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LETTER TO THE EDITOR

An Uncertainty Inequality for Wavelet Sets

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Abstract — The purpose of this note is to present an extension and an alternative proof to Theorem 1.3 from G. Battle (*Appl. Comput. Harmonic Anal.* **4** (1997) 119–146). This extension applies to wavelet Bessel sets which include wavelet Riesz bases for their span, wavelet Riesz bases (including orthogonal and biorthogonal wavelet bases), and wavelet frames.

Let $\Psi \in L^2(\mathbf{R})$ and $a > 1$, $b > 0$ be given data. We denote by

$$W_{\Psi;ab} = \{\Psi_{mn;ab};\ m, n \in \mathbf{Z}\},\ \Psi_{mn;ab}(x) = a^{m/2}\Psi(a^m x - nb) \tag{1}$$

the *wavelet set* associated to the wavelet Ψ and parameters a, b.

DEFINITION We call $W_{\Psi;ab}$ a *wavelet Bessel set* if there exists a constant $B > 0$ such that for every $f \in L^2(\mathbf{R})$:

$$\sum_{m,n} |\langle f, \Psi_{mn;ab}\rangle|^2 \leqslant B\|f\|^2. \tag{2}$$

We shall use the notations of [1] for P, X, $\sigma_\Psi(X)$, $\sigma_\Psi(P)$, $\langle P\rangle_\Psi$, $\langle X\rangle_\Psi$. Then, the main result can be stated as

THEOREM *Suppose $W_{\Psi;ab}$ is a wavelet Bessel set. Then*

$$\|X\Psi\| \cdot \|P\Psi\| \geqslant \tfrac{3}{2}. \tag{3}$$

Furthermore, if $\langle P\rangle_\Psi = 0$ (for instance, when Ψ is real-valued) then

$$\sigma_{\Psi_{mn,ab}}(X)\sigma_{\Psi_{mn,ab}}(P) = \sigma_\Psi(X)\sigma_\Psi(P) \geqslant \tfrac{3}{2}. \tag{4}$$

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Proof. If $X\Psi$, $P\Psi$ do not both lie in $L^2(\mathbf{R})$, then either $\|X\Psi\|$ or $\|P\Psi\|$ is infinite and (3), (4) trivially hold.

Suppose now that both $X\Psi$, $P\Psi$ are in $L^2(\mathbf{R})$, which means, equivalently, $x\Psi$, $\Psi' \in L^2(\mathbf{R})$. Thus Ψ and $\hat{\Psi}$ are integrable (i.e., in $L^1(\mathbf{R})$) and continuous.

On the other hand, the same technique that C. K. Chui and X. Shi used to prove Littlewood–Paley type inequalities for wavelet frames in [2] allows us to obtain these two conditions on Ψ because $W_{\Psi;ab}$ is a Bessel set,

$$\frac{1}{b}\sum_{m\in\mathbf{Z}} |\hat{\Psi}(a^m\xi)|^2 \leqslant B, \tag{5}$$

a.e. $\xi \in \mathbf{R}$ and, since $\hat{\Psi}$ is continuous, it follows that (5) holds for any $\xi \in \mathbf{R}$. By integration from 1 to a we get the second relation,

$$\frac{1}{2b\log a}\int_{-\infty}^{\infty} \frac{|\hat{\Psi}(\xi)|^2}{|\xi|}\, d\xi \leqslant B. \tag{6}$$

Since $\hat{\Psi}$ is continuous we obtain that necessarily $\hat{\Psi}(0) = 0$ which means

$$\int \Psi(x)dx = 0. \tag{7}$$

Consider now two linear spaces (S is the space of the rapidly decreasing functions):

$$S_0 = \{\varphi \in S, \int \varphi(x)dx = 0\}, \tag{8}$$

$$V_0 = \{f \in L^2(\mathbf{R}),\ Xf,\ Pf \in L^2(\mathbf{R}), \text{ and } \int f(x)dx = 0\}. \tag{9}$$

We claim that S_0 is dense in V_0 with respect to the norm $|||f||| = \|f\| + \|Xf\| + \|Pf\|$ (for which, by the way, the space V_0 is closed). To see this, consider $f \in V_0$ and a sequence $\varphi_n \in S$ such that $|||\varphi_n - f||| \to 0$ (i.e., $\|\varphi_n - f\| \to 0$, $\|X\varphi_n - Xf\| \to 0$, $\|P\varphi_n - Pf\| \to 0$). Choose $G \in S$ such that $\int G(x)dx = 1$ and set $c_n = \int \varphi_n(x)dx$. Then $\varphi_n^0 = \varphi_n - c_nG \in S_0$ and $|||\varphi_n^0 - f||| \to 0$, since $c_n \to 0$. Thus S_0 is dense in V_0.

For $\Psi \in S_0$, Battle proved that (3) holds and, when $\langle P\rangle_\Psi = 0$, (4) holds as well. We extend now his result to V_0 by a density argument.

Consider now $\Psi \in V_0$. Choose $\varphi_n \in S_0$ converging to Ψ in norm $|||\cdot|||$. Then, obviously

$$\|X\varphi_n\| \to \|X\Psi\|, \quad \|P\varphi_n\| \to \|P\Psi\| \tag{10}$$

and thus (3) is established.

For (4) we first note that (10) implies $\langle P\rangle_{\varphi n} \to \langle P\rangle_\Psi = 0, \langle X\rangle_{\Psi n} \to \langle X\rangle_\Psi$, and, since $\sigma_\Psi(X) = (\|X\Psi\|^2 - (\langle X\rangle_\Psi)^2)^{1/2}$, $\sigma_\Psi(P) = (\|P\Psi\|^2 - (\langle P\rangle_\Psi)^2)^{1/2}$, we get as well that $\sigma_{\varphi n}(X) \to \sigma_{\Psi}(X)$ and $\sigma_{\varphi n}(P) \to \sigma_{\Psi}(P)$. Finally, as has been observed many times before (for instance in [3]), the uncertainty product $\sigma_{\Psi}(X)\sigma_{\Psi}(P)$ is invariant along the wavelet set. This ends the proof of (4) and of the theorem. ■

Remark We point out that the inequality (3) holds as well for every element of $W_{\Psi;ab}$, i.e.,

$$\|X\Psi_{mn;ab}\| \cdot \|P\Psi_{mn;ab}\| \geqslant \tfrac{3}{2}, \tag{11}$$

since (7) holds for every $\Psi_{mn;ab}$.

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