*Hz||_2} + \frac{||X^*X||_2}{||zX^*||_2}
\end{aligned} \tag{C.22}$$

Up until this point we have not used the fact that $H\mathbb{P}_{\mathrm{Ran}(z)} = H = \mathbb{P}_{\mathrm{Ran}(z)}H$ and $X\mathbb{P}_{\ker(z)} = 0$. We do so now by noting that if $z = U_1\Lambda V^*$ for $U_1 \in \mathbb{C}^{n\times k}$ such that $U_1U_1^* = \mathbb{P}_{\mathrm{Ran}(z)}$, $\Lambda = \mathrm{diag}(\sigma_1(z), \ldots, \sigma_k(z))$ is the diagonal matrix of ordered singular values $\sigma_1(z) \geq \cdots \geq \sigma_k(z) > 0$, and $V_1 \in \mathbb{C}^{r\times k}$ such that $V_1V_1^* = \mathbb{P}_{\ker(z)^\perp}$

then

$$\text{(C.23)}\quad ||z^*H^2z|| = ||\Lambda U_1^* H^2 U_1 \Lambda||_2 \le \sigma_1(z)^2 ||U_1^* H^2 U_1||_2 = \sigma_1(z)^2 \sqrt{\mathrm{tr}\{\mathbb{P}_{\mathrm{Ran}(z)} H^2 \mathbb{P}_{\mathrm{Ran}(z)} H^2\}} = \sigma_1(z)^2 ||H^2||_2$$

$$||z^*Hz|| = ||\Lambda U_1^* H U_1 \Lambda||_2 \ge \sigma_k(z)^2 ||U_1^* H U_1||_2 = \sigma_k(z)^2 \sqrt{\mathrm{tr}\{\mathbb{P}_{\mathrm{Ran}(z)} H \mathbb{P}_{\mathrm{Ran}(z)} H\}} = \sigma_k(z) ||H||_2$$

$$||zX^*||_2 = ||\Lambda V_1^* X^*||_2 = ||\Lambda (XV_1)^*||_2 \ge \sigma_k(z) ||XV_1||_2 = \sigma_k(z) \sqrt{\mathrm{tr}\{X \mathbb{P}_{\ker(z)^\perp} X^*\}} = \sigma_k(z) ||X||_2$$

Thus if $\kappa(z) = \sigma_1(z)/\sigma_k(z)$ is the condition number of z we find

$$\text{(C.24)}\quad \begin{aligned} R(w) &\le 2\kappa(z)^2 \frac{||H^2||_2}{||H||_2} + \sigma_k(z)^{-1} \frac{||X^*X||_2}{||X||_2} \\ &\le 2\kappa(z)^2 ||H||_2 + \sigma_k^{-1}(z) ||X||_2 \\ &\le 2\kappa(z)^2 \sigma_k(z)^{-1} ||Hz||_2 + \sigma_k^{-1}(z) ||X||_2 \\ &\le \frac{\sqrt{2}\max(2\kappa(z)^2, 1)}{\sigma_k(z)} \sqrt{||Hz||_2^2 + ||X||_2^2} \\ &= \underbrace{\frac{2\sqrt{2}\kappa(z)^2}{\sigma_k(z)}}_{C(z)} ||w||_2 \end{aligned}$$

Thus returning to $a_1(z)$ we obtain

$$\text{(C.25)}\quad \begin{aligned} a_1(z) &\le \lim_{R\to 0} \inf_{\substack{w\in H_z \\ ||w||_2 < R}} \frac{\sum_{j=1}^m |\langle zw^* + wz^*, A_j\rangle_{\mathbb{R}}|^2}{||zw^* + wz^*||_2^2} (1 + 2C(z)||w||_2) \\ &= \inf_{\substack{w\in H_z \\ w\neq 0}} \frac{\sum_{j=1}^m |\langle zw^* + wz^*, A_j\rangle_{\mathbb{R}}|^2}{||zw^* + wz^*||_2^2} \\ &= \inf_{\substack{w\in H_{\pi,\hat{z}} \\ \hat{w}\neq 0}} \frac{\sum_{j=1}^m |\langle \hat{z}\hat{w}^* + \hat{w}\hat{z}^*, A_j\rangle_{\mathbb{R}}|^2}{||\hat{z}\hat{w}^* + \hat{w}\hat{z}^*||_2^2} \\ &= \min_{\substack{W\in T_{\pi(\hat{z})}(\mathring{S}^{k,0}(\mathbb{C}^n)) \\ ||W||_2 = 1}} \sum_{j=1}^m |\langle W, A_j\rangle_{\mathbb{R}}|^2 \\ &= a(z) \end{aligned}$$

This proves (5.12). In order to prove (5.14) we will employ an explicit parametrization of $T_{\pi(\hat{z})}(\mathring{S}^{k,0}(\mathbb{C}^n))$ implied by (4.7). The condition on $W \in \mathrm{Sym}(\mathbb{C}^n)$ in (4.7) that $\mathbb{P}_{\mathrm{Ran}(z)^\perp} W \mathbb{P}_{\mathrm{Ran}(z)^\perp} = 0$ implies that

$$\text{(C.26)}\quad W \in T_{\pi(\hat{z})}(\mathring{S}^{k,0}(\mathbb{C}^n)) \iff W = W_1 + \frac{1}{2}(W_2 + W_2^*)$$

For $W_1, W_2 \in \mathbb{C}^{n\times n}$ where $\mathbb{P}_{\mathrm{Ran}(z)} W_1 = W_1 = W_1^*$, $\mathbb{P}_{\mathrm{Ran}(z)} W_2 = 0$, and $W_2 \mathbb{P}_{\mathrm{Ran}(z)} = W_2$. In other words, if $U_1 \in \mathbb{C}^{n\times k}$ and $U_2 \in \mathbb{C}^{n\times n-k}$ are as in Definition 5.4 then

$$\text{(C.27)}\quad T_{\pi(\hat{z})}(\mathring{S}^{k,0}) = \{U_1 A U_1^* + \frac{1}{2}(U_2 B U_1^* + U_1 B^* U_2^*) | A \in \mathrm{Sym}(\mathbb{C}^k), B \in \mathbb{C}^{n-k\times k}\}$$

We will now employ the fact that the maps τ and μ in (5.6) are isometries. Specifically, if $A, B \in \mathrm{Sym}(\mathbb{C}^n)$ then $\langle A, B\rangle_{\mathbb{R}} = \tau(A)^T\tau(B)$ and if $X, Y \in \mathbb{C}^{n\times r}$ then $\langle X, Y\rangle_{\mathbb{R}} = \mu(X)^T\mu(Y)$. With this in mind, we obtain that for $W \in T_{\pi(\hat{z})}(\mathring{S}^{k,0})$

$$\begin{aligned}\sum_{j=1}^m |\langle W, A_j\rangle_{\mathbb{R}}|^2 &= \sum_{j=1}^m |\langle U_1AU_1^* + \frac{1}{2}(U_2BU_1^* + U_1B^*U_2^*), A_j\rangle_{\mathbb{R}}|^2 \\ &= \sum_{j=1}^m |\langle U_1AU_1^*, A_j\rangle_{\mathbb{R}} + \langle U_2BU_1^*, A_j\rangle_{\mathbb{R}}|^2 \\ &= \sum_{j=1}^m |\langle A, U_1^*A_jU_1\rangle_{\mathbb{R}} + \langle B, U_2^*A_jU_1\rangle_{\mathbb{R}}|^2 \\ &= \sum_{j=1}^m \left(\begin{bmatrix}\tau(A)\\ \mu(B)\end{bmatrix}^T \begin{bmatrix}\tau(U_1^*A_jU_1)\\ \mu(U_2^*A_jU_1)\end{bmatrix}\right)^2 \\ &= \begin{bmatrix}\tau(A)\\ \mu(B)\end{bmatrix}^T \left(\sum_{j=1}^m \begin{bmatrix}\tau(U_1^*A_jU_1)\\ \mu(U_2^*A_jU_1)\end{bmatrix}\begin{bmatrix}\tau(U_1^*A_jU_1)\\ \mu(U_2^*A_jU_1)\end{bmatrix}^T\right)\begin{bmatrix}\tau(A)\\ \mu(B)\end{bmatrix} \\ &= \mathcal{W}^TQ_z\mathcal{W}\end{aligned} \tag{C.28}$$

Where $\mathcal{W} = \begin{bmatrix}\tau(A)\\ \mu(B)\end{bmatrix} \in \mathbb{R}^{k^2+2k(n-k)} = \mathbb{R}^{2nk-k^2}$. Meanwhile, again owing to the fact that τ and μ are isometries, we find that for $W \in T_{\pi(\hat{z})}(\mathring{S}^{k,0})$ we have $||W||_2 = ||\mathcal{W}||_2$. Thus returning to our computation of $a(z)$

$$\begin{aligned}a(z) &= \min_{\substack{W\in T_{\pi(\hat{z})}(\mathring{S}^{k,0}(\mathbb{C}^n))\\ ||W||_2=1}} \sum_{j=1}^m |\langle W, A_j\rangle_{\mathbb{R}}|^2 \\ &= \min_{\substack{\mathcal{W}\in\mathbb{R}^{2nk-k^2}\\ ||\mathcal{W}||_2=1}} \mathcal{W}^TQ_z\mathcal{W} \\ &= \lambda_{2nk-k^2}(Q_z)\end{aligned} \tag{C.29}$$

This concludes the proof of $(i)-(iii)$. As for (iv) and (v) note that when $\mathrm{rank}(x) \le k$ then we may find $P \in U(r)$ such that $x = [\hat{x}|0]P$ for $\hat{x} \in \mathbb{C}^{n\times k}$ and moreover $d(x, z) = d(\hat{x}, \hat{z})$ and $xx^* - zz^* = \hat{x}\hat{x}^* - \hat{z}\hat{z}^*$. Thus

$$\begin{aligned}\hat{a}_1(z) &= \lim_{R\to 0} \inf_{\substack{x\in\mathbb{C}^{n\times r}\\ d(z,x)<R\\ \mathrm{rank}(x)\le k}} \frac{\sum_{j=1}^m |\langle xx^* - zz^*, A_j\rangle_{\mathbb{R}}|^2}{d(x, z)^2} \\ &= \lim_{R\to 0} \inf_{\substack{\hat{x}\in\mathbb{C}^{n\times k}\\ d(\hat{x},\hat{z})<R}} \frac{\sum_{j=1}^m |\langle \hat{x}\hat{x}^* - \hat{z}\hat{z}^*, A_j\rangle_{\mathbb{R}}|^2}{d(\hat{x}, \hat{z})^2}\end{aligned} \tag{C.30}$$

