ON A PROBLEM BY HANS FEICHTINGER

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Abstract. In this paper, we solve a spectral problem about positive semi-definite trace-class pseudodifferential operators on modulation spaces which was posed by H. Feichtinger. Later, C. Heil and D. Larson rephrased the problem in the broader setting of positive semi-definite trace-class operators on a separable Hilbert space. Our solution consists in constructing a counterexample that solves Hans Feichtinger’s problem by first solving this second problem.

1. Introduction

In this paper we answer the following question posed by Feichtinger at an Oberwolfach mini-workshop on wavelets [4].

PROBLEM 1.1. Let $T$ be a positive semi-definite trace class operator on $L^2(\mathbb{R})$ given by

$$ T f(x) = \int_{\mathbb{R}} k(x,y) f(y) dy, $$

where $f \in L^2(\mathbb{R})$ and $k \in M^1(\mathbb{R}^2)$, the so-called Feichtinger algebra. Suppose that

$$ T = \sum_{k=1}^{\infty} h_k \otimes \overline{h_k}, $$

where $\{h_k\}_{k=1}^{\infty} \subset L^2(\mathbb{R})$ is a set of orthogonal eigenfunctions of $T$ corresponding to the eigenvalues $\{||h_k||_2^2\}_{k=1}^{\infty}$, such that $||h_k||_{M^1(\mathbb{R})} < \infty$, and the bar denotes the complex conjugation. In particular, $\text{Trace}(T) = \sum_{k=1}^{\infty} ||h_k||_2^2 < \infty$.

Must we have: $\sum_{k=1}^{\infty} ||h_k||_{M^1(\mathbb{R})}^2 < \infty$?

Heil and Larson later put the problem in the broader setting of positive semi-definite trace-class operators on a separable Hilbert space $\mathbb{H}$ [9]. To state this generalization we first set some notations. Let $\mathbb{H}$ be a separable Hilbert space and choose an orthonormal basis $\{w_n\}_{n \geq 1}$ for $\mathbb{H}$. We define a subspace $\mathbb{H}^1$ of $\mathbb{H}$ by

$$ \mathbb{H}^1 = \left\{ f \in \mathbb{H} : ||f|| := \sum_{n=1}^{\infty} |\langle f, w_n \rangle| < \infty \right\}. $$


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It follows that \( \|w_n\| = \|w_n\| = 1 \) for every \( n \), and that if \( f \in \mathbb{H}^1 \) then \( f = \sum_{n=1}^{\infty} \langle f, w_n \rangle w_n \), with convergence of this series in both norms \( \| \cdot \| \) and \( \| \cdot \| \).

We define an operator \( T : \mathbb{H} \to \mathbb{H} \) by

\[
T = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} (w_m \otimes w_n),
\]

where the scalars \( c_{mn} \) are such that

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |c_{mn}| < \infty
\]

and the tensor product \( w_m \otimes w_n \) maps linearly \( \mathbb{H} \) to \( \mathbb{H} \) via

\[
f \in \mathbb{H} \mapsto w_m \otimes w_n(f) = \langle f, w_n \rangle w_m.
\]

It is easy to see that \( T \in \mathcal{A}_1 \), the space of all trace-class operators, with

\[
\|T\|_{\mathcal{A}_1} \leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \|c_{mn} (w_m \otimes w_n)\|_{\mathcal{A}_1} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |c_{mn}| < \infty.
\]

In addition, note that the series defining \( T \) converges not only in the strong operator topology and operator norm, but also in trace-class norm.

Now suppose that the operator \( T \) given by (1.2) is positive semi-definite. Let \( \{h_n\}_{n \geq 1} \) be an orthonormal basis of eigenvectors of \( T \) and \( \{\lambda_n\}_{n \geq 1} \subset [0, \infty) \) be the corresponding eigenvalues. It follows that

\[
T = \sum_{n=1}^{\infty} \lambda_n (h_n \otimes \overline{h_n}) = \sum_{n=1}^{\infty} g_n \otimes \overline{g_n},
\]

where \( g_n = \lambda_n^{1/2} h_n \). In addition,

\[
\|T\|_{\mathcal{A}_1} = \sum_{n=1}^{\infty} \lambda_n = \sum_{n=1}^{\infty} \lambda_n \|h_n\|^2 < \infty.
\]

Heil and Larson's generalization of Problem 1.1 is the following question [9].

