Optimal $l^1$ factorizations of positive semi-definite matrices

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Works:


Let $\text{Sym}^+(\mathbb{C}^n) = \{ A \in \mathbb{C}^{n \times n} : A^* = A \geq 0 \}$. For $A \in \text{Sym}^+(\mathbb{C}^n)$,

$$\gamma_+(A) := \inf_{A=\sum_{k \geq 1} x_k x_k^*} \sum_k \|x_k\|_1^2$$

The *matrix conjecture*: There is a universal constant $C_0$ such that, for every $n \geq 1$ and $A \in \text{Sym}^+(\mathbb{C}^n)$,

$$\gamma_+(A) \leq C_0 \|A\|_1 := C_0 \sum_{k,l=1}^{n} |A_{k,l}|$$
Motivation
A Feichtinger Problem

At a 2004 Oberwolfach meeting, Hans Feichtinger asked the following question: (Q1) Given a positive semi-definite trace-class operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $Tf(x) = \int K(x, y)f(y)dy$, with $K \in M^1(\mathbb{R}^d \times \mathbb{R}^d)$, and its spectral factorization, $T = \sum_k \langle \cdot, h_k \rangle h_k$, must it be $\sum_k \|h_k\|_{M^1}^2 < \infty$?

A modified version of the question is: (Q2) Given $T$ as before, i.e., $T = T^* \geq 0$, $K \in M^1(\mathbb{R}^d \times \mathbb{R}^d)$, is there a factorization $T = \sum_k \langle \cdot, g_k \rangle g_k$ such that $\sum_k \|g_k\|_{M^1}^2 < \infty$?
Consider an infinite matrix $A = (A_{m,n})_{m,n \geq 0}$ so that
\[
\|A\|_\wedge := \|A\|_1 := \sum_{m,n \geq 0} |A_{m,n}| < \infty.
\]
This implies that $A$ acts on $l^2(\mathbb{N})$ as a trace-class compact operator.
Assume additionally $A = A^* \geq 0$ as a quadratic form.
Let $(e_k)_{k \geq 0}$ denote an orthogonal set of eigenvectors normalized so that
$A = \sum_{k \geq 0} e_k e^*_k$. It is easy to check that $e_k \in l^1(\mathbb{N})$, for each $k$.
Equivalent reformulations of the two problems (Heil, Larson ‘08):
Consider an infinite matrix $A = (A_{m,n})_{m,n \geq 0}$ so that
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Equivalent reformulations of the two problems (Heil, Larson ‘08):
**Q1:** Does it hold $\sum_{k \geq 0} \|e_k\|_1^2 < \infty$?
Problem Reformulation
Matrix Language

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**Q1:** Does it hold \( \sum_{k \geq 0} \|e_k\|_1^2 < \infty \)? Answer: Negative in general! (see [1])
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**Q1:** Does it hold $\sum_{k \geq 0} \|e_k\|_1^2 < \infty$? Answer: Negative in general! (see [1])

**Q2:** Is there a factorization $A = \sum_{k \geq 0} f_k f_k^*$ so that $\sum_{k \geq 0} \|f_k\|_1^2 < \infty$?
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Consider an infinite matrix $A = (A_{m,n})_{m,n \geq 0}$ so that
$$\|A\|_\wedge := \|A\|_1 := \sum_{m,n \geq 0} |A_{m,n}| < \infty.$$ This implies that $A$ acts on $l^2(\mathbb{N})$ as a trace-class compact operator.
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Equivalent reformulations of the two problems (Heil, Larson ‘08):
Q1: Does it hold $\sum_{k \geq 0} \|e_k\|_1^2 < \infty$? Answer: Negative in general! (see [1])
Q2: Is there a factorization $A = \sum_{k \geq 0} f_k f_k^*$ so that $\sum_{k \geq 0} \|f_k\|_1^2 < \infty$?

Using previous equivalence and some functional analysis arguments:

Proposition
If (Q2) is answered affirmatively, then the matrix conjecture must be true.
Recall the setup. Take $A \in \text{Sym}^+(\mathbb{C}^n) := \{ A \in \mathbb{C}^{n \times n} , A^* = A \geq 0 \}$. We are interested in this quantity:

$$\gamma_+(A) := \inf_{A = \sum_{k \geq 1} x_k x_k^*} \sum_k \|x_k\|_1^2$$

Recall definitions of norms:

$$\|A\|_1 = \sum_{k,l=1}^n |A_{k,l}| , \|A\|_{op} = \max_{\|x\|_2 = 1} \|Ax\|_2 = s_{max}(A)$$

The matrix conjecture: There is a universal constant $C_0$ such that, for every $n \geq 1$ and $A \in \text{Sym}^+(\mathbb{C}^n)$,

$$\gamma_+(A) \leq C_0 \|A\|_1$$
The infimum is achieved:

\[
\gamma_+(A) := \inf_{A=\sum_{k \geq 1} x_k x_k^*} \sum_k \|x_k\|_1^2 = \min_{A=\sum_{k=1}^{n^2} x_k x_k^*} \sum_k \|x_k\|_1^2.
\]