The constraint $\mathrm{rank}(x) \le k$ is therefore equivalent to the assumption that $z \in \mathbb{C}_*^{n\times k}$. Hence, in order to avoid a plethora of hats we will assume $z \in \mathbb{C}_*^{n\times k}$. This assumption simplifies the situation considerably since in this case $\Delta_z = H_{\pi,z}$. As we shall see, if the Γ_z component of Δ_z were to be non-trivial, the local lower bounds $\hat{a}_1(z)$ and $\hat{a}_2(z)$ would be zero. We next note that $d(x, z) = ||x - z||_2||x + z||_2$ precisely when

$x^*z = z^*x \geq 0$, which may be achieved without loss of generality in $\hat{a}_1(z)$ via choice of representative for x. Thus, keeping in mind that $z \in \mathbb{C}_*^{n\times k}$, we find

(C.31)
$$\begin{aligned}\hat{a}_1(z) &= \lim_{R\to 0} \inf_{\substack{x\in\mathbb{C}^{n\times k}\\ d(z,x)<R}} \frac{\sum_{j=1}^m |\langle xx^* - zz^*, A_j\rangle_{\mathbb{R}}|^2}{d(x,z)^2} \\ &= \lim_{R\to 0} \inf_{\substack{x\in\mathbb{C}^{n\times k}\\ ||x-z||_2\cdot||x+z||_2<R \\ x^*z=z^*x\geq 0}} \frac{\sum_{j=1}^m |\langle z(x-z)^* + (x-z)z^* + (x-z)(x-z)^*, A_j\rangle_{\mathbb{R}}|^2}{||x-z||_2^2 \cdot ||x+z||_2^2}\end{aligned}$$

In analogy with our analysis of $a_1(z)$ we change variables from x to $w = x - z$ and are thus able to linearize the infimization constraint, since for $||w||_2 < \sigma_k(z)$ we have that $z^*(z+w) \geq 0$ if and only if $z^*w = w^*z$, or in other words if and only if $z \in \Delta_z \iff z \in H_{\pi,z}$ (the vertical component of Δ_z, namely Γ_z, is trivial for $z \in \mathbb{C}_*^{n\times k}$). We also exploit the fact that D and d generate the same topology and therefore instead of $||w||_2||2z+w||_2 < R$ we may simply take $||w||_2 < R$.

(C.32)
$$\begin{aligned}\hat{a}_1(z) &= \lim_{R\to 0} \inf_{\substack{w\in H_{\pi,z}\\ ||w||_2<R}} \frac{\sum_{j=1}^m |\langle zw^* + wz^* + ww^*, A_j\rangle_{\mathbb{R}}|^2}{||w||_2^2||2z+w||_2^2} \\ &= \frac{1}{4||z||_2^2} \lim_{R\to 0} \inf_{\substack{w\in H_{\pi,z}\\ ||w||_2<R}} \frac{1}{||w||_2^2} \sum_{j=1}^m |\langle zw^* + wz^*, A_j\rangle_{\mathbb{R}}|^2 (1 + O(||w||_2^2)) \\ &= \frac{1}{4||z||_2^2} \inf_{\substack{w\in H_{\pi,z}\\ ||w||_2=1}} \sum_{j=1}^m |\langle zw^* + wz^*, A_j\rangle_{\mathbb{R}}|^2 \\ &= \frac{1}{4||z||_2^2} \hat{a}(z)\end{aligned}$$

We now consider $\hat{a}_2(z)$. In a manner precisely analogous to (C.30) the constraint in $\hat{a}_2(z)$ that $\mathrm{rank}(x) \leq k$ and $\mathrm{rank}(y) \leq k$ is equivalent to the assumption that $z \in \mathbb{C}_*^{n\times k}$. We first employ the unitary freedom of x and y to note that

(C.33)
$$\begin{aligned}\hat{a}_2(z) &= \lim_{R\to 0} \inf_{\substack{x,y\in\mathbb{C}^{n\times k}\\ d(x,z)<R \\ d(y,z)<R}} \frac{\sum_{j=1}^m |\langle xx^* - yy^*, A_j\rangle_{\mathbb{R}}|^2}{d(x,y)^2} \\ &= \lim_{R\to 0} \inf_{\substack{x,y\in\mathbb{C}^{n\times k}\\ ||x-z||_2||x+z||_2<R \\ ||y-z||_2||y+z||_2<R \\ x^*z=z^*x\geq 0 \\ y^*z=z^*y\geq 0}} \frac{\sum_{j=1}^m |\langle xx^* - yy^*, A_j\rangle_{\mathbb{R}}|^2}{d(x,y)^2} \\ &= \lim_{R\to 0} \inf_{\substack{x,y\in\mathbb{C}^{n\times k}\\ ||x-z||_2<R \\ ||y-z||_2<R \\ x^*z=z^*x \\ y^*z=z^*y}} \frac{\sum_{j=1}^m |\langle xx^* - yy^*, A_j\rangle_{\mathbb{R}}|^2}{d(x,y)^2}\end{aligned}$$

We now weaken the infimization constraints and obtain a lower bound. We note that $x^*z = z^*x$ and $y^*z = z^*y$ taken together imply that $(x-y)^*z = z^*(x-y)$, and also that the denominator $d(x,y)^2 \le ||x-y||_2^2 ||x+y||_2^2$. Thus, changing variables to $\xi = x - z$ and $\eta = y - z$ we obtain

(C.34)

$$\begin{aligned}
\hat{a}_2(z) &\ge \lim_{R\to 0} \inf_{\substack{\xi,\eta\in\mathbb{C}^{n\times k}\\ ||\xi||_2<R\\ ||\eta||_2<R\\ z^*(\xi-\eta)=(\xi-\eta)^*z}} \frac{\sum_{j=1}^m |\langle z(\xi-\eta)^* + (\xi-\eta)z^* + \xi\xi^* - \eta\eta^*, A_j\rangle_{\mathbb{R}}|^2}{||\xi-\eta||_2^2 ||2z+\xi+\eta||_2^2} \\
&= \frac{1}{4||z||_2^2} \lim_{R\to 0} \inf_{\substack{\xi,\eta\in\mathbb{C}^{n\times k}\\ ||\xi||_2<R\\ ||\eta||_2<R\\ z^*(\xi-\eta)=(\xi-\eta)^*z}} \frac{\sum_{j=1}^m |\langle z(\xi-\eta)^* + (\xi-\eta)z^*, A_j\rangle_{\mathbb{R}}|^2}{||\xi-\eta||_2^2}(1+O(||\xi||_2^2+||\eta||_2^2)) \\
&= \frac{1}{4||z||_2^2} \lim_{R\to 0} \inf_{\substack{\xi,\eta\in\mathbb{C}^{n\times k}\\ ||\xi||_2<R\\ ||\eta||_2<R\\ z^*(\xi-\eta)=(\xi-\eta)^*z}} \frac{\sum_{j=1}^m |\langle z(\xi-\eta)^* + (\xi-\eta)z^*, A_j\rangle_{\mathbb{R}}|^2}{||\xi-\eta||_2^2} \\
&= \frac{1}{4||z||_2^2} \lim_{R\to 0} \inf_{\substack{\xi,\eta\in\mathbb{C}^{n\times k}\\ ||\xi-\eta||_2<2R\\ z^*(\xi-\eta)=(\xi-\eta)^*z}} \frac{\sum_{j=1}^m |\langle z(\xi-\eta)^* + (\xi-\eta)z^*, A_j\rangle_{\mathbb{R}}|^2}{||\xi-\eta||_2^2}
\end{aligned}$$

∎

The last line is an equality rather than an inequality owing to homogeneity in $\xi - \eta$. Changing variables once more to $w = \xi - \eta$ and using the fact that for $z \in \mathbb{C}_*^{n\times k}$ $z^*w = w^*z \iff w \in \Delta_z \iff w \in H_{\pi,z}(\mathbb{C}_*^{n\times k})$ gives

(C.35)

$$\begin{aligned}
\hat{a}_2(z) &\ge \frac{1}{4||z||_2^2} \lim_{R\to 0} \inf_{\substack{w\in H_{\pi,z}(\mathbb{C}_*^{n\times k})\\ ||w||_2<2R}} \frac{\sum_{j=1}^m |\langle zw^* + wz^*, A_j\rangle_{\mathbb{R}}|^2}{||w||_2^2} \\
&= \frac{1}{4||z||_2^2} \inf_{\substack{w\in H_{\pi,z}(\mathbb{C}_*^{n\times k})\\ ||w||_2=1}} \sum_{j=1}^m |\langle zw^* + wz^*, A_j\rangle_{\mathbb{R}}|^2 \\
&= \hat{a}(z) = \hat{a}_1(z)
\end{aligned}$$

The reverse inequality $\hat{a}_2(z) \le \hat{a}_1(z)$ is immediate from the definitions of $\hat{a}_1(z)$ and $\hat{a}_2(z)$, thus (5.15) is proved. We now turn to explicit computation of $\hat{a}(z)$ as the smallest non-zero eigenvalue of $\hat{Q}_z$. As with the computation of $a(z)$ we rely on several embeddings. Specifically we define

(C.36)

$$l : \mathbb{C}^{n\times k} \to \mathbb{R}^{2n\times k} \qquad\qquad j : \mathbb{C}^{n\times k} \to \mathbb{R}^{2n\times 2k}$$

$$l(X) = \begin{bmatrix} \Re X \\ \Im X \end{bmatrix} \qquad\qquad j(X) = \begin{bmatrix} \Re X & -\Im X \\ \Im X & \Re X \end{bmatrix}$$

Note that j is an injective homomorphism and moreover that

(C.37)
$$j(X) = \begin{bmatrix} l(X) & Jl(X) \end{bmatrix}$$

where $J \in \mathbb{R}^{2n\times 2n}$ is the symplectic form

(C.38)
$$J = \begin{bmatrix} 0 & -\mathbb{I}_{n\times n} \\ \mathbb{I}_{n\times n} & 0 \end{bmatrix}$$

Note that $Jj(X) = j(X)J$ for all $X \in \mathbb{C}^{n\times n}$.The embedding l is isometric, and the embedding j is isometric up to a constant since for $X, Y \in \mathbb{C}^{n\times k}$ we have $\langle X, Y\rangle_{\mathbb{R}} = \langle l(X), l(Y)\rangle_{\mathbb{R}} = \frac{1}{2}\langle j(X), j(Y)\rangle_{\mathbb{R}}$. The embedding j is furthermore a structure preserving homomorphism since for $p \in \mathbb{C}^{n\times k}, q \in \mathbb{C}^{k\times l}$ we have that $j(p)l(q) = l(pq)$, $j(pq) = j(p)j(q)$, and $j(p^*) = j(p)^T$. We will also employ the isometric embedding vec defined in the obvious way in (5.8). We will need the fact that if $A \in \mathbb{R}^{n\times k}$ and $B \in \mathbb{R}^{k\times l}$ then

$$\text{(C.39)} \qquad \operatorname{vec}(AB) = (\mathbb{I}_{l\times l} \otimes A)\operatorname{vec}(B)$$