**Problem 1.2.** With the above notations, must we have

\[
\sum_{n=1}^{\infty} \lambda_n \|h_n\|^2 < \infty?
\]
2. Preliminaries

In this section we recall the definition of the modulation spaces and some of their properties. In the second half of the section, we introduce two classes of trace-class operators that capture the behaviors of the operators in Problems 1.1 and 1.2.

2.1. Modulation spaces

Let \( g \in \mathcal{S}'(\mathbb{R}) \) be a function in the Schwartz space of smooth and rapidly decaying functions, e.g., \( g(x) = e^{-\pi x^2} \), and let \( 1 \leq p \leq \infty \). We say that a tempered distribution \( f \) is in the modulation space \( M^p(\mathbb{R}) \) if and only if

\[
\|f\|_{M^p}^p := \iint_{\mathbb{R}^2} |V_g f(x, \omega)|^p \, dx \, d\omega < \infty,
\]

with the usual modification for \( p = \infty \), where

\[
V_g f(x, \omega) = \int_{\mathbb{R}} f(t) g(t-x) e^{-2\pi i \omega t} \, dt
\]

is the short-time Fourier transform (STFT) of a function \( f \) with respect to \( g \). A simple application of the Plancherel formula shows that if \( f \in L^2(\mathbb{R}) \) then

\[
\|V_g f\|_{L^2(\mathbb{R}^2)}^2 = \iint_{\mathbb{R}^2} |V_g f(x, \omega)|^2 \, dx \, d\omega = \|g\|_2^2 \|f\|_2^2.
\]

Consequently, \( V_g \) is a multiple of an isometry from \( L^2(\mathbb{R}) \) into \( L^2(\mathbb{R}^2) \) and \( M^p(\mathbb{R}) = L^2(\mathbb{R}) \), \( [7] \). The other modulation space that will be of interest in the sequel is \( M^1(\mathbb{R}) \), which is also known as the Feichtinger algebra \([5, 7]\). In particular, we note that

\[
\mathcal{S}(\mathbb{R}) \subset M^1(\mathbb{R}) \subset M^2(\mathbb{R}) = L^2(\mathbb{R}) \subset M^\infty(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}).
\]

We also need a discrete characterization of \( L^2 \) and \( M^1 \). Such a characterization exists for all the modulation spaces in terms of the so-called Wilson basis, see \([2, 6, 12]\). In particular, it is known that there exists an orthonormal basis \( \mathcal{W} := \{w_n\}_{n \geq 1} \) for \( L^2(\mathbb{R}) \) where for each \( n \geq 1 \), \( w_n \in M^1(\mathbb{R}) \). In addition, for \( 1 \leq p \leq \infty \) and for all \( f \in M^p \),

\[
f = \sum_{n \geq 1} \langle f, w_n \rangle w_n,
\]

where the series converges unconditionally in the norm of \( M^p \) if \( 1 \leq p < \infty \), and is weak* convergent if \( p = \infty \). Moreover,

\[
\|f\|_{M^p} = \left( \sum_{n \geq 1} |\langle f, w_n \rangle|^p \right)^{1/p}
\]

is an equivalent norm for \( M^p \); we refer to \([7, Theorem 8.5.1]\) for details. In the sequel, we shall only be interested in \( p = 1 \), and \( p = 2 \). In the latter case, \( \{w_n\}_{n \geq 1} \) is an orthonormal basis for \( L^2(\mathbb{R}) \).
It is trivial to extend these characterizations to modulation spaces defined on \( \mathbb{R}^d \). In particular, one defines a Wilson orthonormal basis for \( L^2(\mathbb{R}^2) \) by taking the tensor product of 1-dimensional Wilson ONBs. For example, \( \{W_{n,m} : n, m \geq 1\} \subset L^2(\mathbb{R}^2) \) is given by
\[
W_{n,m}(x,y) := w_n \otimes w_m(x,y) = w_n(x)w_m(y), \quad n, m \geq 1,
\]
and it acts by
\[
W_{n,m}(f) = (f, w_m)w_n = \left( \int_{\mathbb{R}} f(y)w_m(y)dy \right)w_n.
\]
In addition, \( \{W_{n,m} : n, m \geq 1\} \) is an unconditional basis for \( M^1(\mathbb{R}^2) \).

