Lipschitz Embeddings and Riemannian Properties of Spaces of Low-Rank Symmetric Matrices

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Overview

1 Introduction

2 Lipschitz Embeddings

3 Geometry of $S^{r,0}(\mathbb{C}^n)$

4 Stability Bounds

5 Criteria for Phase Retrievability
The Complex Phase Retrieval Problem: Variants

- Continuous Fourier/Windowed Fourier: Recover $f \in \mathcal{B} \subset \{ f \in S'(\mathbb{R})| \hat{f} \in L^1_{loc}(\mathbb{R}) \}$ from $|\hat{f}|$ or $|V_g f|$ (for some known window $g$). Only possible if $\mathcal{B}$ is sufficiently restrictive - for example if $\hat{f}$ is taken to have compact support or is supported in the half line \cite{KST95, Jam14, AW21, GL22}.

- Discrete Fourier/Windowed Fourier: Recover $f = (f[0], \ldots, f[n-1]) \in \mathbb{C}^n$ from the (typically squared) magnitudes of its DFT coefficients $y[k] = |\sum_{j=0}^{n} y[j] e^{2\pi i kj/n}|^2$ \cite{Fie82, Hay82, IPSV20, IMP19, PS19}.

- Separable Hilbert space: Take $H$ a separable complex Hilbert space. Recover $z \in H$ from $(|\langle z, f_k \rangle|)_{k \in I}$ where $(f_k)_{k \in I} \subset H$ is a frame for $H$ \cite{CCD16}.

- Finite Hilbert space: Recover $z \in H = \mathbb{C}^n$ from $(|\langle z, f_k \rangle|)_{k=1}^{m}$ where $(f_k)_{k=1}^{m}$ is a frame for $\mathbb{C}^n$ [...].

- Phase Retrieval with generalized frames: Recover $z \in H = \mathbb{C}^n$ from $\langle z, A_j z \rangle$ where $(A_j)_{j=1}^{m}$ is a generalized frame of Hermitian matrices (termed measurement matrices). Note that $A_j = f_j f_j^*$ gives the finite Hilbert space case. \cite{WX19}

In all such cases recovery is only ever possible up to an overall phase - that is to say modulo the action of $U(1)$.
Applications

- Inverse Problem in Potential Scattering - Determine potential / surface structure from (typically x-ray or neutron) scattering matrix. [KST95]
- Thin film optics - Inferring dielectric permittivity $\epsilon(z)$ of medium from the frequency dependence of the ratio $R(k)$ of the strength of transmitted and reflected tangential components. [KST95]
- Coherent Diffraction Imaging - infer shape of object in imaging plane from the diffraction pattern it produces under a coherent beam. [HTSL20]
- X-ray crystallography - infer electron density function $\rho(r) = \sum_{i=1}^{N} r_i \delta(r - r_i)$ of a single crystal cell from the measured diffraction pattern. [KH91]
- Speech recognition - the human ear is quite reliably “phase deaf,” determining what has been said only from the magnitude spectrum of a signal. [DJPH93]
- Pure state quantum tomography - inferring the state of a quantum system (represented by a vector in a Hilbert space) from potentially noisy measurements. [BBCE09][KW15]
Motivating Application: Mixed Quantum Tomography

A mixed state quantum system is modeled as a statistical ensemble over pure quantum states living in a Hilbert space $H$. The standard example is unpolarized light. In such cases, all of the measurable information in the system is contained in a density matrix:

$$\rho = \sum_{j \in \mathcal{I}} p_j \psi_j \psi_j^*$$

- $p_j$ - ensemble probability of being in pure state $\psi_j$: $\sum_{i \in \mathcal{I}} p_j = 1$.
- $\psi_j \in H$ - a pure state: Given an observable (Hermitian matrix) $A$ with eigenpair $(\nu, \lambda)$ we have $\Pr_{\psi_j}[A \text{ takes value } \lambda] = |\langle \nu, \psi_j \rangle|^2$.

If we take $H = \mathbb{C}^n$ and $|\mathcal{I}| = r$ then $\rho$ is a positive semi-definite matrix of rank at most $r$ and having unit trace, we write $\rho \in S^{r,0}(\mathbb{C}^n) \cap \{x \in \text{Sym}(\mathbb{C}^n) | \text{tr}\{x\} = 1\}$, where $S^{r,0}(\mathbb{C}^n)$ denotes the set of PSD matrices of rank at most $r$. The goal of quantum tomography is to infer $\rho$ from measurements $(\text{tr}\{\rho A_j\})_{j \in [m]}$ given by a collection of observables $(A_j)_{j=1}^m$. 
Motivating Application: Mixed Quantum Tomography

The expectation of an observable $A_j$ in mixed state $\rho$ is

$$
\mathbb{E}_\rho[A_j] = \sum_{k=1}^{r} p_k \langle \psi_k, A_j \psi_k \rangle = \sum_{k=1}^{r} p_k \text{tr}\{\psi_k \psi_k^* A_j\} = \text{tr}\{\rho A_j\} = \langle \rho, A_j \rangle
$$