Note that this further implies that for $x, y \in \mathbb{R}^{n\times k}$ and $F \in \mathbb{R}^{n\times n}$ we have that

$$\text{(C.40)} \qquad \operatorname{vec}(x)^T(\mathbb{I}_{k\times k} \otimes F)\operatorname{vec}(y) = \operatorname{vec}(x)^T\operatorname{vec}(Fy) = \langle x, Fy\rangle_{\mathbb{R}} = \operatorname{tr}\{x^T F y\}$$

With this in mind we find that for $z \in \mathbb{C}^{n\times k}_*$ and $w \in H_{\pi,z}(\mathbb{C}^{n\times k}_*)$

$$\text{(C.41)} \qquad \begin{aligned} |\langle D\pi(z)(w), A_j\rangle_{\mathbb{R}}|^2 &= 4|\langle wz^*, A_j\rangle_{\mathbb{R}}|^2 \\ &= \langle j(wz^*), A_j\rangle^2 \\ &= \langle j(w), A_j j(z)\rangle^2 \\ &= \Big(\operatorname{vec}(j(w))^T \operatorname{vec}(j(A_j)j(z))\Big)^2 \\ &= \Big(\operatorname{vec}(j(w))^T (\mathbb{I}_{2k\times 2k} \otimes j(A_j))\operatorname{vec}(j(z))\Big)^2 \\ &= 4\Big(\operatorname{vec}(l(w))^T (\mathbb{I}_{k\times k} \otimes j(A_j))\operatorname{vec}(l(z))\Big)^2 \\ &= 4W^T F_j Z Z^T F_j W \end{aligned}$$

where $W = \mu(w)$, $Z = \mu(z)$ and $F_j = \mathbb{I}_{k\times k} \otimes j(A_j)$. This should not be too surprising since in fact

$$\text{(C.42)} \qquad \begin{aligned} \beta_j(z) &= \langle zz^*, A_j\rangle_{\mathbb{R}} \\ &= \langle z, A_j z\rangle_{\mathbb{R}} \\ &= \frac{1}{2}\langle j(z), j(A_j)j(z)\rangle \\ &= \frac{1}{2}\operatorname{vec}(j(z))^T \operatorname{vec}(j(A_j)j(z)) \\ &= \frac{1}{2}\operatorname{vec}(j(z))^T (\mathbb{I}_{2k\times 2k} \otimes j(A_j))\operatorname{vec}(j(z)) \\ &= \operatorname{vec}(l(z))^T (\mathbb{I}_{k\times k} \otimes j(A_j))\operatorname{vec}(l(z)) = Z^T F_j Z \end{aligned}$$

Thus when β_j is viewed as map from $\mathbb{R}^{2nk}$ to $\mathbb{R}$ we find that $|D\beta_j(Z)(W)|^2 = 4W^T F_j Z Z^T F_j W$. Returning to $a(z)$ we first note that the constraint $w \in H_{\pi,z}(\mathbb{C}^{n\times k}_*)$ precisely avoids the "trivial" kernel of dimension k^2 common to each $F_j Z Z^T F_j$. Specifically, we note that $Z^T F_j V = 0$ for $V \in \mathcal{V}_z \subset \mathbb{R}^{2nk}$ where

$$\text{(C.43)} \qquad \mathcal{V}_z = \{\operatorname{vec}(Jl(z)S + l(z)A) | S \in \operatorname{Sym}(\mathbb{R}^k), A \in \operatorname{Asym}(\mathbb{R}^k)\}$$

Namely if $V \in \mathcal{V}_z$ and $\eta = Jl(z)S + l(z)A \in \mathbb{R}^{2n\times r}$ for $A \in \mathrm{Asym}(\mathbb{R}^k)$ and $S \in \mathrm{Sym}(\mathbb{R}^k)$ so that $V = \mathrm{vec}(\eta)$ then

$$\begin{aligned} Z^T F_j V &= \mathrm{vec}(l(z))^T(\mathbb{I}_{k\times k} \otimes j(A_j))\mathrm{vec}(\eta) \\ &= \mathrm{tr}\{l(z)^T j(A_j)\eta\} \\ &= \mathrm{tr}\{l(z)^T j(A_j)(Jl(z)S + l(z)A)\} \\ &= \mathrm{tr}\{l(z)^T j(A_j)Jl(z)S\} + \mathrm{tr}\{l(z)^T j(A_j)l(z)A\} \\ &= 0 \end{aligned} \tag{C.44}$$

The last line follows from the fact that $j(A_j)$ is symmetric and $j(A_j)J$ is anti-symmetric since $(j(A_j)J)^* = -Jj(A_j) = -j(A_j)J$. The reason that $w \in H_{\pi,z}(\mathbb{C}_*^{n\times k})$ avoids this common kernel is that in fact $\mathcal{V}_z = \mu(V_{\pi,z}(\mathbb{C}_*^{n\times k}))$. Recall that

$$V_{\pi,z}(\mathbb{C}_*^{n\times k}) = \{zK | K \in \mathrm{Asym}(\mathbb{C}^k)\} \tag{C.45}$$

We may decompose $K \in \mathrm{Asym}(\mathbb{C}^n)$ as $K = A + iS$ where $A \in \mathrm{Asym}(\mathbb{R}^n)$ and $S \in \mathrm{Sym}(\mathbb{R}^n)$. Hence if $u \in V_{\pi,z}(\mathbb{C}_*^{n\times k})$ then on the one hand $j(u) = [l(u)|Jl(u)]$ and on the other

(C.46)

$$j(u) = j(zK) = j(z)j(K) = [l(z)|Jl(z)]\begin{bmatrix} A & -S \\ S & A \end{bmatrix} = [l(z)A + Jl(z)S| - l(z)S + Jl(z)A]$$

From which we may clearly identify $l(u) = l(z)A + Jl(z)S$, thus

$$\mathcal{V}_z = \{\mu(u) | u \in V_{\pi,z}(\mathbb{C}_*^{n\times k})\} \tag{C.47}$$

The map μ is an isometry, so if $w \in H_{\pi,z}(\mathbb{C}_*^{n\times k})$ then the image $W = \mu(w)$ lies precisely in the orthogonal complement of $\mathcal{V}_z$. Thus

$$\begin{aligned} \hat{a}(z) &= \min_{\substack{w\in H_{\pi,\hat{z}}(\mathbb{C}_*^{n\times k}) \\ ||w||_2=1}} \sum_{j=1}^m |\langle D\pi(\hat{z})(w), A_j\rangle_{\mathbb{R}}|^2 \\ &= \min_{\substack{W\in\mathbb{R}^{2nk} \\ W\perp\mathcal{V}_z \\ ||W||_2=1}} W^T(4\sum_{j=1}^m F_j Z Z^T F_j)W \\ &= \lambda_{2nk-k^2}(\hat{Q}_z) \end{aligned} \tag{C.48}$$

Note that at this point the hats return and $Z = \mu(\hat{z})$. Eigenvalues are continuous with respect to matrix entries, and $\hat{Q}_z$ is manifestly continuous with respect to z. As a result of this and the fact that $k \mapsto 2nk - k^2$ is monotone increasing for $k \le n$ we conclude that $\hat{a}(z)$ approaches zero whenever z approaches a drop in rank. Indeed, $\hat{a}(z)$ jumps discontinuously to a non-zero value once the surface of lower rank is actually reached, but this cannot prevent $\inf_{z\in\mathbb{C}^{n\times r}} \hat{a}(z)$ from being zero, thus there is no hope of defining a non-zero global lower bound $\hat{a}_0$. This concludes the proof of claims (iv)-(vi).

Claim (vii) gives local control of $a(z)$ in terms of $\hat{a}(z)$. We first prove that the

the inequality (5.17) holds. To do so we consider the following operators:

$$\text{(C.49)}\qquad \begin{aligned}&\Pi_1(\hat z) : (T_{\pi(\hat z)}(\mathring{S}^{k,0}(\mathbb{C}^n)), ||\cdot||_2) \to (\mathbb{R}^m, ||\cdot||_2)\\ &\Pi_1(\hat z)(W) = (\operatorname{tr}\{WA_j\})_{j=1}^m\end{aligned}$$

$$\text{(C.50)}\qquad \begin{aligned}&\Pi_2(\hat z) : (H_{\pi,\hat z}(\mathbb{C}_*^{n\times k}), ||\cdot||_2) \to (\mathbb{R}^m, ||\cdot||_2)\\ &\Pi_2(\hat z)(w) = (\operatorname{tr}\{(\hat z w^* + w\hat z^*)A_j\})_{j=1}^m = \Pi_1(\hat z)D\pi(\hat z)w\end{aligned}$$

Note that $a(z)$ and $\hat a(z)$, defined respectively in (5.3) and (5.4), are expressible in terms of the operator norms of the pseudo-inverses of $\Pi_1(\hat z)$ and $\Pi_2(\hat z)$.

$$\text{(C.51)}\qquad \begin{aligned}a(z) &= ||\Pi_1(\hat z)^\dagger||_*^{-2}\\ \hat a(z) &= ||\Pi_2(\hat z)^\dagger||_*^{-2}\end{aligned}$$

We may therefore obtain operator-theoretic inequalities relating $a(z)$ and $\hat a(z)$, namely

$$\text{(C.52)}\qquad \begin{aligned}||\Pi_2(\hat z)^\dagger||_* &= ||D\pi(\hat z)^{-1}\Pi_1(\hat z)^\dagger||_* \le ||D\pi(\hat z)^{-1}||_*||\Pi_1(\hat z)^\dagger||_*\\ ||\Pi_1(\hat z)^\dagger||_* &= ||D\pi(\hat z)\Pi_2(\hat z)^\dagger||_* \le ||D\pi(\hat z)||_*||\Pi_2(\hat z)^\dagger||_*\end{aligned}$$

Hence

$$\text{(C.53)}\qquad ||D\pi(\hat z)||_*^{-2}\hat a(z) \le a(z) \le ||D\pi(\hat z)^{-1}||_*^2 \hat a(z)$$

It remains only to compute appropriate bounds for $||D\pi(\hat z)||_*^{-2}$ and $||D\pi(z)^{-1}||_*^2$ in order to prove (5.17). First note that

(C.54)
$$||D\pi(\hat z)^{-1}||_*^2 = \sup_{W\in\mathbb{T}_{\pi(\hat z)}(\mathring{S}^{k,0}(\mathbb{C}^n))\setminus\{0\}} \frac{||D\pi(\hat z)^{-1}(W)||_2^2}{||W||_2^2} = \left(\inf_{w\in H_{\pi,\hat z}(\mathbb{C}_*^{n\times k})\setminus\{0\}} \frac{||\hat z w^* + w\hat z^*||_2^2}{||w||_2^2}\right)^{-1}$$