Let \( T : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) be a compact integral operator associated with the kernel \( k \in M^1(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \) and defined by
\[
Tf(x) = \int_{\mathbb{R}} k(x,y)f(y)dy.
\]
Then, \( T \) is a trace-class operator \([9]\), and
\[
k = \sum_{m,n \geq 1} \langle k, W_{m,n} \rangle W_{m,n}, \tag{2.1}
\]
with convergence of the series in the \( M^1 \)-norm. In addition,
\[
||k||_{M^1} = \sum_{m,n \geq 1} |\langle k, W_{m,n} \rangle| < \infty. \tag{2.2}
\]
It now follows that for \( f \in L^2(\mathbb{R}) \),
\[
Tf = \sum_{m,n \geq 1} \langle k, W_{m,n} \rangle (w_m \otimes w_n)(f) = \sum_{m,n \geq 1} \langle k, W_{m,n} \rangle (W_{m,n})(f).
\]
The discrete version of the integral operator \( T \) is given by the matrix \( K = (\langle k, W_{m,n} \rangle)_{m,n \geq 1} \), or equivalently
\[
T = \sum_{m,n \geq 1} \langle k, W_{m,n} \rangle W_{m,n}. \tag{2.3}
\]
Suppose in addition that \( T \) is positive semi-definite. Then, by the spectral theorem,
\[
T = \sum_{k=1}^{\infty} \lambda_k f_k \otimes \overline{f_k} = \sum_{k=1}^{\infty} h_k \otimes \overline{h_k},
\]
where \( \{\lambda_k\}_{k=1}^{\infty} \subset (0, \infty) \) is the set of eigenvalues of \( T \) and \( \{f_k\}_{k=1}^{\infty} \) is an orthonormal basis of corresponding eigenfunctions, and \( h_k = \sqrt{\lambda_k} f_k \) for each \( k \geq 1 \). It was proved in \([1, 9]\) that \( h_k \in M^1(\mathbb{R}) \).
2.2. Type A and type B operators

Let \( \mathcal{H} \) denote an infinite-dimensional separable Hilbert space, with norm \( \| \cdot \| \) and inner product \( \langle \cdot , \cdot \rangle \). Let \( \mathcal{S}_1 \subset \mathcal{B}(\mathcal{H}) \) be the subspace of trace-class operators. A positive semi-definite operator \( T \) belongs to \( \mathcal{S}_1 \) if and only if

\[
\| T \|_{\mathcal{S}_1} = \sum_{n=1}^{\infty} \lambda_n(T) < \infty,
\]

where \( \{ \lambda_n(T) \}_{n \geq 1} \) is the set of eigenvalues of \( T \) arranged in a decreasing order and repeated according to multiplicity. For a detailed study on trace-class operators see [3, 10].

We fix now an orthonormal basis \( \{ w_n \}_{n \geq 1} \) for \( \mathcal{H} \), once and for all. This basis induces the norm \( \| \cdot \| \) on the dense subset \( \mathcal{H}^1 \) introduced in (1.1), and repeated here for the convenience of the reader:

\[
\| f \| = \sum_{n=1}^{\infty} |\langle f, w_n \rangle|, \quad \mathcal{H}^1 = \left\{ f \in \mathcal{H} : \sum_{n=1}^{\infty} |\langle f, w_n \rangle| < \infty \right\}.
\]

**Definition 2.1.** An operator \( T \) given by (1.2) is of Type A with respect to the orthonormal basis \( \{ w_n \}_{n \geq 1} \) if, for an orthogonal set of eigenvectors \( \{ g_n \}_{n \geq 1} \) of \( T \) such that \( T = \sum_{n=1}^{\infty} g_n \otimes g_n \), with convergence in the strong operator topology, we have that

\[
\sum_{n=1}^{\infty} \| g_n \|^2 < \infty.
\]

**Definition 2.2.** An operator \( T \) given by (1.2) is of Type B with respect to the orthonormal basis \( \{ w_n \}_{n \geq 1} \) if there is some sequence of vectors \( \{ v_n \}_{n \geq 1} \) in \( \mathcal{H} \) such that \( T = \sum_{n=1}^{\infty} v_n \otimes v_n \) with convergence in the strong operator topology and we have that

\[
\sum_{n=1}^{\infty} \| v_n \|^2 < \infty.
\]

It is clear that if \( T \) is of Type A then it is of Type B. However, it was shown in [9, Example 2.2] that not every positive trace-class operator is of Type A or Type B, even when the operator is finite-rank.