By repeatedly measuring our observables and allowing the system to “relax” we may obtain these expectations to within a small error. Since $\rho \in S^{r,0}(\mathbb{C}^n)$ we may write via Cholesky factorization for some $z \in \mathbb{C}^{n \times r}$

$$
\rho = zz^*
$$

Note $\rho$ is unchanged by $z \mapsto zU$ for $U \in U(r)$, so the problem becomes to stably recover $z$ modulo $U(r)$ (a “unitary phase”) from $(\langle zz^*, A_j \rangle)_{j=1}^m$. In particular we would like the following map to be injective (and indeed lower Lipschitz):

$$
\beta : \mathbb{C}^{n \times r} / U(r) \to \mathbb{R}^m
$$

$$
\beta(z) = (\langle zz^*, A_j \rangle)_{j=1}^m
$$

A generalized frame $(A_j)_{j=1}^m$ for which $\beta$ is injective is called $U(r)$ phase retrievable.
**U(r) phase retrievability**

A generalized frame $\mathcal{A} = (A_j)_{j=1}^m$ for which $\beta$ is injective is called *U(r)* phase retrievable.

- As for $U(1)$, *U(r)* phase retrievability is a stronger condition than being a generalized frame for $\mathbb{C}^{n \times r}$.
- If $\mathcal{A}$ is a frame for $\text{Sym}(\mathbb{C}^n)$ itself then it is automatically *U(r)* phase retrievable.
- If $\mathcal{A}$ is *U(r)* phase retrievable then it is *U(k)* phase retrievable for any $1 \leq k \leq r$, in particular it is phase retrievable.

Thus the concept of being *U(r)* phase retrievable is an intermediate between being phase retrievable for $\mathbb{C}^n$ and being a frame for $\text{Sym}(\mathbb{C}^n)$. Another way to think about *U(r)* phase retrieval is as low rank positive semi-definite matrix recovery [Xu18].

In analogy with the pure state case in which one is also interested in the stable recovery properties of the non-linear measurement map $\alpha_j(x) = |\langle x, f_j \rangle|$ we define

$$\alpha : \mathbb{C}^{n \times r}/U(r) \to \mathbb{R}^m$$

$$\alpha(z) = (\langle zz^*, A_j \rangle^{1/2})_{j=1}^m$$
The problem

\[ \beta, \alpha : \mathbb{C}^{n \times r} / U(r) \to \mathbb{R}^m \]
\[ \beta(z) = (\langle zz^*, A_j \rangle)^m_{j=1} \]
\[ \alpha(z) = (\langle zz^*, A_j \rangle^{\frac{1}{2}})^m_{j=1} \]

The problem is then to

- Identify appropriate distances on \( \mathbb{C}^{n \times r} / U(r) \) to use for stability analysis of \( \alpha \) and \( \beta \).
- Find out whether \( \beta \) (resp. \( \alpha \)) is lower Lipschitz on its range whenever \( (A_j)^m_{j=1} \) is \( U(r) \) phase retrievable.
- If so, provide a means of computing the lower Lipschitz constant for \( \beta \) (resp. \( \alpha \)).
- Give if and only if criteria for a given frame of observables to be phase retrievable.
Metric Space Structures

We define the equivalence relation $\sim$ on $\mathbb{C}^{n \times r}$ via

$$x \sim y \iff \exists U \in U(r) | x = yU$$

and denote by $[x]$ the equivalence class of $x \in \mathbb{C}^{n \times r}$, and by $\mathbb{C}^{n \times r}/U(r)$ the set of equivalence classes $\{[x] | x \in \mathbb{C}^{n \times r}\}$. We define $D, d, \delta : \mathbb{C}^{n \times r} \times \mathbb{C}^{n \times r} \to \mathbb{R}$:

$$D(x, y) = \min_{U \in U(r)} ||x - yU||_2 = \sqrt{||x||^2_2 + ||y||^2_2 - 2||x^*y||_1}$$

$$d(x, y) = \min_{U \in U(r)} ||x - yU||_2||x + yU||_2 = \sqrt{(||x||^2_2 + ||y||^2_2)^2 - 4||x^*y||^2_1}$$

$$\delta(x, y) = ||xx^* - yy^*||_2$$

- $D$ is known as the Bures-Wasserstein distance, or the ”natural” distance. Note for $\lambda \in \mathbb{C}$, $D(\lambda x, \lambda y) = |\lambda|D(x, y)$, so $D$ is appropriate for analyzing the $\alpha$ map.
- $d$ scales like $d(\lambda x, \lambda y) = |\lambda|^2d(x, y)$ and is appropriate for analyzing $\beta$ and so is $\delta$, the matrix norm induced distance.
- $d, D, \delta$ are not Lipschitz equivalent but they do generate the same topology on $\mathbb{C}^{n \times r}/U(r)$. 