Next note that for $w = H\hat z + X \in H_{\pi,\hat z}(\mathbb{C}_*^{n\times k})$ we have $||w||_2^2 = ||H\hat z||_2^2 + ||X||_2^2$ and $||\hat z w^* + w\hat z||_2^2 = 2(||\hat z^* H\hat z||_2^2 + ||\hat z\hat z^* H||_2^2 + ||\hat z X^*||_2^2)$ thus

$$\text{(C.55)}\qquad \begin{aligned}||D\pi(\hat z)^{-1}||_*^{-2} &= \inf_{w\in H_{\pi,\hat z}(\mathbb{C}_*^{n\times k})\setminus\{0\}} \frac{||\hat z w^* + w\hat z^*||_2^2}{||w||_2^2}\\ &= 2\inf_{\substack{H\in\mathrm{Sym}(\mathbb{C}^n),\mathbb{P}_{\mathrm{Ran}(\hat z)}H=H\\ X\in\mathbb{C}^{n\times k},\mathbb{P}_{\mathrm{Ran}(\hat z)}X=0}} \frac{||\hat z^* H\hat z||_2^2 + ||\hat z\hat z^* H||_2^2 + ||\hat z X^*||_2^2}{||H\hat z||_2^2 + ||X||_2^2}\\ &\ge 2\inf_{\substack{H\in\mathrm{Sym}(\mathbb{C}^n),\mathbb{P}_{\mathrm{Ran}(\hat z)}H=H\\ X\in\mathbb{C}^{n\times k},\mathbb{P}_{\mathrm{Ran}(\hat z)}X=0}} \frac{||\hat z^* H\hat z||_2^2 + ||\hat z X^*||_2^2}{||H\hat z||_2^2 + ||X||_2^2}\\ &\ge 2\sigma_k(\hat z)^2 \inf_{\substack{H\in\mathrm{Sym}(\mathbb{C}^n),\mathbb{P}_{\mathrm{Ran}(\hat z)}H=H\\ X\in\mathbb{C}^{n\times k},\mathbb{P}_{\mathrm{Ran}(\hat z)}X=0}} \frac{||H\hat z||_2^2 + ||X||_2^2}{||H\hat z||_2^2 + ||X||_2^2}\\ &= 2\sigma_k(z)^2\end{aligned}$$

Hence $||D\pi(\hat{z})^{-1}||_*^2 \leq \frac{1}{2\sigma_k(z)^2}$. For the opposing bound note that

$$\begin{aligned} ||D\pi(\hat{z})||_*^2 &= \sup_{w \in H_{\pi,\hat{z}}(\mathbb{C}_*^{n\times k})\setminus\{0\}} \frac{||\hat{z}w^* + w\hat{z}^*||_2^2}{||w||_2^2} \\ &\leq \sup_{w \in H_{\pi,\hat{z}}(\mathbb{C}_*^{n\times k})\setminus\{0\}} \frac{||\hat{z}w^* + w\hat{z}^*||_1^2}{||w||_2^2} \\ &\leq \sup_{w \in H_{\pi,\hat{z}}(\mathbb{C}_*^{n\times k})\setminus\{0\}} \frac{4||\hat{z}w^*||_1^2}{||w||_2^2} \\ &\leq 4||z||_2^2 \end{aligned} \tag{C.56}$$

Hence $||D\pi(\hat{z})||_*^{-2} \geq \frac{1}{4||z||_2^2}$, proving (5.17). We note that choosing $w = \hat{z} \in H_{\pi,\hat{z}}(\mathbb{C}_*^{n\times k})$ proves that in fact $||D\pi(\hat{z})||_{2\to 1} = \frac{1}{2||z||_2}$. Finally, the claimed bounds in (5.17) are tight in the case $\text{rank}(z) = 1$, since in this case the inequality is equivalent to the norm inequality for $W \in \mathbb{C}^{n\times n}$

$$\frac{1}{\sqrt{\text{rank}(W)}}||W||_1 \leq ||W||_2 \leq ||W||_1 \tag{C.57}$$

Specifically if $W \in T_{\pi(z)}(\mathring{S}^{1,0}(\mathbb{C}^n))$ for $z \in \mathbb{C}_*^n$ then $W = zw^* + wz^*$ for some $w \in H_{\pi,z}(\mathbb{C}_*^n) \subset \mathbb{C}^n$ and has rank at most 2. Moreover we have that

$$||W||_1 = ||zw^* + wz^*||_1 = \frac{1}{2}||(z+w)(z+w)^* - (z-w)(z-w)^*||_1 \tag{C.58}$$

Recall (3.8) that for $x, y \in \mathbb{C}^n$ we have that $||xx^* - yy^*||_1 = d(x,y)$ and that $d(x,y) = ||x-y||_2||x+y||_2$ when $x^*y \geq 0$. Let $x = z + w$ and $y = z - w$, and note that in this case $w \in H_{\pi,z}(\mathbb{C}_*^n)$ implies $x^*y = z^*z + w^*z - z^*w - w^*w = z^*z - w^*w \geq 0$ for $||w||_2$ sufficiently small. Thus for $||w||_2$ or equivalently $||W||_2$ sufficiently small,

$$||W||_1 = \frac{1}{2}||(z+w) - (z-w)||_2||(z+w) + (z-w)||_2 = 2||z||_2||w||_2 \tag{C.59}$$

The condition that $||W||_2$ be sufficiently small is of no issue since the ratio in $a(z)$ is homogeneous in $||W||_2$, hence recalling that $\text{rank}(W) \leq 2$ (C.57) implies

$$\sqrt{2}||z||_2||w||_2 \leq ||W||_2 \leq 2||z||_2||w||_2 \tag{C.60}$$

Thus for $\text{rank}(z) = 1$ the inequality (C.57) is equivalent to

$$\frac{1}{4||z||_2^2}\hat{a}(z) \leq a(z) \leq \frac{1}{2||z||_2^2}\hat{a}(z) \tag{C.61}$$

which is recognizable as (5.17) since if $\text{rank}(z) = 1$ then $||z||_2^2 = \sigma_1(z)^2$ and hence since (C.57) is tight so too is (5.17). This concludes the proof of (*vii*).

To prove (*viii*) we combine (5.11) and (5.14) to obtain the following formula for computing a_0:

$$a_0 = \min_{k=1,\ldots,r} \min_{\substack{U \in U(n) \\ U = [U_1|U_2] \\ U_1 \in \mathbb{C}^{n\times k} \\ U_2 \in \mathbb{C}^{n\times(n-k)}}} \lambda_{2nk-k^2}(Q_U) \tag{C.62}$$

Recalling that

$$Q_{[U_1|U_2]} = \sum_{j=1}^{m} \begin{bmatrix} \tau(U_1^* A_j U_1) \\ \mu(U_2^* A_j U_1) \end{bmatrix} \begin{bmatrix} \tau(U_1^* A_j U_1) \\ \mu(U_2^* A_j U_1) \end{bmatrix}^T \tag{C.63}$$

Finally, we need to prove that the minimum over k in fact occurs at $k = r$. We may write

$$a_0 = \min_{k=1,\dots,r} \inf_{z\in\mathbb{C}_*^{n\times k}} \min_{W\in T_{\pi(z)}(\mathring{S}^{k,0}(\mathbb{C}^n))} \frac{1}{||W||_2^2} \sum_{j=1}^{m} |\langle W, A_j\rangle_{\mathbb{R}}|^2 \tag{C.64}$$

Then note that if $\hat{z} \in \mathbb{C}_*^{n\times k}$ and $\tilde{z} \in \mathbb{C}_*^{n\times(r-k)}$ is such that $\hat{z}^*\tilde{z} = 0$ then $z = [\hat{z}|\tilde{z}] \in \mathbb{C}_*^{n\times r}$ and moreover, recalling the parametrization of the tangent space (4.7) (or alternately that the stratification is a-regular), we find that $T_{\pi(z)}(\mathring{S}^{r,0}(\mathbb{C}^n)) \supset T_{\pi(\hat{z})}(\mathring{S}^{k,0}(\mathbb{C}^n))$ since $\mathrm{Ran}(z)^\perp = \mathrm{Ran}(\hat{z})^\perp \cap \mathrm{Ran}(\tilde{z})^\perp$. Thus, in fact

$$a_0 = \min_{\substack{U\in U(n)\\ U=[U_1|U_2]\\ U_1\in\mathbb{C}^{n\times r}\\ U_2\in\mathbb{C}^{n\times(n-r)}}} \lambda_{2nr-r^2}(Q_U) \tag{C.65}$$

We now set out to prove (ix), specifically to control a_0 using an infimization of $\hat{a}(z)$ rather than of $a(z)$ by including the additional constraint that $z^*z = \mathbb{I}_{r\times r}$. With this constraint we may write any $w \in H_{\pi,z}(\mathbb{C}_*^{n\times r})$ as $w = z\tilde{H} + X$ where $\tilde{H} \in \mathrm{Sym}(\mathbb{C}^r)$ and $X \in \mathbb{C}^{n\times r}$ satisfies $\mathbb{P}_{\mathrm{Ran}(z)}X = 0$ (equivalently X satisfies $z^*X = 0$). We note that for z satisfying the constraint

$$||w||_2^2 = ||\tilde{H}||_2^2 + ||X||_2^2 \tag{C.66}$$

$$||zw^* + wz^*||_2^2 = 4||\tilde{H}||_2^2 + 2||X||_2^2 \tag{C.67}$$

Hence referring to (5.3) and (5.4) we find that for $z^*z = \mathbb{I}_{r\times r}$

$$\frac{1}{4}\hat{a}(z) \le a(z) \le \frac{1}{2}\hat{a}(z) \tag{C.68}$$

Note that a direct application of (5.17) to the case where z has orthonormal columns would lead to the lower constant being $\frac{1}{4r}$ rather than $\frac{1}{4}$.The form (5.18) for a_0 tells us that $a(z)$ depends only on the range of z, and that we may obtain a_0 via

$$a_0 = \inf_{\substack{z\in\mathbb{C}_*^{n\times r}\\ z^*z=\mathbb{I}_{r\times r}}} a(z) \tag{C.69}$$

Thus

$$\frac{1}{4} \inf_{\substack{z\in\mathbb{C}_*^{n\times r}\\ z^*z=\mathbb{I}_{r\times r}}} \hat{a}(z) \le a_0 \le \frac{1}{2} \inf_{\substack{z\in\mathbb{C}_*^{n\times r}\\ z^*z=\mathbb{I}_{r\times r}}} \hat{a}(z) \tag{C.70}$$

This concludes the proof of (ix) and Theorem 5.6. □

Remark C.3. For $r = 1$ the inequality (5.17) tells us that

$$\frac{1}{4||z||_2^2}\hat{a}(z) \le a(z) \le \frac{1}{2||z||_2^2}\hat{a}(z) \tag{C.71}$$