Upper bounds:

\[
\gamma_+(A) \leq n \text{trace}(A) \leq n \|A\|_1 = n \sum_{k,j} |A_{k,j}|
\]

\[
\gamma_+(A) \leq n \text{trace}(A) \leq n^2 \|A\|_{op}
\]

Lower bounds:

\[
\|A\|_1 = \min_{A=\sum_{k \geq 1} x_k y_k^*} \sum_k \|x_k\|_1 \|y_k\|_1 \leq \gamma_+(A)
\]

Convexity: for \( A, B \in \text{Sym}^+(\mathbb{C}^n) \) and \( t \geq 0 \),

\[
\gamma_+(A + B) \leq \gamma_+(A) + \gamma_+(B), \quad \gamma_+(tA) = t \gamma_+(A)
\]
Current Status of the Matrix Conjecture [2]

Lower bound is achieved:

1. If $A = xx^*$ is of rank one, then $\gamma_+(A) = \|x\|_1^2 = \|A\|_1$.
2. If $A \geq 0$ is diagonally dominant matrix, then $\gamma_+(A) = \|A\|_1$.

Continuity and Lipschitz:

1. Let $\text{Sym}^{++}(\mathbb{C}^n) = \{A = A^* > 0\}$. Then $\gamma_+|_{\text{Sym}^{++}} : \text{Sym}^{++}(\mathbb{C}^n) \rightarrow \mathbb{R}$ is continuous.
2. If $A, B \in \text{Sym}^{++}(\mathbb{C}^n)$, $\text{trace}(A), \text{trace}(B) \leq 1$ and $A, B \geq \delta I$ then

$$|\gamma_+(A) - \gamma_+(B)| \leq \left(\frac{n}{\delta^2} + n^2\right) \|A - B\|_{op}$$

hence Lipschitz continuous.

Maximum of $\sum_k \|x_k\|_1^2 / \|A\|_1$ over 30 random noise realizations, where $x_k'$s are obtained from the eigendecomposition, or the LDL factorization.
Two New Results

Optimal Factorization from a Measure Theory Perspective

Let $S_1 = \{x \in \mathbb{C}^n, \|x\|_1 = 1\}$ denote the compact unit sphere with respect to the $l^1$ norm, and let $\mathcal{B}(S_1)$ denote the set of Borel measures over $S_1$. For $A \in \text{Sym}(\mathbb{C}^n)^+(\mathbb{C}^n)$ consider the optimization problem:

$$(p^*, \mu^*) = \inf_{\mu \in \mathcal{B}(S_1)} \int_{S_1} xx^* d\mu(x) = A \quad \mu(S_1) \quad (M)$$

Theorem (Optimal Measure)

For any $A \in \text{Sym}^+(\mathbb{C}^n)$ the optimization problem $(M)$ is convex and its global optimum (minimum) is achieved by

$$p^* = \gamma_+(A), \quad \mu^*(x) = \sum_{k=1}^{m} \lambda_k \delta(x - g_k)$$

where $A = \sum_{k=1}^{m} (\sqrt{\lambda_k} g_k)(\sqrt{\lambda_k} g_k)^*$ is an optimal decomposition that achieves

$$\gamma_+(A) = \sum_{k=1}^{m} \lambda_k.$$
Super-resolution and Convex Optimizations

\[
\gamma_+ (A) = \min_{x_1, \ldots, x_m : \ A = \sum_{k=1}^{m} x_k x_k^*} \sum_{k=1}^{m} \|x_k\|_2^2, \ m = n^2 \quad (P)
\]

\[
p^* = \inf_{\mu \in \mathcal{B}(S_1) : \ A = \int_{S_1} xx^* d\mu(x)} \int_{S_1} d\mu(x) \quad (M)
\]

Remarks

1. The optimization problem \((P)\) is non-convex, but finite-dimensional. The optimization problem \((M)\) is convex, but infinite-dimensional.

2. If \(g_1, \ldots, g_m \in S_1\) in the support of \(\mu^*\) are known so that \(\mu^* = \sum_{k=1}^{m} \lambda_k \delta(x - g_k)\), then the optimal \(\lambda_1, \ldots, \lambda_m \geq 0\) are determined by a linear program. More general, \((M)\) is an infinite-dimensional linear program.

3. Finding the support of \(\mu^*\) is an example of a super-resolution problem. One possible approach is to choose a redundant dictionary (frame) that includes the support of \(\mu^*\), and then solve the induced linear program.
Theorem (The Continuity Property)

The map $\gamma_+: (\text{Sym}^+(\mathbb{C}^n), \| \cdot \|) \to \mathbb{R}$ is continuous.

Remarks

1. This statement extends the continuity result from $\text{Sym}^{++}(\mathbb{C}^n) = \{A = A^* > 0\}$ to $\text{Sym}^+(\mathbb{C}^n) = \{A = A^* \geq 0\}$.