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Embeddings of Low-Rank Symmetric Matrices  
September 12, 2022
The new (and mysterious) distance $d$

It is easy to show that $D, d(x, y) = \min_{U \in U(r)} \|x - yU\|_2$ is a (semi)distance. For $d, d(x, y) = \min_{U \in U(r)} \|x - yU\|_2 \|x + yU\|_2$, it is easy to show positivity and symmetry. The tricky part is the triangle inequality. For $r = 1$ real-case see [EM14, BCMN14, Sal19] (the last paper analyzes the complex-case as well).

However it has never been explicitly mentioned it is a metric. In the real-case the triangle inequality reduces to a statement about the analytic geometry of parallepipeds in $\mathbb{R}^3$, namely that the sum of the products of face diagonals on any two sides sharing a vertex exceeds the product of the third side sharing the vertex.

On the other hand, in the real-case, for $x, y \in \mathbb{R}^n$, 
\[ \|x - y\|_2 \|x + y\|_2 = \|xx^T - yy^T\|_1; \] in the complex case, for $x, y \in \mathbb{C}^n$,
\[ \min_{\varphi} \|x - e^{i\varphi}y\|_2 \|x + e^{i\varphi}y\|_2 = \|xx^* - yy^*\|_1. \]

This identity implies the triangle inequality in the case $r = 1$. For $r > 1$, the identity $\min_{U \in U(r)} \|x - yU\|_2 \|x + yU\|_2 = \|xx^* - yy^*\|_1$ fails in general for $x, y \in \mathbb{C}^{n \times r}$! However both sides define (inequivalent) distances!
Lipschitz Embeddings

We would like to embed the metric spaces \((\mathbb{C}^{n\times r}/U(r), D \text{ or } d)\) into \((\text{Sym}(\mathbb{C}^n), \| \cdot \|_2)\) in a (bi)Lipschitz way. Defining invertible maps (modulo \(\sim\))

\[
\theta, \pi, \psi : \mathbb{C}^{n\times r} \rightarrow S^{r,0}(\mathbb{C}^n)
\]

\[
\theta(x) = (xx^*)^{\frac{1}{2}} \quad \pi(x) = xx^* \quad \psi(x) = \| x \|_2 (xx^*)^{\frac{1}{2}}.
\]

**Theorem ([BD22])**

(i) \(\theta : (\mathbb{C}^{n\times r}/U(r), D) \rightarrow (S^{r,0}(\mathbb{C}^n), \| \cdot \|_2)\) is a bi-Lipschitz map:

\[
\frac{1}{\sqrt{2}} \| \theta(x) - \theta(y) \|_2 \leq D(x, y) \leq \| \theta(x) - \theta(y) \|_2 \quad \forall x, y \in \mathbb{C}^{n\times r}/U(r)
\]

(ii) \(\pi, \psi : (\mathbb{C}^{n\times r}/U(r), d) \rightarrow (S^{r,0}(\mathbb{C}^n), \| \cdot \|_1)\) are upper and lower Lipschitz:

\[
\| \pi(x) - \pi(y) \|_1 \leq d(x, y) \leq 2\| \psi(x) - \psi(y) \|_2 \quad \forall x, y \in \mathbb{C}^{n\times r}/U(r)
\]

(iii) For \(r = 1\) we have \(d(x, y) = \| \pi(x) - \pi(y) \|_1\)

(iv) For \(r > 1\), there is no constant \(C\) satisfying \(d(x, y) \leq C\| \pi(x) - \pi(y) \|_2\) for all \(x, y \in \mathbb{C}^{n\times r}/U(r)\) (hence the use of the alternate embedding \(\psi\)).
Lipschitz Constants

With these embeddings in mind we define lower Lipschitz bounds:

\[
a_0 = \inf_{x, y \in \mathbb{C}^{n \times r}, [x] \neq [y]} \frac{||\beta(x) - \beta(y)||_2^2}{||\pi(x) - \pi(y)||_2^2} = \inf_{x, y \in \mathbb{C}^{n \times r}, [x] \neq [y]} \frac{\sum_{j=1}^{m} (\langle xx^*, A_j \rangle_{\mathbb{R}} - \langle yy^*, A_j \rangle_{\mathbb{R}})^2}{||xx^* - yy^*||_2^2}
\]

\[
A_0 = \inf_{x, y \in \mathbb{C}^{n \times r}, [x] \neq [y]} \frac{||\alpha(x) - \alpha(y)||_2^2}{||\theta(x) - \theta(y)||_2^2} = \inf_{x, y \in \mathbb{C}^{n \times r}, [x] \neq [y]} \frac{\sum_{j=1}^{m} (\langle xx^*, A_j \rangle_{\mathbb{R}}^{\frac{1}{2}} - \langle yy^*, A_j \rangle_{\mathbb{R}}^{\frac{1}{2}})^2}{||(xx^*)^{\frac{1}{2}} - (yy^*)^{\frac{1}{2}}||_2^2}
\]

Assume \((A_j)_{j \in [m]}\) is \(U(r)\) phase retrievable for \(\mathbb{C}^{n \times r}\). Then we showed that:

1. The bound \(a_0 > 0\) and provided a means of computing it for any \(r \geq 1\); hence \(\beta : (\mathbb{C}^{n \times r} / U(r), \delta) \to (\text{Sym}(\mathbb{C}^n), || \cdot ||_2)\) is bi-Lipschitz, where \(\delta(x, y) = ||xx^* - yy^*||_2\).