But in fact, as was proved in [6], more is true. Namely if the nuclear norm is used in the definition of a_0 instead of the Frobenius norm so that

$$a_0^1 = \inf_{\substack{x,y\in\mathbb{C}^{n\times r}\\ x\neq y}} \frac{\sum_{j=1}^{m}(\langle xx^*, A_j\rangle_{\mathbb{R}} - \langle yy^*, A_j\rangle_{\mathbb{R}})^2}{||xx^* - yy^*||_1^2} \tag{C.72}$$

And similarly in the definition of $a(z)$ so that

$$a^1(z) = \min_{\substack{W\in T_{\pi(\hat{z})}(\mathring{S}^{k,0}(\mathbb{C}^n))\\ ||W||_1=1}} \sum_{j=1}^{m} |\langle W, A_j\rangle_{\mathbb{R}}|^2 \tag{C.73}$$

then

$$a_0^1 = \inf_{z\in\mathbb{C}^{n\times r}\setminus\{0\}} a^1(z) \tag{C.74}$$

$$a^1(z) = \frac{1}{4||z||_2^2}\hat{a}(z) \tag{C.75}$$

Remark C.4. For $r=1$, Q_z is orthogonally equivalent to the restriction of $\hat{Q}_z$ to the orthogonal complement of its null space, giving a correspondence between (5.14) and (3.5) in [2] when the frame is positive semidefinite ($A_j = f_j f_j^*$). Specifically, if $r=1$ then we may take $U_1 = \frac{z}{||z||_2} =: e_1$ and $U_2 = [e_2,\dots,e_n]$ where $e_1,\dots,e_n$ forms an orthonormal basis for $\mathbb{C}^n$ with respect to the complex inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$. Thus

$$\begin{aligned} \tau(U_1^*A_jU_1) &= \frac{|\langle z, f_j\rangle_{\mathbb{C}}|^2}{||z||_2^2} = \frac{1}{||z||_2}\langle e_1, f_j\rangle_{\mathbb{C}}\langle f_j, z\rangle_{\mathbb{C}} \\ \mu(U_2^*A_jU_1) &= \frac{1}{||z||_2} l(\begin{bmatrix}\langle e_2, f_j\rangle_{\mathbb{C}}\langle f_j, z\rangle_{\mathbb{C}}\\ \vdots \\ \langle e_n, f_j\rangle_{\mathbb{C}}\langle f_j, z\rangle_{\mathbb{C}}\end{bmatrix}) \end{aligned} \tag{C.76}$$

Note that $\tau(U_1^*A_jU_1)$ is real, hence if we insert a single 0 in the middle of $\mu(U_2^*A_jU_1)$ between $\mathrm{vec}(\Re(U_2^*A_jU_1))$ and $\mathrm{vec}(\Im(U_2^*A_jU_1))$ we obtain

(C.77)
$$\begin{bmatrix}\tau(U_1^*A_jU_1)\\ \mathrm{vec}(\Re(U_2^*A_jU_1))\\ 0\\ \mathrm{vec}(\Im(U_2^*A_jU_1))\end{bmatrix} = \frac{1}{||z||_2} l(\begin{bmatrix}\langle e_1, f_j\rangle_{\mathbb{C}}\langle f_j, z\rangle_{\mathbb{C}}\\ \vdots \\ \langle e_n, f_j\rangle_{\mathbb{C}}\langle f_j, z\rangle_{\mathbb{C}}\end{bmatrix}) = \frac{1}{||z||_2} l(U^*A_jz) = \frac{1}{||z||_2} j(U)^T j(A_j) l(z)$$

Where in the last inequality the algebraic properties of l and j are employed. Thus (up to a row and column of zeros)

$$Q_z = j(U)^T\left\{\frac{1}{||z||_2^2}\sum_{j=1}^{m} j(A_j)l(z)l(z)^T j(A_j)\right\}j(U) \tag{C.78}$$

In accordance with the notation of [2] we denote $\xi = l(z)$, $\phi_j = l(f_j)$, and $\Phi_j = j(A_j) = \phi_j\phi_j^T + J\phi_j\phi_j^T J^T$ so that the above becomes

$$Q_z = j(U)^T\left\{\frac{1}{||\xi||_2^2}\sum_{j=1}^{m} \Phi_j\xi\xi^T\Phi_j\right\}j(U) \tag{C.79}$$

Finally note that the column of $j(U)$ corresponding to the the row and column of zeros on the left hand side is $Jl(z)/||z||_2 = J\xi/||\xi||_2$, thus if we multiply on the left by $j(U)$ and on the right by $j(U)^T$ we obtain

$$j(U)Q_z j(U)^T = (\mathbb{I} - \mathbb{P}_{J\xi})\left\{\frac{1}{||\xi||_2^2}\sum_{j=1}^{m}\Phi_j\xi\xi^T\Phi_j\right\}(\mathbb{I} - \mathbb{P}_{J\xi}) \tag{C.80}$$

C.3. Proof of Theorem 5.9.

Proof. As was the case for $\hat{a}_1(z)$ and $\hat{a}_2(z)$ the rank constraints in $A_1(z)$, $A_2(z)$, $\hat{A}_1(z)$, and $\hat{A}_2(z)$ allow us to assume that $z \in \mathbb{C}_*^{n\times k}$ rather than $\mathbb{C}^{n\times r}$. As before, this is done because without this assumption the resulting lower bounds would be zero for every z not full rank. We begin with the analysis of $\hat{A}_1(z)$, the simpler of the local lower bounds (we will show (x) that $A_i(z)$ differ from $\hat{A}_i(z)$ only by a constant factor, and hence will not analyze them separately). As we have done several times before we will employ the right hand unitary freedom of the variable x to require that $z^*x \geq 0$, and then make the change of variables from x to $w = x - z$.

(C.81)

$$\begin{aligned}
\hat{A}_1(z) &= \lim_{R\to 0} \inf_{\substack{x\in\mathbb{C}^{n\times k}\\ xx^*\neq zz^*\\ D(x,z)<R}} \frac{1}{D(x,z)^2}\sum_{j=1}^{m}|\langle xx^*, A_j\rangle^{\frac{1}{2}} - \langle zz^*, A_j\rangle^{\frac{1}{2}}|^2 \\
&= \lim_{R\to 0} \inf_{\substack{w\in\mathbb{C}^{n\times k}\\ zw^*+wz^*+ww^*\neq 0\\ ||w||_2<R\\ z^*(z+w)\geq 0}} \frac{1}{||w||_2^2}\sum_{j=1}^{m}|\langle (z+w)(z+w)^*, A_j\rangle^{\frac{1}{2}} - \langle zz^*, A_j\rangle^{\frac{1}{2}}|^2 \\
&= \lim_{R\to 0} \inf_{\substack{w\in\mathbb{C}^{n\times k}\\ zw^*+wz^*+ww^*\neq 0\\ ||w||_2<R\\ w\in\Delta_z}} \frac{1}{||w||_2^2}\left\{\sum_{j\in I_0(z)}\langle ww^*, A_j\rangle_{\mathbb{R}} + \sum_{j\in I(z)}\frac{|\langle zw^*+wz^*+ww^*, A_j\rangle_{\mathbb{R}}|^2}{|\langle (z+w)(z+w)^*, A_j\rangle^{\frac{1}{2}} + \langle zz^*, A_j\rangle^{\frac{1}{2}}|^2}\right\}
\end{aligned}$$

Where $I_0(z) = \{j \in \{1,\ldots,m\} | \alpha_j(z) = 0\}$ are the indices for which α_j is zero (and hence not differentiable) and $I(z) = \{j \in \{1,\ldots,m\} | \alpha_j(z) \neq 0\}$ are the indices for which α_j is not zero (and hence is differentiable). Thus, since z is full rank we know that $\Delta_z = H_{\pi,z}(\mathbb{C}_*^{n\times k})$ and since $zw^* + wz^* + ww^* \neq 0 \iff w \neq 0$ for $w \in H_{\pi,z}(\mathbb{C}_*^{n\times k})$ and sufficiently small in norm, we obtain

(C.82)

$$\begin{aligned}
\hat{A}_1(z) &= \lim_{R\to 0} \inf_{\substack{w\in H_{\pi,z}(\mathbb{C}_*^{n\times k})\\ 0<||w||_2<R}} \frac{1}{||w||_2^2}\left\{\sum_{j\in I_0(z)}\langle ww^*, A_j\rangle_{\mathbb{R}} + \sum_{j\in I(z)}\frac{|\langle zw^*+wz^*+ww^*, A_j\rangle_{\mathbb{R}}|^2}{|\langle (z+w)(z+w)^*, A_j\rangle^{\frac{1}{2}} + \langle zz^*, A_j\rangle^{\frac{1}{2}}|^2}\right\} \\
&= \lim_{R\to 0} \inf_{\substack{w\in H_{\pi,z}(\mathbb{C}_*^{n\times k})\\ 0<||w||_2<R}} \frac{1}{||w||_2^2}\left\{\sum_{j\in I_0(z)}\langle ww^*, A_j\rangle_{\mathbb{R}} + \sum_{j\in I(z)}\frac{|\langle zw^*+wz^*, A_j\rangle_{\mathbb{R}}|^2}{4\langle zz^*, A_j\rangle} + O(||w||^3)\right\} \\
&= \min_{\substack{w\in H_{\pi,z}(\mathbb{C}_*^{n\times k})\\ ||w||_2=1}} \frac{1}{||w||_2^2}\left\{\sum_{j\in I_0(z)}\langle ww^*, A_j\rangle_{\mathbb{R}} + \sum_{j\in I(z)}\frac{|\langle zw^*+wz^*, A_j\rangle_{\mathbb{R}}|^2}{4\langle zz^*, A_j\rangle}\right\}
\end{aligned}$$

Now recall from (C.41) and (C.42) respectively that $|\langle zw^*+wz^*, A_j\rangle_{\mathbb{R}}|^2 = |\langle D\pi(z)(w), A_j\rangle_{\mathbb{R}}|^2 =$

$4W^T F_j Z Z^T F_j W$ and $\langle ww^*, A_j\rangle = \beta_j(w) = W^T F_j W$. Thus the above is

$$\hat{A}_1(z) = \min_{\substack{W\in\mathbb{R}^{2nk}\\ W\perp\mathcal{V}_z\\ ||W||_2=1}} W^T\Big\{\sum_{j\in I_0(z)} F_j + \sum_{j\in I(z)} \frac{F_j Z Z^T F_j}{Z^T F_j Z}\Big\}W \tag{C.83}$$

As has already been noted in (C.44) the null space of each $F_j Z Z^T F_j$ contains $\mathcal{V}_z$, but in fact so does the null space of each F_j for $j \in I_0(z)$ since in this case $F_j\mu(zK) = (\mathbb{I}_{k\times k}\otimes j(A_j))\mathrm{vec}(l(zK)) = \mathrm{vec}(j(A_j)l(zk)) = \mathrm{vec}(l(A_j zK)) = 0$. Thus we obtain finally that

$$\hat{A}_1(z) = \lambda_{2nk-k^2}\Big(\sum_{j\in I_0(z)} F_j + \sum_{j\in I(z)} \frac{F_j\mu(\hat{z})\mu(\hat{z})^T F_j}{\mu(\hat{z})^T F_j \mu(\hat{z})}\Big) \tag{C.84}$$