**Problem 1.2** can now be reformulated as follows.

**Problem 2.3.** If \( T \) is of Type B with respect to an orthonormal basis \( \{ w_n \}_{n \geq 1} \), must it be of Type A with respect to the same ONB \( \{ w_n \}_{n \geq 1} \)?

3. Main results

We answer negatively Problems 1.2 and 2.3 by constructing a counterexample for the complex Hilbert space \( \mathbb{H} \), in Proposition 3.1. This example is then modified to generate an example when the Hilbert space \( \mathbb{H} \) is over the real field, in Proposition 3.3. From there, we answer the Feichtinger original problem in Theorem 3.4.
Proposition 3.1. Let $\mathbb{H} = \ell^2\left(\{1,2,\ldots\}\right)$, and choose $p > 1$. Let $\{w_{\ell}\}_{\ell=1}^{\infty}$ denote the standard orthonormal basis of $\mathbb{H}$, i.e., $w_{\ell} = \delta_{\ell \ell}$. Then $\mathbb{H}^1 = \ell^1\left(\{1,2,\ldots\}\right)$.

For each $n \geq 1$, let $\{e_{n,k}\}_{k=0}^{n-1}$ be the Fourier ONB of $\mathbb{C}^n$ defined by

$$e_{n,k} = \frac{1}{\sqrt{n}} \left( e^{-\frac{2\pi ik}{n}} \right)_{k=0}^{n-1} = \frac{1}{\sqrt{n}} \left( 1, e^{-\frac{2\pi i k}{n}}, e^{-\frac{4\pi i k}{n}}, \ldots, e^{-\frac{2\pi i (n-1) k}{n}} \right)^T,$$

and consider the $n \times n$ matrix $T_n$ given by

$$T_n = \sum_{k=0}^{n-1} \lambda_{n,k} (e_{n,k} \otimes \overline{e_{n,k}}) = \frac{1}{n^3} \sum_{k=0}^{n-1} \left( 1 + \frac{k}{n^p} \right) (e_{n,k} \otimes \overline{e_{n,k}}) \in \mathbb{C}^{n \times n},$$

where $\lambda_{n,k} = \frac{1}{n^3} \left( 1 + \frac{k}{n^p} \right)$. We define an infinite block-diagonal matrix $T$ by

$$T = T_1 \oplus T_2 \oplus \ldots \oplus T_n \oplus \ldots$$

Then, $T$ is a positive semi-definite trace-class operator of Type B but not of Type A with respect to the orthonormal basis $\{w_{\ell}\}$.

Proof. By construction, the blocks $T_n$ that make up $T$ are pairwise orthogonal. Furthermore, for each $n \geq 1$, the spectrum of $T_n$ consists of simple eigenvalues $\lambda_{n,k}$ with corresponding eigenvectors $e_{n,k}$ for $k = 0, \ldots, n-1$. Consequently, for each $n \geq 1$, and each $k \in \{0,\ldots,n-1\}$, $e_{n,k}$ generates a one-dimensional eigenspace of $T$ corresponding to the eigenvalue $\lambda_{n,k}$. It is clear that $T$ is positive semi-definite.

Since $\|e_{n,k}\|_2 = 1$ and $T = \bigoplus_{n=1}^{\infty} \sum_{k=0}^{n-1} \lambda_{n,k} (e_{n,k} \otimes \overline{e_{n,k}})$, we see that

$$\|T\|_{op} \leq \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{n^3} \left( 1 + \frac{k}{n^p} \right) \|e_{n,k} \otimes \overline{e_{n,k}}\|_{op}$$

$$= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{n^3} \left( 1 + \frac{k}{n^p} \right) \|e_{n,k}\|$$

$$= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{n^3} \left( 1 + \frac{k}{n^p} \right) < \infty.$$ 