2. However \(A_0 = 0\) for \(r > 1\)! Thus the \(\alpha\) map is not Lipschitz invertible for \(r > 1\) with respect to any of the three metrics \(d, D\) or \(\delta\), nor matrix norm induced distances via \(\theta, \pi, \psi\).
Geometry of $S^{r,0}(\mathbb{C}^n)$

To compute $a_0$ and $A_0$ we need to linearize $\pi$ and $\theta$ and find their actions on the “tangent bundle” of $S^{r,0}(\mathbb{C}^n)$. $S^{r,0}(\mathbb{C}^n)$ is only a semi-algebraic variety, however, so we need to understand its singular structure and whether the linearized problem suffices when “boundary manifolds” are encountered. We showed that $S^{r,0}(\mathbb{C}^n)$ has a Whitney stratification over the smooth Riemannian manifolds $\tilde{S}^{i,0}(\mathbb{C}^n)$ (PSD matrices of rank exactly $i$) for $i = 0, \ldots, r$ having real dimension $2ni - i^2$.

Definition (a-regular, b-regular, from [Kal00])

Let $V_i, V_j$ be disjoint real manifolds embedded in $\mathbb{R}^d$ such that $\dim V_j > \dim V_i$ and $V_i \cap \overline{V_j}$ non-empty. Let $x \in V_i \cap \overline{V_j}$. Then a triple $(V_j, V_i, x)$ is called $a$– (resp. $b$–) regular if

(a) If a sequence $(y_n)_{n \geq 1} \subset V_j$ converges to $x$ in $\mathbb{R}^d$ and $T_{y_n}(V_j)$ converges in the Grassmannian $\text{Gr}_{\dim V_j}(\mathbb{R}^d)$ to a subspace $\tau_x$ of $\mathbb{R}^d$ then $T_x(V_i) \subset \tau_x$.

(b) If sequences $(y_n)_{n \geq 1} \subset V_j$ and $(x_n)_{n \geq 1} \subset V_i$ converge to $x$ in $\mathbb{R}^d$, the unit vector $(x_n - y_n)/\|x_n - y_n\|_2$ converges to a vector $v \in \mathbb{R}^d$, and $T_{y_n}(V_j)$ converges in the Grassmannian $\text{Gr}_{\dim V_j}(\mathbb{R}^d)$ to a subspace $\tau_x$ of $\mathbb{R}^d$ then $v \in \tau_x$. 

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Geometry of $S^{r,0}(\mathbb{C}^n)$

Definition (Whitney stratification, from [Kal00])

Let $V$ be a real semi-algebraic variety. A disjoint decomposition

$$V = \bigsqcup_{i \in I} V_i, \quad V_i \cap V_j = \emptyset \text{ for } i \neq j$$

into smooth manifolds $\{V_i\}_{i \in I}$, termed strata, is a Whitney stratification if

- Each point has a neighborhood intersecting only finitely many strata
- The boundary sets $\overline{V_j} \setminus V_j$ of each stratum $V_j$ are unions of other strata.
- Every triple $(V_j, V_i, x)$ such that $x \in V_i \subset \overline{V_j}$ is $a$-regular and $b$-regular.

The point is that there is a compatibility between the stratifying manifolds - if you are in the tangent space of lower dimensional strata you are in a limiting sense also in the tangent space of higher strata. This gives the semi-algebraic variety more structure, and as we’ll see in this case enables us to find what almost looks like a Riemannian geometry on the whole of $S^{r,0}(\mathbb{C}^n)$ (even though it isn’t a manifold).
Geometry of $S^{r,0}(\mathbb{C}^n)$

We will stratify $S^{r,0}(\mathbb{C}^n)$ as $\bigsqcup_{i=0}^{r} S^{i,0}(\mathbb{C}^n)$, where $S^{i,0}(\mathbb{C}^n)$ is the set of positive semi-definite matrices of rank exactly $i$.

Theorem ([BD22]; see also [BJL19])

Let $D$ be the Bures-Wasserstein (a.k.a., the natural) distance. Then

(i) $S^{p,q}(\mathbb{C}^n)$ is a real analytic manifold with
$$\dim_{\mathbb{R}}(S^{p,q}(\mathbb{C}^n)) = 2n(p + q) − (p + q)^2.$$

(ii) $\pi : \mathbb{C}_*^{n \times r} \to S^{r,0}(\mathbb{C}^n)$ can be made into a Riemannian submersion by choosing the unique Riemannian metric on $S^{r,0}(\mathbb{C}^n)$, for $Z_1, Z_2 \in T_X(S^{r,0}(\mathbb{C}^n))$:
$$h_X'(Z_1, Z_2) = \text{tr}\{Z_2^\Vert \int_0^\infty e^{-uX} Z_1^\Vert e^{-uX} du\} + \Re \text{tr}\{Z_1^\bot * Z_2^\bot X^\dagger\}$$

where $Z_i^\Vert = \mathbb{P} \text{Ran}(X) Z_i \mathbb{P} \text{Ran}(X), Z_i^\bot = \mathbb{P} \text{Ran}(X) \bot Z_i \mathbb{P} \text{Ran}(X)$ and the real Hilbert-Schmidt inner product as metric on $T_{\pi^{-1}(X)}(\mathbb{C}_*^{n \times r})$.