Note that in addition to proving (5.24) this also proves $(viii)$ as this form makes clear that, owing to continuity of eigenvalues, infimizing $\hat{A}_1(z)$ over z will give zero (and hence so too will infimizing $\hat{A}_2(z)$ over z since $\hat{A}_2(z) \le \hat{A}_1(z)$). Specifically the number of possibly non-zero eigenvalues of $\hat{R}_z + \hat{T}_z$ is $2nk - k^2$ and is thus monotone increasing in rank, and thus a sequence $(z_i)_{i\ge 1} \subset \mathbb{C}_*^{n\times r}$ approaching a surface of lower rank k will have $\lambda_{2nr-r^2}(\hat{R}_z + \hat{T}_z)$ approach zero. Somewhat more remarkably, (C.84) actually gives us $\hat{A}_2(z)$ as an eigenvalue problem also. Specifically, we prove that the "differentiable" terms in $\hat{A}_2(z)$ are equal to those in $\hat{A}_1(z)$ and that in fact these are the only terms which contribute to $\hat{A}_2(z)$. We define

$$\begin{aligned}
\hat{A}_2^I(z) &= \lim_{R\to 0} \inf_{\substack{x,y\in\mathbb{C}^{n\times r}\\ D(x,z)<R\\ D(y,z)<R\\ \mathrm{rank}(x)\le k\\ \mathrm{rank}(y)\le k}} \frac{\sum_{k\in I(z)} |\alpha_k(x)-\alpha_k(y)|^2}{D(x,y)^2}\\
\hat{A}_2^{I_0}(z) &= \lim_{R\to 0} \inf_{\substack{x,y\in\mathbb{C}^{n\times r}\\ D(x,z)<R\\ D(y,z)<R\\ \mathrm{rank}(x)\le k\\ \mathrm{rank}(y)\le k}} \frac{\sum_{k\in I_0(z)} |\alpha_k(x)-\alpha_k(y)|^2}{D(x,y)^2}\\
\hat{A}_1^I(z) &= \lim_{R\to 0} \inf_{\substack{x\in\mathbb{C}^{n\times r}\\ D(z,x)<R\\ \mathrm{rank}(x)\le k}} \frac{\sum_{k\in I(z)} |\alpha_k(x)-\alpha_k(z)|^2}{D(x,z)^2}\\
\hat{A}_1^{I_0}(z) &= \lim_{R\to 0} \inf_{\substack{x\in\mathbb{C}^{n\times r}\\ D(z,x)<R\\ \mathrm{rank}(x)\le k}} \frac{\sum_{k\in I_0(z)} |\alpha_k(x)-\alpha_k(z)|^2}{D(x,z)^2}
\end{aligned} \tag{C.85}$$

So that $\hat{A}_2(z) \ge \hat{A}_2^{I_0}(z) + \hat{A}_2^I(z) \ge \hat{A}_2^I(z)$, $\hat{A}_2^I(z) \le \hat{A}_1^I(z)$, and $\hat{A}_2^{I_0}(z) \le \hat{A}_1^{I_0}(z)$. Applying the mean value theorem to the functions $g_k : [0,1] \to \mathbb{R}, g_k(c) = \alpha_k((1-c)x + cy)$ for $k \in I(z)$ we see that there exist $c_k \in [0,1]$ so that $\alpha_k(y) - \alpha_k(x) = g(1) - g(0) = g'(c_k) = D\alpha_k((1-c_k)x + c_k y)(y-x)$ (recall that these are precisely the k for which said differential exists, and the differential is taken with respect to the real vector space structure). Hence, replacing the rank constraints with the assumption that $z \in \mathbb{C}_*^{n\times k}$ and aligning both x and y with z so that $z^*x \geq 0$ and $z^*y \geq 0$ we have:

$$\hat{A}_2^I(z) = \lim_{R\to 0} \inf_{\substack{x,y\in\mathbb{C}^{n\times k} \\ ||x-z||<R \\ ||y-z||<R \\ z^*x\geq 0 \\ z^*y\geq 0}} \frac{\sum_{k\in I(z)} |D\alpha_k((1-c_k)x + c_k y)(y-x)|^2}{D(x,y)^2} \tag{C.86}$$

Using the fact that $D(x,y) \leq ||y-x||_2$ and writing $x = z+\xi$ and $y = z+\eta$ we obtain that

$$\hat{A}_2^I(z) \geq \lim_{R\to 0} \inf_{\substack{\eta,\xi\in\Delta_z \\ ||\xi||<R \\ ||\eta||<R}} \frac{\sum_{k\in I(z)} |D\alpha_k(z+(1-c_k)\xi + c_k\eta)(\eta-\xi)|^2}{||\eta-\xi||_2^2} \tag{C.87}$$

The trick of linearizing the conic constraints here to $\xi,\eta \in \Delta_z$ is crucial since it allows us to strictly weaken the constraints in the infimum by taking $w = \eta - \xi$ so that, after using the continuity of $D\alpha_k$ (α_k is continuously differentiable when differentiable)

$$\begin{aligned}
\hat{A}_2^I(z) &\geq \lim_{R\to 0} \inf_{\substack{\eta,\xi\in\Delta_z \\ ||\xi||_2<R \\ ||\eta||_2<R}} \frac{\sum_{k\in I(z)} |D\alpha_k(z+(1-c_k)\xi + c_k\eta)(\eta-\xi)|^2}{||\eta-\xi||_2^2} \\
&= \lim_{R\to 0} \inf_{\substack{\eta,\xi\in\Delta_z \\ ||\xi||_2<R \\ ||\eta||_2<R}} \frac{\sum_{k\in I(z)} |D\alpha_k(z)(\eta-\xi)|^2}{||\eta-\xi||_2^2} + O(||\xi||_2^2 + ||\eta||_2^2) \\
&\geq \lim_{R\to 0} \inf_{\substack{w\in\Delta_z \\ ||w||_2<2R}} \frac{\sum_{k\in I(z)} |D\alpha_k(z)(w)|^2}{||w||_2^2} \\
&= \min_{\substack{w\in H_{\pi,z}(\mathbb{C}_*^{n\times k}) \\ ||w||_2=1}} \sum_{k\in I(z)} |D\alpha_k(z)(w)|^2 \\
&= \lambda_{2nk-k^2}\Big(\sum_{j\in I(z)} \frac{F_j\mu(\hat{z})\mu(\hat{z})^T F_j}{\mu(\hat{z})^T F_j \mu(\hat{z})}\Big) = \hat{A}_1^I(z)
\end{aligned} \tag{C.88}$$

We already had the reverse inequality $\hat{A}_2^I(z) \leq \hat{A}_1^I(z)$, hence $\hat{A}_2^I(z) = \hat{A}_1^I(z)$. Moreover, assuming this minimum is achieved by $w_0 \in H_{\pi,z}(\mathbb{C}_*^{n\times k})$ then if we put $x = z + \frac{1}{2}w_0$ $y = z - \frac{1}{2}w_0$ we see that the $\hat{A}_2^{I_0}(z)$ term vanishes and $\hat{A}_2^I(z)$ is achieved, hence $\hat{A}_2(z) \leq \hat{A}_2^I(z)$. We already had the reverse inequality, so we conclude that $\hat{A}_2(z) = \hat{A}_2^I(z) = \hat{A}_1^I(z)$ and $\hat{A}_2^{I_0}(z) = 0$. In summary

$$\begin{aligned}
\hat{A}_2(z) &= \min_{\substack{W\in\mathbb{R}^{2nk} \\ W\perp\mathcal{V}_z \\ ||W||_2=1}} W^T \Big\{ \sum_{j\in I(z)} \frac{F_j Z Z^T F_j}{Z^T F_j Z} \Big\} W \\
&= \lambda_{2nk-k^2}\Big(\sum_{j\in I(z)} \frac{F_j Z Z^T F_j}{Z^T F_j Z}\Big)
\end{aligned} \tag{C.89}$$

Thus claims (i) and (ii) are proven. Claim (iii) follows immediately from the inequality (3.6). This concludes the proof of the Theorem 5.9. ∎

Remark C.5. If z were not assumed full rank in (C.81) then $w \in \Delta_z$ would possibly have a non-zero component w_Γ in $\Gamma_z \subset V_{\pi,z}(\mathbb{C}_*^{n\times k})$. As a result, it would be possible to obtain a sequence (with the horizontal space component of w converging to zero) for which the second sum in the last line of (C.81) is eventually fourth order in $||w||_2$, thus $A_1(z)$ would be zero wherever α is differentiable (almost everywhere in measure). The rank constraint in the definition of $\hat{A}_1(z)$ that $\text{rank}(x) \leq k$ avoids this, since it allows us to assume that z is full rank and hence that Γ_z is trivial.

C.4. Proof of Theorem 5.13.

Proof. The proof of (i) is essentially identical to the proof of the analogous eigenvalue formula for the lower bound a_0 in Theorem 5.6. One first changes coordinates to $z = \frac{1}{2}(x+y)$ and $w = x - y$ and repeats the computation (C.6) to obtain

$$b_0 = \sup_{z\in\mathbb{C}^{n\times r}} \max_{\substack{W\in T_{\pi(\hat{z})}(\mathring{S}^{k,0}(\mathbb{C}^n))\\ ||W||_2=1}} \sum_{j=1}^{M} |\langle W, A_j\rangle_{\mathbb{R}}|^2 \tag{C.90}$$

At this point we note that

$$b_0 \leq \sup_{W\in\text{Sym}(\mathbb{C}^n)} \frac{||\mathcal{A}(W)||_2^2}{||W||_2^2} = ||\mathcal{A}||_{2\to 2}^2 \tag{C.91}$$

As before we observe that it suffices to take $z \in \mathbb{C}_*^{n\times r}$ since if $\hat{z} \in \mathbb{C}_*^{n\times k}$ and $\tilde{z} \in \mathbb{C}_*^{n\times(r-k)}$ and $z = [\hat{z}|\tilde{z}]$ with $\tilde{z}^*\hat{z} = 0$ then $T_{\pi(z)}(\mathring{S}^{r,0}(\mathbb{C}^n)) \supset T_{\pi(\hat{z})}(\mathring{S}^{k,0})$. One then employs the tangent space parametrization (C.27) and repeats the computation (C.28) to obtain

$$b_0 = \sup_{z\in\mathbb{C}_*^{n\times r}} \lambda_1(Q_z) = \max_{\substack{U\in U(n)\\ U=[U_1|U_2]\\ U_1\in\mathbb{C}^{n\times r}, U_2\in\mathbb{C}^{n\times n-r}}} \lambda_1(Q_{[U_1|U_2]}) \tag{C.92}$$

This concludes the proof of (i). To prove (ii) we will employ the following lemma.