Furthermore, since $p > 1$, we see that

$$\|T\|_{\mathcal{S}_1} = \text{trace}(T) = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{n^3} \left( 1 + \frac{k}{n^p} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^3} \left( n + \frac{n(n-1)}{2n^p} \right)$$

$$< \infty.$$ 

Hence $T$ is a well-defined trace-class operator on $\mathbb{H}$.
We now show that $T$ is of Type $B$. To this end we observe that for each $n \geq 1$, \( \sum_{k=0}^{n-1} c_{n,k} \otimes \overline{c_{n,k}} = I_n \), where $I_n$ denotes the identity of order $n$. Then
\[
T_n = \frac{1}{n^3} \sum_{k=0}^{n-1} \left(1 + \frac{k}{n^p}\right) (c_{n,k} \otimes \overline{c_{n,k}})
\]
\[
= \frac{1}{n^3} \sum_{k=0}^{n-1} (c_{n,k} \otimes \overline{c_{n,k}}) + \frac{1}{n^3+p} \sum_{k=0}^{n-1} k (c_{n,k} \otimes \overline{c_{n,k}})
\]
\[
= \frac{1}{n^3} I_n + \frac{1}{n^3+p} \sum_{k=0}^{n-1} k (c_{n,k} \otimes \overline{c_{n,k}}).
\]
Thus $T$ can be written as
\[
T = \bigoplus_{n \geq 1} T_n = \bigoplus_{n \geq 1} \left( \frac{1}{n^3} I_n + \frac{1}{n^3+p} \sum_{k=0}^{n-1} k (c_{n,k} \otimes \overline{c_{n,k}}) \right)
\]
\[
= \bigoplus_{n \geq 1} \left( \frac{1}{n^3} I_n \right) + \bigoplus_{n \geq 1} \frac{1}{n^3+p} \sum_{k=0}^{n-1} k (c_{n,k} \otimes \overline{c_{n,k}})
\]
\[
= \bigoplus_{n \geq 1} \frac{1}{n^3} \sum_{k=1}^{n} (w_{n(n-1) + k} \otimes \overline{w_{n(n-1) + k}}) + \bigoplus_{n \geq 1} \frac{1}{n^3+p} \sum_{k=0}^{n-1} k (c_{n,k} \otimes \overline{c_{n,k}}).
\]
Then we have
\[
\| w_{n(n-1) + k} \| = 1, \quad \| c_{n,k} \| = \sqrt{n},
\]
and
\[
\sum_{n \geq 1} \frac{1}{n^3} \sum_{k=1}^{n} 1^2 + \sum_{n \geq 1} \frac{1}{n^3+p} \sum_{k=0}^{n-1} k \cdot (\sqrt{n})^2
\]
\[
= \sum_{n \geq 1} \left( \frac{1}{n^2} + \frac{n - 1}{2n^3 + p} \right) < \infty, \quad \text{for any } p > 1.
\]
Hence, $T$ is of Type $B$ with respect to $\{w_k\}_{k \geq 1}$.

We now show that $T$ is not of Type $A$ with respect to $\{w_k\}_k$. The key point is that $T$ has only one-dimensional eigenspaces, so
\[
\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \lambda_{n,k} (c_{n,k} \otimes \overline{c_{n,k}}) = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{n^3} \sum_{k=0}^{n-1} \left(1 + \frac{k}{n^p}\right) (c_{n,k} \otimes \overline{c_{n,k}})
\]
is the unique decomposition of $T$ as a sum of rank one projections generated by orthogonal eigenfunctions of $T$. Note again that $\| c_{n,k} \| = \sqrt{n}$, and
\[
\lambda_{n,k} \| c_{n,k} \| = \frac{1}{n^3} \left(1 + \frac{k}{n^p}\right) \cdot \sqrt{n} < \infty.
\]
However,
\[
\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \lambda_{n,k} \left\| e_{n,k} \right\|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=0}^{n-1} \left( 1 + \frac{k}{n^p} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} \left( n + \frac{n(n-1)}{2n^p} \right) \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty. \quad \square
\]