(iii) $(S^{r,0}(\mathbb{C}^n), h^r)$ is a Riemannian manifold with $D \circ (\pi^{-1} \times \pi^{-1})$ as its geodesic distance.

(iv) $S^{r,0}(\mathbb{C}^n)$ admits as a Whitney stratification $(S^{i,0})_{i=0}^r$. 

Radu Balan (UMD)
Embeddings of Low-Rank Symmetric Matrices
September 12, 2022 16 / 30
Geometry of $S^{r,0}(\mathbb{C}^n)$

We will stratify $S^{r,0}(\mathbb{C}^n)$ as $\bigsqcup_{i=0}^r \hat{S}^{i,0}(\mathbb{C}^n)$, where $\hat{S}^{i,0}(\mathbb{C}^n)$ is the set of positive semi-definite matrices of rank exactly $i$.

**Theorem ([BD22])**

The geometry associated to $h$ is compatible with the Whitney stratification in the following sense: If $(A_i)_{i \geq 1}, (B_i)_{i \geq 1} \subset \hat{S}^{p,0}$ have limits $A$ and $B$ respectively in $\hat{S}^{q,0}$ for $q < p$ and if $\gamma_i : [0, 1] \to \hat{S}^{p,0}$ are geodesics in $\hat{S}^{p,0}$ connecting $A_i$ to $B_i$ chosen in such a way that the limiting curve $\delta : [0, 1] \to \hat{S}^{p,0}$ given by

$$\delta(t) = \lim_{i \to \infty} \gamma_i(t)$$

exists, then the image of $\delta$ lies in $\hat{S}^{q,0}$ and is a geodesic curve in $\hat{S}^{q,0}$ connecting $A$ to $B$.

Another way to look at this is if $0 \leq q \leq p \leq r$ and $X = xx^* \in \hat{S}^{p,0}$, $Y = yy^* \in \hat{S}^{q,0}$ and $\gamma_{X_1,X_2} : [0, 1] \to \hat{S}^{p,0}$ is the geodesic connecting $X_1$ to $X_2$ then

$$D(x, y)^2 = \min_{U \in U(r)} \|x - yU\|_2^2 = \lim_{Z \to Y} \int_0^1 h^p_{\gamma_X, Z(t)}(\gamma'_X, Z(t), \gamma'_X, Z(t)) dt$$
Geometry of $S^{r,0}(\mathbb{C}^n)$ via $\mathbb{C}^{n \times r}$

We may view $S^{r,0}(\mathbb{C}^n)$ as the image under $\pi : x \mapsto \pi(x) = xx^*$ of $\mathbb{C}^{n \times r}$, and each stratifying manifold $\check{S}^{i,0}(\mathbb{C}^n)$ as the image of $[\mathbb{C}_{*}^{n \times i} | 0^{n \times (r-i)}]$ (the $*$ means full rank). This is surjective, but not injective owing to the ambiguity $U(r)$. We can compute the differential $D\pi(z)(w) = zw^* + wz^*$, its kernel $V_{\pi,x}(\mathbb{C}_{*}^{n \times r})$ (the vertical space), and the orthogonal complement of its kernel $H_{\pi,x}(\mathbb{C}_{*}^{n \times r})$ (the horizontal space) which maps one to one onto the tangent space of $\check{S}^{i,0}(\mathbb{C}^n)$. 

\[ Z(\det x^*x) \]
\[ T_x(\mathbb{C}^{n \times r}) = \mathbb{C}^{n \times r} \]
\[ [x] \]
\[ V_{\pi,x} \]
\[ H_{\pi,x} \]
\[ \pi(x) \]
\[ \check{S}^{r,0}(\mathbb{C}^n) \]
\[ T_{\pi(x)}(\check{S}^{r,0}) \]
\[ 0 = D\pi(x)(V_{\pi,x}) \]
\[ D\pi(x) \]
\[ D\pi(x)^\dagger \]
The spaces $V_{\pi,x}(\mathbb{C}^{n\times r})$, $H_{\pi,x}(\mathbb{C}^{n\times r})$ and $T_{\pi(x)}(\hat{S}^{r,0}(\mathbb{C}^n))$ may be computed as