LEMMA C.6. *Let $|||\cdot|||$ be any norm. Then*

$$||\mathcal{A}||_{1\to|||\cdot|||} = \sup_{\substack{x\in\mathbb{C}^n\\ ||x||_2=1}} |||\mathcal{A}(xx^*)||| \tag{C.93}$$

In other words the operator norm $||\mathcal{A}||_*$ of $\mathcal{A} : (Sym(\mathbb{C}^n)(\mathbb{C}^n), ||\cdot||_1) \to (\mathbb{R}^m, |||\cdot|||)$ is achieved on a matrix of rank 1.

Proof. Let $R \in \text{Sym}(\mathbb{C}^n)$ be non-zero such that $||R||_1 = 1$ and $|||\mathcal{A}(R)||| = ||\mathcal{A}||_*||R||_1$. Write $R = \sum_{j=1}^n r_j e_j e_j^*$ and note that $||R||_1 = 1$ implies $\sum_{j=1}^n |r_j| = 1$. Then

(C.94)

$$||\mathcal{A}||_* = ||\mathcal{A}||_*||R||_1 = |||\sum_{j=1}^n r_j\mathcal{A}(e_je_j^*)||| \leq (\sum_{j=1}^n |r_j|) \max_{j=1,\dots,n} |||\mathcal{A}(e_je_j^*)||| = \max_{j=1,\dots,n} |||\mathcal{A}(e_je_j^*)|||$$

Let $x_0 = e_{j_0}$ where j_0 is the index that achieves the maximum. Then $||x_0||_2 = 1$ and $||A||_* \leq |||\mathcal{A}(x_0x_0^*)|||$, but of course this bound is achievable by just plugging in $x_0x_0^*$ into $\mathcal{A}$. Thus the operator norm of $\mathcal{A}$ is achieved on a matrix of rank 1 and the lemma holds. ∎

Next note that

$$
\begin{aligned}
b_{0,1} &= \sup_{\substack{x,y\in\mathbb{C}^{n\times r}\\ [x]\neq[y]}} \frac{\sum_{j=1}^m |\langle xx^*-yy^*, A_j\rangle_{\mathbb{R}}|^2}{||xx^*-yy^*||_1^2} \\
&= \sup_{z\in\mathbb{C}_*^{n\times r}} \sup_{W\in T_{\pi(z)}(\mathring{S}^{r,0}(\mathbb{C}^n))} \frac{||\mathcal{A}(W)||_2^2}{||W||_1^2} \\
&\leq \sup_{\substack{W\in\mathrm{Sym}(\mathbb{C}^n)\\ ||W||_1=1}} ||\mathcal{A}(W)||_2^2 \\
&= ||\mathcal{A}||_{1\to 2}^2
\end{aligned}
\tag{C.95}
$$

Note that by an identical computation $b_0 \leq ||\mathcal{A}||_{2\to 2}$. By the Lemma $||\mathcal{A}||_{1\to 2} = \sup_{x\in\mathbb{C}^n, ||x||_2=1} ||\mathcal{A}(xx^*)||_2^2$, hence

$$
\begin{aligned}
b_{0,1} &\leq \sup_{x\in\mathbb{C}^n} \frac{||\mathcal{A}(xx^*)||_2^2}{||xx^*||_1^2} \\
&\leq \sup_{x\in\mathbb{C}^{n\times r}} \frac{||\mathcal{A}(xx^*)||_2^2}{||xx^*||_1^2} \\
&= \frac{||\mathcal{A}(x_0x_0^*)||_2^2}{||x_0x_0^*||_1^2} \\
&\leq \sup_{\substack{U_2\in\mathbb{C}^{n\times n-k}\\ U_2^*U_2=\mathbb{I}_{n-k\times n-k}\\ k=1,\ldots,r}} \sup_{\substack{W\in\mathrm{Sym}(\mathbb{C}^n)\\ U_2^*WU_2=0}} \frac{||\mathcal{A}(W)||_2^2}{||W||_1^2} \\
&= b_0
\end{aligned}
\tag{C.96}
$$

Where in the second to last equality we note that it suffices to take U_2 such that $U_2U_2^* = \mathbb{P}_{\mathrm{Ran}(x_0)^\perp}$ and in the last equality we use the implicit parametrization of the tangent space (4.7). Thus

$$
b_{0,1} = ||\mathcal{A}||_{1\to 2} = \sup_{x\in\mathbb{C}^n} \frac{||\mathcal{A}(xx^*)||_2^2}{||xx^*||_1^2} = \sup_{x\in\mathbb{C}^{n\times r}} \frac{||\mathcal{A}(xx^*)||_2^2}{||xx^*||_1^2}
\tag{C.97}
$$

We now seek an operator $T_r : \mathbb{C}^{n\times r} \to (\mathbb{C}^{n\times r})^m$, an integer q, and a norm $|||\cdot|||$ so that for $x \in \mathbb{C}^{n\times r}$

$$
|||T_r(x)|||^q = ||\mathcal{A}(xx^*)||_2^2
\tag{C.98}
$$

We find that if $A_j \geq 0$ for all j then

$$
||\mathcal{A}(xx^*)||_2^2 = \sum_{j=1}^m |\langle xx^*, A_j\rangle_{\mathbb{R}}|^2 = \sum_{j=1}^m ||A_j^{\frac{1}{2}}x||_2^4
\tag{C.99}
$$

So we let T_r be as in Definition 5.12, $|||X||| = |||X|||_{2,4}$ and $q = 4$ and find $b_0 = ||T_r||_{2\to(2,4)}^4 = ||T_1||_{2\to(2,4)}^4$. This concludes the proof of (ii). To prove (iii) note that by (3.5) $||(xx^*)^{\frac{1}{2}} - (yy^*)^{\frac{1}{2}}||_2 \geq D(x,y)$ hence

$$
B_0 \leq \sup_{\substack{x,y\in\mathbb{C}^{n\times r}\\ [x]\neq[y]}} \frac{||\alpha(x)-\alpha(y)||_2^2}{D(x,y)^2}
\tag{C.100}
$$

Thus

$$
\begin{aligned}
B_0 &\leq \sup_{\substack{x,y\in\mathbb{C}^{n\times r}\\ [x]\neq[y]}} \frac{1}{D(x,y)^2}\sum_{j=1}^{m} |\langle xx^*, A_j\rangle^{\frac{1}{2}} - \langle yy^*, A_j\rangle^{\frac{1}{2}}|^2 \\
&= \sup_{\substack{x,y\in\mathbb{C}^{n\times r}\\ x^*y\geq 0}} \frac{1}{||x-y||_2^2}\sum_{j=1}^{m} \frac{|\langle xx^*-yy^*, A_j\rangle_{\mathbb{R}}|^2}{(\langle xx^*, A_j\rangle^{\frac{1}{2}} + \langle yy^*, A_j\rangle^{\frac{1}{2}})^2}
\end{aligned}
\tag{C.101}
$$

We now make the change of coordinates $z = \frac{1}{2}(x+y)$, $w = x-y$ so that $x = z + \frac{1}{2}w$, $y = z - \frac{1}{2}w$. As before let $I_0(z)$ be the subset of $\{1,\dots,m\}$ for which $A_j z = 0$ and $I(z)$ its complement in $\{1,\dots,m\}$. In this case we note that if $j \in I_0(z)$ then $0\langle zw^* + wz^*, A_j\rangle_{\mathbb{R}} = \langle xx^* - yy^*, A_j\rangle$. Thus, employing the triangle inequality via $\langle xx^*, A_j\rangle^{\frac{1}{2}} + \langle yy^*, A_j\rangle^{\frac{1}{2}} = ||A_j^{\frac{1}{2}}x||_2 + ||A_j^{\frac{1}{2}}y||_2 \geq 2||A_j^{\frac{1}{2}}z||_2 = 2\langle zz^*, A_j\rangle^{\frac{1}{2}}$ we find that

$$
B_0 \leq \sup_{\substack{x,y\in\mathbb{C}^{n\times r}\\ x^*y\geq 0}} \frac{1}{||x-y||_2^2}\sum_{j\in I(z)}^{m} \frac{|\langle xx^*-yy^*, A_j\rangle_{\mathbb{R}}|^2}{(\langle xx^*, A_j\rangle^{\frac{1}{2}} + \langle yy^*, A_j\rangle^{\frac{1}{2}})^2}
\tag{C.102}
$$

$$
\leq \sup_{\substack{z\in\mathbb{C}^{n\times r}\\ z\neq 0}} \sup_{\substack{w\in\mathbb{C}^{n\times r}\\ z^*z-\frac{1}{4}w^*w+\frac{1}{2}(w^*z-z^*w)\geq 0}} \frac{1}{||w||_2^2}\sum_{j\in I(z)} \frac{|\langle zw^*+wz^*, A_j\rangle_{\mathbb{R}}|^2}{4\langle zz^*, A_j\rangle}
\tag{C.103}
$$

Next note that the condition $z^*z - \frac{1}{4}w^*w + \frac{1}{2}(w^*z - z^*w) \geq 0$ holds if and only if $z^*w = w^*z$ and $w^*w \leq 4z^*z$. Moreover, since w only appears as $w/||w||_2$ we may scale w so that $\sigma_1(w) \leq \sigma_k(z)$ (where z has rank k), thus the latter non-linear criterion becomes the linear criterion that $w\mathbb{P}_{\ker(z)} = 0$. Taken together, these these criterion hold if and only if $w \in H_z$. Thus, with reference to the computations (C.41) and (C.42) we find that

$$
B_0 \leq \sup_{\substack{z\in\mathbb{C}^{n\times r}\\ z\neq 0}} \sup_{w\in H_z} \frac{1}{||w||_2^2}\sum_{j\in I(z)} \frac{|\langle zw^*+wz^*, A_j\rangle_{\mathbb{R}}|^2}{4\langle zz^*, A_j\rangle}
\tag{C.104}
$$

$$
= \sup_{\substack{z\in\mathbb{C}^{n\times r}\\ z\neq 0}} \max_{\substack{W\in\mathbb{R}^{2nk}\\ W\perp\mathcal{V}_Z\\ ||W||_2=1}} W^T\left(\sum_{j\in I(z)} \frac{F_j\mu(\hat{z})\mu(\hat{z})^T F_j}{\mu(\hat{z})^T F_j \mu(\hat{z})}\right)W
\tag{C.105}
$$

$$
= \sup_{\substack{z\in\mathbb{C}^{n\times r}\\ z\neq 0}} \lambda_1(\hat{T}_z)
\tag{C.106}
$$