We can modify the counterexample in Proposition 3.1 to deal with the case of a real Hilbert space $H$. This amounts to using a real-valued ONB for $\mathbb{R}^n$ instead of the Fourier ONB $\{ e_{n,k} \}_{k=0}^{n-1}$. For this let $\{ h_{n,k} \}_{k=0}^{n-1}$ denote the Hartley ONB basis for $\mathbb{R}^n$ (see [11]), where
\[
h_{n,k} = \frac{1}{\sqrt{n}} \left( \cos \left( \frac{2\pi kl}{n} \right) + \sin \left( \frac{2\pi kl}{n} \right) \right)_{l=0}^{n-1} = \sqrt{\frac{2}{n}} \left( \cos \left( \frac{2\pi kl}{n} - \frac{\pi}{4} \right) \right)_{l=0}^{n-1}.
\]
Thus
\[
\sum_{k=0}^{n-1} h_{n,k} \otimes h_{n,k} = \sum_{k=0}^{n-1} h_{n,k} \otimes h_{n,k} = I_n,
\]
where $I_n$ denotes the identity of order $n$ in $\mathbb{R}^n$.

**Lemma 3.2.** For a fixed $n \geq 1$ and each $0 \leq k \leq n-1$ we have
\[
\sqrt{\frac{n}{2}} \leq \left\| h_{n,k} \right\| = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \left| \cos \left( \frac{2\pi kl}{n} \right) + \sin \left( \frac{2\pi kl}{n} \right) \right| \leq \sqrt{n}. \quad (3.1)
\]

**Proof.** Denote by $S_n$ the set
\[
S_n := \left\{ \frac{2\pi k}{n} : 0 \leq k \leq n-1 \right\}.
\]
It is easy to see that for each $0 \leq l \leq n-1$ we have
\[
S_n = \left\{ \frac{2\pi kl}{n} \pmod{2\pi} : 0 \leq k \leq n-1 \right\} = \left\{ -\frac{2\pi k}{n} \pmod{2\pi} : 0 \leq k \leq n-1 \right\}.
\]
Let $E := \sum_{x \in S_n} \left| \cos x + \sin x \right|$. Then
\[
2E = \sum_{x \in S_n} \left| \cos x + \sin x \right| + \sum_{-x \in S_n} \left| \cos x + \sin x \right| = \sqrt{2} \sum_{k=0}^{n-1} \left| \cos \left( \frac{2\pi k}{n} - \frac{\pi}{4} \right) \right| + \sqrt{2} \sum_{k=0}^{n-1} \left| \cos \left( \frac{2\pi k}{n} + \frac{\pi}{4} \right) \right| = \sqrt{2} \sum_{k=0}^{n-1} \left[ \left| \cos \left( \frac{2\pi k}{n} - \frac{\pi}{4} \right) \right| + \left| \sin \left( \frac{2\pi k}{n} - \frac{\pi}{4} \right) \right| \right]. \quad (3.2)
\]
Now for each \( x \in \mathbb{R} \),
\[
(|\sin x| + |\cos x|)^2 = |\sin x|^2 + |\cos x|^2 + 2|\sin x\cos x| = 1 + |\sin 2x| \geq 1,
\]
\[\Rightarrow \sqrt{2} \geq |\sin x| + |\cos x| \geq 1.
\]

It follows from (3.2) that \( n \geq E \geq \frac{n}{\sqrt{2}} \) and therefore (3.1). \( \square \)

**Proposition 3.3.** Let \( H = \ell^2([1, 2, \ldots]) \), and choose \( p > 1 \). Let \( \{w_\ell\}_{\ell=1}^n \) denote the standard orthonormal basis of \( H \), i.e., \( w_\ell = \delta_\ell \). For each \( n \geq 1 \) let \( T_n \) denote the \( n \times n \) matrix given by
\[
T_n = \frac{1}{n^2} \sum_{k=0}^{n-1} \left( 1 + \frac{k}{n^p} \right) (h_{n,k} \otimes h_{n,k}) \in \mathbb{R}^{n \times n}.
\]

We define an infinite block-diagonal matrix \( T \) by
\[
T = T_1 \oplus T_2 \oplus \ldots \oplus T_n \oplus \ldots
\]

Then, \( T \) is a positive semi-definite trace-class operator of Type B but not of Type A with respect to the orthonormal basis \( \{w_\ell\}_{\ell>1} \).

**Proof.** The proof is almost identical to that of Proposition 3.1 where the Fourier ONB vectors \( e_{n,k} \) are replaced by the Hartley ONB vectors \( h_{n,k} \) and the estimate \( \|e_{n,k}\| = \sqrt{n} \) is replaced by \( \sqrt{n/2} \leq \|h_{n,k}\| \leq \sqrt{n} \), cf. Lemma 3.2. \( \square \)

We can now give an answer to Feichtinger's question, i.e., Problem 1.2.