**Lemma ([BD22])**

Let $\pi : \mathbb{C}^{n\times r} \to \hat{S}^{r,0}(\mathbb{C}^n)$ be as before and let $V_{\pi,x}(\mathbb{C}^{n\times r})$ and $H_{\pi,x}(\mathbb{C}^{n\times r})$ denote the vertical and horizontal spaces of the manifold $\mathbb{C}^{n\times r}$ at $x$ with respect to the embedding $\pi$. Let $T_{\pi(x)}(\hat{S}^{r,0}(\mathbb{C}^n))$ denote the tangent space of $\hat{S}^{r,0}(\mathbb{C}^n)$ at $\pi(x)$. Then

\[
V_{\pi,x}(\mathbb{C}^{n\times r}) = \{ xK | K \in \mathbb{C}^{r\times r}, K^* = -K \}
\]
\[
H_{\pi,x}(\mathbb{C}^{n\times r}) = \{ Hx + X | H \in \mathbb{C}^{n\times n}, H^* = H = \mathbb{P} \text{Ran}(x) H, X \in \mathbb{C}^{n\times r}, \mathbb{P} \text{Ran}(x) X = 0 \}
\]
\[
T_{\pi(x)}(\hat{S}^{r,0}(\mathbb{C}^n)) = \{ W \in \text{Sym}(\mathbb{C}^n) | \mathbb{P} \text{Ran}(x) \perp W \mathbb{P} \text{Ran}(x) \perp = 0 \}
\]
\[
= D_{\pi(x)}(H_{\pi,x}(\mathbb{C}^{n\times r}))
\]

Note that $\dim_{\mathbb{R}}(V_{\pi,x}(\mathbb{C}^{n\times r})) = r^2$ and $\dim_{\mathbb{R}}(T_{\pi(x)}(\hat{S}^{r,0}(\mathbb{C}^n))) = \dim_{\mathbb{R}}(H_{\pi,x}(\mathbb{C}^{n\times r})) = 2nr - r^2$. 

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The tangent space Lipschitz bounds

In our effort to obtain or at least control the global Lipschitz constant $a_0$ we define the following local lower Lipschitz constants:

$$a_1(z) = \lim_{R \to 0} \inf_{x \in \mathbb{C}^n \times r, \ ||\pi(x) - \pi(z)||_2 < R} \frac{||\beta(x) - \beta(z)||_2^2}{||\pi(x) - \pi(z)||_2^2}$$

$$a_2(z) = \lim_{R \to 0} \inf_{x, y \in \mathbb{C}^n \times r, \ ||\pi(x) - \pi(z)||_2 < R, \ ||\pi(y) - \pi(z)||_2 < R} \frac{(||\beta(x) - \beta(y)||_2^2)}{||\pi(x) - \pi(y)||_2^2}$$

As well as the following geometric constant

$$a(z) := \min_{W \in T_{\pi(\hat{z})}(S^k, \mathbb{C}^n)} \sum_{j=1}^{m} \left| \langle W, A_j \rangle_{\mathbb{R}} \right|^2$$

Where here $\hat{z} \in \mathbb{C}^{n \times k}_*$ is such that $z = [\hat{z}|0]U$ for some $U \in U(r)$ ($\hat{z} = z$ if $z$ has rank $r$, and moreover the tangent space doesn’t depend on the choice of $\hat{z}$).
The tangent space Lipschitz bounds

Given $z \in \mathbb{C}^{n \times r}$ having rank $k > 0$ define $Q_z \in \mathbb{R}^{(2nk-k^2) \times (2nk-k^2)}$ as follows. Let $U_1 \in \mathbb{C}^{n \times k}$ be a matrix whose columns are left singular vectors of $z$ corresponding to non-zero singular values of $z$, so that $U_1 U_1^* = \mathbb{P} \text{ Ran}(z)$. Let $U_2 \in \mathbb{C}^{n \times (n-k)}$ be a matrix whose columns are left singular vectors of $z$ corresponding to the zero singular values of $z$, so that $U_2 U_2^* = \mathbb{P} \text{ Ran}_z \perp$. Then

$$Q_z := Q[u_1 | u_2] = \sum_{j=1}^{m} \begin{bmatrix} \tau(U_1^* A_j U_1) \\ \mu(U_2^* A_j U_1) \end{bmatrix}^T$$

where the isometric isomorphisms $\tau$ and $\mu$ are given by

$$\tau : \text{Sym}(\mathbb{C}^k) \to \mathbb{R}^{k^2} \quad \mu : \mathbb{C}^{p \times q} \to \mathbb{R}^{2pq}$$

$$\tau(X) = \begin{bmatrix} D(X) \\ \sqrt{2} T(\Re X) \\ \sqrt{2} T(\Im X) \end{bmatrix} \quad \mu(X) = \text{vec}(\begin{bmatrix} \Re X \\ \Im X \end{bmatrix})$$

where if $A \in \text{Sym}(\mathbb{R}^n)$ $D(A)$ is the vectorization of its diagonal and and $T(A)$ is the vectorization of its upper triangular part.
The tangent space Lipschitz bounds

Theorem ([BD22])