Moreover note that by setting $y = 0$ in the definition of B_0 and observing that $||(xx^*)^{\frac{1}{2}}||_2 = ||x||_2$ and that $\langle xx^*, A_j\rangle \geq 0$ we obtain that

$$
B_0 \geq \sup_{x\in\mathbb{C}^{n\times r}} \frac{1}{||x||_2^2}\sum_{j=1}^{m}\langle xx^*, A_j\rangle = B
\tag{C.107}
$$

Meanwhile by Cauchy-Schwartz $\langle zw^*, A_j\rangle \leq ||A_j^{\frac{1}{2}}w||_2||A_j^{\frac{1}{2}}z||_2 = \langle ww^*, A_j\rangle^{\frac{1}{2}}\langle zz^*, A_j\rangle^{\frac{1}{2}}$ $\blacksquare$

(similarly for $\langle wz^*, A_j\rangle$). Hence

$$
\begin{aligned}
B_0 &\leq \sup_{\substack{z\in\mathbb{C}^{n\times r}\\ z\neq 0}} \lambda_1(\hat{T}_z) \\
&= \sup_{\substack{z\in\mathbb{C}^{n\times r}\\ z\neq 0}} \sup_{w\in H_z} \frac{1}{||w||_2^2} \sum_{j\in I(z)} \frac{|\langle zw^* + wz^*, A_j\rangle_{\mathbb{R}}|^2}{4\langle zz^*, A_j\rangle} \\
&\leq \sup_{w\in H_z} \frac{1}{||w||_2^2} \sum_{j\in I(z)} \langle ww^*, A_j\rangle \\
&\leq \sup_{w\in\mathbb{C}^{n\times r}} \frac{1}{||w||_2^2} \sum_{j=1}^{m} \langle ww^*, A_j\rangle_{\mathbb{R}} = B
\end{aligned}
\tag{C.108}
$$

Thus $B \leq B_0 \leq \sup_{\substack{z\in\mathbb{C}^{n\times r}\\ z\neq 0}} \lambda_1(\hat{T}_z) \leq B$ and hence all three are equal. This concludes the proof of (iii) and of Theorem 5.13. ∎

C.5. Proof of Theorem 5.14.

Proof. It is shown in Proposition 5.1 that the map β is injective if and only if it is lower Lipschitz, that is if and only if $a_0 > 0$. This gives equivalence of (i) to (ii) immediately since we proved in Theorem 5.6 that

$$
a_0 = \min_{\substack{U_1\in\mathbb{C}^{n\times r}\\ U_2\in\mathbb{C}^{n\times(n-r)}\\ [U_1|U_2]\in U(n)}} \lambda_{2nr-r^2}(Q_{[U_1|U_2]})
\tag{C.109}
$$

Similarly, it is evident from (C.70) that $a_0 > 0$ if and only if $\hat{a}(z) > 0$ whenever $z^*z = \mathbb{I}_{r\times r}$. It is proved in Theorem 5.6 that $\hat{a}(z) = \lambda_{2nr-r^2}(\hat{Q}_z)$, and also that the null space of $\hat{Q}_z$ includes the r^2 dimension $\mathcal{V}_z$. Thus the frame is generalized phase retrievable if and only if the null space $\hat{Q}_z$ does not extend beyond $\mathcal{V}_z$ for any z of orthonormal columns, proving equivalence of (i) to (iii). We prove equivalence of (ii) to (iv) by noting that $Q_{[U_1|U_2]}$ is invertible if and only if

$$
\mathrm{span}_{\mathbb{R}}\{\begin{bmatrix}\tau(U_1^*A_jU_1)\\ \mu(U_2^*A_jU_1)\end{bmatrix}\}_{j=1}^{m} = \mathbb{R}^{2nr-r^2}
\tag{C.110}
$$

Noting that $\tau^{-1}(\mathbb{R}^{r^2}) = \mathrm{Sym}(\mathbb{C}^r)$ and $\mu^{-1}(\mathbb{R}^{2nr-2r^2}) = \mathbb{C}^{n-r\times r}$, thus $Q_{[U_1|U_2]}$ is invertible if and only if there exist $c_1,\ldots,c_m \in \mathbb{R}$ so that (5.39a) and (5.39b) are satisfied. To prove equivalence with (v) note that (5.39a) and (5.39b) both hold if and only if for all $U = [U_1|U_2]$ we have

$$
\begin{aligned}
\mathrm{span}_{\mathbb{R}}\{A_jU_1\} &= \{U\begin{bmatrix}H\\ B\end{bmatrix}|H\in\mathrm{Sym}(\mathbb{R}^n), B\in\mathbb{C}^{(n-r)\times r}\} \\
&= \{U_1K|K\in\mathbb{C}^{r\times r}, K^* = -K\}^{\perp}
\end{aligned}
\tag{C.111}
$$

Finally note that while (v) trivially implies (vi) it is also the case that $\langle A_jU_1, U_1K\rangle_{\mathbb{R}} = \langle U_1^*A_jU_1, K\rangle_{\mathbb{R}} = 0$ for every U_1 and every K since $U_1^*A_jU_1$ is Hermitian and K is skew-Hermitian, hence it is automatically true that $\mathrm{span}_{\mathbb{R}}\{A_jU_1\} \subset \{U_1K|K \in \mathbb{C}^{r\times r}, K^* = -K\}^{\perp}$. Thus we also obtain (vi) implies (v).

This concludes the proof of Theorem 5.14. ∎

Acknowledgments. This work was supported in part by NSF under Grant DMS-1816608.

REFERENCES

[1] R. Balan, *Stability of frames which give phase retrieval*, Houston Journal of Mathematics, (2015).

[2] R. Balan, *Frames and phaseless reconstruction*, Finite Frame Theory: A Complete Introduction to Overcompleteness, 93 (2016), p. 175.

[3] R. Balan, *Reconstruction of signals from magnitudes of redundant representations: The complex case*, Foundations of Computational Mathematics, 16 (2016), pp. 677–721.

[4] R. Balan, P. Casazza, and D. Edidin, *On signal reconstruction without phase*, Applied and Computational Harmonic Analysis, 20 (2006), pp. 345–356.

[5] R. Balan and Y. Wang, *Invertibility and robustness of phaseless reconstruction*, Applied and Computational Harmonic Analysis, 38 (2015), pp. 469–488.

[6] R. Balan and D. Zou, *On lipschitz analysis and lipschitz synthesis for the phase retrieval problem*, Linear Algebra and its Applications, 496 (2016), pp. 152–181.

[7] A. S. Bandeira, J. Cahill, D. G. Mixon, and A. A. Nelson, *Saving phase: Injectivity and stability for phase retrieval*, Applied and Computational Harmonic Analysis, 37 (2014), pp. 106–125.

[8] R. Bhatia, T. Jain, and Y. Lim, *On the bures–wasserstein distance between positive definite matrices*, Expositiones Mathematicae, 37 (2019), pp. 165–191.

[9] R. Bhatia and F. Kittaneh, *Notes on matrix arithmetic–geometric mean inequalities*, Linear Algebra and Its Applications, 308 (2000), pp. 203–211.

[10] J. Cahill, P. Casazza, and I. Daubechies, *Phase retrieval in infinite-dimensional hilbert spaces*, Transactions of the American Mathematical Society, Series B, 3 (2016), pp. 63–76.

[11] E. Candés, Y. Eldar, T. Strohmer, and V. Voroninski, *Phase retrieval via matrix completion problem*, SIAM J. Imag. Sci., 6 (2013), pp. 199–225.

[12] E. J. Candes, X. Li, and M. Soltanolkotabi, *Phase retrieval via wirtinger flow: Theory and algorithms*, IEEE Transactions on Information Theory, 61 (2015), pp. 1985–2007.

[13] X. Chen, D. P. Hardin, and E. B. Saff, *On the search for tight frames of low coherence*, Journal of Fourier Analysis and Applications, 27 (2021), pp. 1–27.

[14] Y. C. Eldar and S. Mendelson, *Phase retrieval: Stability and recovery guarantees*, Applied and Computational Harmonic Analysis, 36 (2014), pp. 473–494.

[15] E. Esser and F. Herrmann, *Application of a convex phase retrieval method to blind seismic deconvolution*, in 76th EAGE Conference and Exhibition 2014, European Association of Geoscientists & Engineers, 2014, pp. 1–5.

[16] S. T. Flammia, D. Gross, Y.-K. Liu, and J. Eisert, *Quantum tomography via compressed sensing: error bounds, sample complexity and efficient estimators*, New Journal of Physics, 14 (2012), p. 095022.

[17] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, vol. 2, Springer, 1990.

[18] C. G. Gibson, *Singular points of smooth mappings*, vol. 105, Pitman London, 1979.

[19] M. A. Hasankhani Fard and S. Moazeni, *Signal reconstruction without phase by norm retrievable frames*, Linear and Multilinear Algebra, 69 (2021), pp. 1484–1499.

[20] V. Kaloshin, *A geometric proof of the existence of whitney stratifications*, arXiv preprint math/0010144, (2000).

[21] F. Krahmer and Y.-K. Liu, *Phase retrieval without small-ball probability assumptions*, IEEE Transactions on Information Theory, 64 (2017), pp. 485–500.

[22] J. Li and T. Zhou, *On gradient descent algorithm for generalized phase retrieval problem*, arXiv preprint arXiv:1607.01121, (2016).

[23] J. Li, T. Zhou, and C. Wang, *On global convergence of gradient descent algorithms for generalized phase retrieval problem*, Journal of Computational and Applied Mathematics, 329 (2018), pp. 202–222.

[24] J. Mather, *Notes on topological stability*, Bulletin of the American Mathematical Society, 49 (2012), pp. 475–506.

[25] P. Salanevich, *Stability of phase retrieval problem*, in 2019 13th International conference on Sampling Theory and Applications (SampTA), IEEE, 2019, pp. 1–4.

[26] B. Vandereycken, P.-A. Absil, and S. Vandewalle, *Embedded geometry of the set of symmetric positive semidefinite matrices of fixed rank*, in 2009 IEEE/SP 15th Workshop on Statistical Signal Processing, IEEE, 2009, pp. 389–392.

[27] Y. Wang and Z. Xu, *Generalized phase retrieval: measurement number, matrix recovery and*

beyond, Applied and Computational Harmonic Analysis, 47 (2019), pp. 423–446.

[28] Y. Wang and Z. Xu, *Generalized phase retrieval: Measurement number, matrix recovery and beyond*, Applied and Computational Harmonic Analysis, 47 (2019), pp. 423–446.

[29] H. Whitney, *Local properties of analytic varieties*, in Hassler Whitney Collected Papers, Springer, 1992, pp. 497–536.

[30] Z. Xu, *The minimal measurement number for low-rank matrix recovery*, Applied and Computational Harmonic Analysis, 44 (2018), pp. 497–508.

[31] Z. Zhuang, *On stability of generalized phase retrieval and generalized affine phase retrieval*, Journal of Inequalities and Applications, 2019 (2019), pp. 1–13.