**Theorem 3.4.** Suppose that \( \{w_n\}_{n \geq 1} \) is a Wilson orthonormal basis for \( L^2(\mathbb{R}) \) with \( g \in M^1(\mathbb{R}) \). Let \( p > 1 \), and for each \( n \geq 1 \) set \( \lambda_{n,k} = \frac{1}{n^p} (1 + \frac{k}{n^p}) \).

For fixed \( n \geq 1 \) and each \( 0 \leq k \leq n-1 \), let \( h_{n,k} \in L^2(\mathbb{R}) \) where
\[
h_{n,k} = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \left( \cos \left( \frac{2\pi kl}{n} \right) + \sin \left( \frac{2\pi kl}{n} \right) \right) w_{n(n-1)+l+1}.
\]

Let \( T \) be the operator defined by
\[
T = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \lambda_{n,k} h_{n,k} \otimes h_{n,k}.
\]

The following statements hold:

(i) \( \{h_{n,k} : 0 \leq k \leq n-1, n \geq 1 \} \) is an orthonormal basis for \( L^2(\mathbb{R}) \).

(ii) \( T \) is a positive semi-definite trace-class operator on \( L^2(\mathbb{R}) \) that provides a counterexample to Problem 1.2.
Proof. (i) It is easy to see that for each \( n \geq 1 \), \( \{h_{n,k}\}_{k=0}^{n-1} \) is an orthogonal set in \( L^2(\mathbb{R}) \). Indeed, \( \langle h_{n,k}, h_{n,k'} \rangle = 0 \), for \( n \neq n' \). Furthermore, since \( \langle w_n, w_m \rangle = \delta_{n,m} \) we have that \( \|h_{n,k}\| = 1 \) for all \( n \geq 1 \), and \( k \in \{0,1,\ldots,n-1\} \).

(ii) It is also easy to see that \( T \) is a well-defined operator on \( L^2(\mathbb{R}) \). In fact, the series defining \( T \) converges in the operator norm. Furthermore, since \( \|h_{n,k} \otimes h_{n,k}\|_{\mathcal{S}_1} = 1 \), it follows that

\[
\|T\|_{\mathcal{S}_1} = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \lambda_{n,k} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=0}^{n-1} \left(1 + \frac{k}{n^p}\right) = \sum_{n=1}^{\infty} \frac{1}{n^3} \left(n + \frac{n(n-1)}{2n^p}\right) < \infty.
\]

Consequently, \( T \) is a trace-class operator.

By Lemma 3.2,

\[
\|h_{n,k}\|_{M^1} = \sum_{m=1}^{\infty} |\langle h_{n,k}, w_m \rangle| = \frac{1}{\sqrt{n}} \sum_{m=1}^{\infty} \left| \sum_{l=0}^{n-1} \left( \cos \left(\frac{2\pi kl}{n}\right) + \sin \left(\frac{2\pi kl}{n}\right) \right) w_{n(n-1)+l} w_m \right| = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \left| \cos \left(\frac{2\pi kl}{n}\right) + \sin \left(\frac{2\pi kl}{n}\right) \right| \geq \sqrt{\frac{n}{2}}.
\]

Also each term

\[
\lambda_{n,k} \|h_{n,k}\|_{M^1} = \frac{1}{n^3} \left(1 + \frac{k}{n^p}\right) \cdot \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \left| \cos \left(\frac{2\pi kl}{n}\right) + \sin \left(\frac{2\pi kl}{n}\right) \right| \leq \frac{1}{n^3} \left(1 + \frac{k}{n^p}\right) \cdot \sqrt{n} < \infty.
\]

However,

\[
\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \lambda_{n,k} \|h_{n,k}\|_{M^1}^2 \geq \sum_{n=1}^{\infty} \frac{1}{2n^2} \sum_{k=0}^{n-1} \left(1 + \frac{k}{n^p}\right) = \sum_{n=1}^{\infty} \frac{1}{2n^2} \left(n + \frac{n(n-1)}{2n^p}\right) \geq \sum_{n=1}^{\infty} \frac{1}{2n} = \infty. \quad \square
\]

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REFERENCES


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