- \((A_j)_{j=1}^m\) is \(U(r)\) phase retrievable if and only if \(a_0 > 0\).
- The global lower bound \(a_0\) is given as \(a_0 = \inf_{z \in \mathbb{C}^{n \times r} \setminus \{0\}} a(z)\).
- The local lower bounds \(a_1(z)\) and \(a_2(z)\) are squeezed between \(a_0 \leq a_2(z) \leq a_1(z) \leq a(z)\) so that in particular \(a_0 = \inf_{z \in \mathbb{C}^{n \times r} \setminus \{0\}} a_i(z)\).
- The infimization problem in \(a(z)\) may be reformulated as an eigenvalue problem. Let \(Q_z\) be as above. Then

\[
a(z) = \lambda_{2nk-k^2}(Q_z)
\]

Corollary

Fix any \(U_2 \in \mathbb{C}^{n \times n-r}\) with orthonormal columns. We may compute \(a_0\) as

\[
a_0 = \min_{U_1 \in \mathbb{C}^{n \times r}} \lambda_{2nr-r^2}(Q[U_1|U_2])
\]

\[
U = [U_1|U_2] \in U(n)
\]
The horizontal space Lipschitz bounds

An alternate method of controlling $a_0$ is to use the (new) distance $d$. We define for $z \in \mathbb{C}^{n \times r}$ with rank $k$ the local lower Lipschitz constants

\[
\hat{a}_1(z) = \lim_{R \to 0} \inf_{x \in \mathbb{C}^{n \times r}} \frac{||\beta(x) - \beta(z)||_2^2}{d(x, z)^2} \quad \text{subject to} \quad \begin{cases} d(x, z) < R \\ \text{rank}(x) \leq k \end{cases}
\]

\[
\hat{a}_2(z) = \lim_{R \to 0} \inf_{x, y \in \mathbb{C}^{n \times r}} \frac{||\beta(x) - \beta(y)||_2^2}{d(x, y)^2} \quad \text{subject to} \quad \begin{cases} d(x, z) < R \\ d(y, z) < R \\ \text{rank}(x) \leq k \\ \text{rank}(y) \leq k \end{cases}
\]

Unfortunately the rank constraints are necessary here - without them the constants would be zero. We also define the geometric constant

\[
\hat{a}(z) = \min_{w \in H, \hat{z} \in \mathbb{C}^{n \times k}} \sum_{j=1}^{m} |\langle D_{\pi}(\hat{z})(w), A_j \rangle_R|^2
\]
Given \( z \in \mathbb{C}^{n \times r} \) having rank \( k > 0 \) define \( \hat{Q}_z \in \mathbb{R}^{2nk \times 2nk} \) as follows. Let \( F_j = \mathbb{I}_{k \times k} \otimes j(A_j) \in \mathbb{R}^{2nk \times 2nk} \) where

\[
j : \mathbb{C}^{m \times n} \to \mathbb{R}^{2m \times 2n}
\]

\[
j(X) = \begin{bmatrix} \Re X & -\Im X \\ \Im X & \Re X \end{bmatrix}
\]

is an injective homomorphism. Then

\[
\hat{Q}_z := 4 \sum_{j=1}^{m} F_j \mu(\hat{z}) \mu(\hat{z})^T F_j
\]
The horizontal space Lipschitz bounds

Theorem ([BD22])

- For $r = 1$ $\hat{a}(z)$ differs from $a(z)$ by a constant factor hence $\inf_{\hat{\mathbf{z}} \in \mathbb{C}^{n \times r} \setminus \{0\}} \hat{a}(z) > 0$. For $r > 1$ this infimum is zero and there is no non-trivial global lower bound $\hat{a}_0$ analogous to $a_0$ for the natural metric $d$.

- The local lower bounds with respect to the alternate metric $d$ satisfy

$$\hat{a}_1(z) = \hat{a}_2(z) = \frac{1}{4||z||_2^2} \hat{a}(z)$$

- The infimization problem in $\hat{a}(z)$ may be reformulated as an eigenvalue problem. Let $\hat{Q}_z$ be as above. Then $\hat{a}(z)$ is directly computable as

$$\hat{a}(z) = \lambda_{2nk-k^2}(\hat{Q}_z)$$

- We have the following local inequality relating $a(z)$ and $\hat{a}(z)$.

$$\frac{1}{4||z||_2^2} \hat{a}(z) \leq a(z) \leq \frac{1}{2\sigma_k(z)^2} \hat{a}(z)$$
The horizontal space Lipschitz bounds

**Theorem ([BD22])**

While the fact that $\hat{a}_0 = 0$ makes clear that $a_0$ cannot be upper bounded by $\inf_{z \in \mathbb{C}^{n \times r} \setminus \{0\}} \hat{a}(z)$, we can achieve a similar end by constraining $z$ to have orthonormal columns. Namely

$$\frac{1}{4} \inf_{z \in \mathbb{C}^{n \times r} \atop z^* z = I_{r \times r}} \hat{a}(z) \leq a_0 \leq \frac{1}{2} \inf_{z \in \mathbb{C}^{n \times r} \atop z^* z = I_{r \times r}} \hat{a}(z)$$
Phase retrievability criteria

The last two theorems give criteria for a frame to be $U(r)$ phase retrievable.

**Theorem ([BD22])**

Let $\{A_j\}_{j=1}^m$ be a frame for $\mathbb{C}^{n \times r}$. Then the following are equivalent:

(i) $\{A_j\}_{j=1}^m$ is $U(r)$ phase retrievable.

(ii) For all $U_1 \in \mathbb{C}^{n \times r}$, $U_2 \in \mathbb{C}^{n \times (n-r)}$ such that $[U_1|U_2] \in U(n)$ the matrix

$$Q_{[U_1|U_2]} = \sum_{j=1}^m \begin{bmatrix} \tau(U_1^* A_j U_1) \\ \mu(U_2^* A_j U_1) \end{bmatrix}^T \begin{bmatrix} \tau(U_1^* A_j U_1) \\ \mu(U_2^* A_j U_1) \end{bmatrix}$$

is invertible.

(iii) For all $z \in \mathbb{C}^{n \times r}$ such that $z$ has orthonormal columns, the matrix

$$\hat{Q}_z = 4 \sum_{j=1}^m (I_{k \times k} \otimes j(A_j))\mu(z)\mu(z)^T(I_{k \times k} \otimes j(A_j))$$

has as its null space the $r$-dimensional $V_z = \{\mu(u) | u \in V_{\pi}, z{:} \mathbb{C}^{n \times r}\}$.
Phase retrievability criteria

Theorem ([BD22])

(Continued)

1. For all \( U_1 \in \mathbb{C}^{n \times r} \), \( U_2 \in \mathbb{C}^{n \times (n-r)} \) such that \([U_1|U_2] \in U(n)\), \( H \in \text{Sym}(\mathbb{C}^r)\), \( B \in \mathbb{C}^{(n-r) \times r} \) there exist \( c_1, \ldots, c_m \in \mathbb{R} \) such that

\[
U_1^* \left( \sum_{j=1}^{m} c_j A_j \right) U_1 = H \quad (1a)
\]

\[
U_2^* \left( \sum_{j=1}^{m} c_j A_j \right) U_1 = B \quad (1b)
\]

2. For all \( U_1 \in \mathbb{C}^{n \times r} \) with orthonormal columns

\[
\text{span}_{\mathbb{R}} \{ A_j U_1 \}_{j=1}^{m} = \{ U_1 K | K \in \mathbb{C}^{r \times r}, K^* = -K \}^\perp
\]

The second criterion is a generalization of the result in [BCMN14] which says that a frame \( (\phi_j)_{i=1}^{m} \) for \( \mathbb{C}^n \) is phase retrievable iff
Other results in the paper

- We give a purely topological proof that \((A_j)_{j=1}^m\) phase retrievable implies \(a_0 > 0\) (we do this before computing \(a_0\)).
- We prove using continuity of eigenvalues with respect to matrix entries that \(A_0 = 0\) for \(r > 1\).
- We compute local lower Lipschitz constants for \(\alpha\).
- We compute Lipschitz upper bounds \(b_0\) and \(B_0\).
- We show that our results reduce to those in [BZ16] for the case \(r = 1\).
- We verify the lower Lipschitz theorems numerically.
### Summary of differences between mixed and pure state case

<table>
<thead>
<tr>
<th>$r = 1$ (pure state case)</th>
<th>$r &gt; 1$ (mixed state case)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase ambiguity is scalar $e^{i\theta}$</td>
<td>Phase ambiguity is in $U(r)$</td>
</tr>
<tr>
<td>$(z_i)_{i \geq 1} \subset \mathbb{C}^n/U(1)$ with $|z_i|_2 = 1$ cannot approach zero</td>
<td>$(z_i)_{i \geq 1} \subset \mathbb{C}^{n \times r}/U(r)$ with $|z_i|_2 = 1$ can come $\epsilon$ close to dropping rank</td>
</tr>
<tr>
<td>$d(x, y) = |xx^* - yy^*|_1$</td>
<td>$\beta$ is bi-Lipschitz with $|xx^* - yy^*|_p$</td>
</tr>
<tr>
<td>$\beta$ is bi-Lipschitz wrt. $d$ and $\delta(x, y) = |xx^* - yy^*|_2$</td>
<td>$\beta$ is bi-Lipschitz wrt. $|xx^* - yy^*|_2$ Only locally lower Lipschitz wrt. $D$</td>
</tr>
<tr>
<td>$A_0 &gt; 0$, $\alpha$ is bi-Lipschitz wrt. $D$ and $|(xx^<em>)^{\frac{1}{2}} - (yy^</em>)^{\frac{1}{2}}|_2$</td>
<td>$A_0 = 0$, $\alpha$ is locally lower Lipschitz wrt. $D$ and $|(xx^<em>)^{\frac{1}{2}} - (yy^</em>)^{\frac{1}{2}}|_2$</td>
</tr>
</tbody>
</table>
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