2. Proof is based on a (new?) comparison result between non-negative operators.

3. Global Lipschitz is still open.
Thank you for listening!

QUESTIONS?
Proof of the Optimal Measure Result

Recall: we want to show the following problems admit same solution:

\[
\gamma_+ (A) = \min_{x_1, \ldots, x_m : A = \sum_{k=1}^m x_k x_k^*} \sum_{k=1}^m \|x_k\|_1^2 , \ m = n^2 \quad (P)
\]

\[
p^* = \inf_{\mu \in \mathcal{B}(S_1) : A = \int_{S_1} x x^* d\mu(x)} \int_{S_1} d\mu(x) \quad (M)
\]

a. Assume \( A = \sum_{k=1}^m x_k x_k^* \) is a global minimum for \((P)\). Then
\[
\mu(x) = \sum_{k=1}^m \|x_k\|_1^2 \delta(x - \frac{x_k}{\|x_k\|_1})\]
is a feasible solution for \((M)\). This shows \( p^* \leq \gamma_+ (A) \).

b. For reverse: Let \( \mu^* \) be an optimal measure in \((M)\). Fix \( \varepsilon > 0 \). Construct a disjoint partition \((U_l)_{1 \leq l \leq L}\) of \( S_1 \) so that each \( U_l \) is included in some ball \( B_\varepsilon (z_l) \) of radius \( \varepsilon \) with \( \|z_l\|_1 = 1 \). Thus \( U_l \subset B_\varepsilon (z_l) \cap S_1 \).

For each \( l \), compute \( x_l = \frac{1}{\mu^*(U_l)} \int_{U_l} x d\mu^*(x) \in B_\varepsilon (z_l) \). Let \( g_l = \sqrt{\mu^*(U_l)} x_l \).
Proof: The Optimal Measure Result (cont)

Key inequality:

\[ 0 \leq R_l := \int_{U_l} (x - x_l)(x - x_l)^* \, d\mu^*(x) = \int_{U_l} xx^* \, d\mu^*(x) - \mu^*(U_l) x_l x_l^* \]

Sum over \( l \) and with \( R = \sum_{l=1}^{L} R_l \) get

\[ A = \int_{S_1} xx^* \, d\mu^*(x) \leq \sum_{l=1}^{L} g_l g_l^* + R \]

By sub-additivity and homogeneity:

\[ \gamma_+(A) \leq \sum_{l=1}^{L} \|g_l\|_1^2 + \gamma_+(R) \leq \sum_{l=1}^{L} \mu^*(U_l) \|x_l\|_1^2 + n \text{ trace}(R) \]

But \( \|x_l - z_l\|_1 \leq \varepsilon \) and \( \|x - x_l\|_1 \leq 2\varepsilon \) for every \( x \in U_l \). Hence \( \|x_l\|_1 \leq 1 + \varepsilon \) and \( \text{trace}(R_l) \leq 4\mu^*(U_l)\varepsilon^2 \).
Proof: The Optimal Measure Result (end)

Thus:

$$\gamma_+(A) \leq \mu^*(S_1) + (2\varepsilon + \varepsilon^2 + 4n\varepsilon^2)\mu^*(S_1)$$

Since $\varepsilon > 0$ is arbitrary, it follows

$$\gamma_+(A) \leq \mu^*(S_1) = p^*$$

This ends the proof of the measure result. $\square$
The Continuity Property

The proof is based on the following two lemmas:

**Lemma (L1)**

Let $A \in \text{Sym}^+(\mathbb{C}^n)$ of rank $r > 0$. Let $\lambda_r > 0$ denote the $r^{th}$ eigenvalue of $A$, and let $P_{A,r}$ denote the orthogonal projection onto the range of $A$. For any $0 < \varepsilon < 1$ and $B \in \text{Sym}^+(\mathbb{C}^n)$ such that $\|A - B\|_{Op} \leq \frac{\varepsilon \lambda_r}{1 - \varepsilon}$, the following holds true:

$$A - (1 - \varepsilon) P_{A,r} B P_{A,r} \geq 0 \quad (1)$$

**Lemma (L2)**

Let $A \in \text{Sym}^+(\mathbb{C}^n)$ of rank $r > 0$. Let $\lambda_r > 0$ denote the $r^{th}$ eigenvalue of $A$. For any $0 < \varepsilon < \frac{1}{2}$ and $B \in \text{Sym}^+(\mathbb{C}^n)$ such that $\|A - B\|_{Op} \leq \varepsilon \lambda_r$, the following holds true:

$$B - (1 - \varepsilon) P_{B,r} A P_{B,r} \geq 0 \quad (2)$$

where $P_{B,r}$ denotes the orthogonal projection onto the top $r$ eigenspace of $B$. 
**Proof of Continuity of $\gamma_+$**

Fix $A \in \text{Sym}^+(\mathbb{C}^n)$. Let $(B_j)_{j \geq 1}$, $B_j \in \text{Sym}^+(\mathbb{C}^n)$, be a convergent sequence to $A$. We need to show $\gamma_+(B_j) \to \gamma_+(A)$.

Let $A = \sum_{k=1}^{n^2} x_k x_k^*$ be the optimal decomposition of $A$ such that

$$\gamma_+(A) = \sum_{k=1}^{n^2} \|x_k\|_1^2.$$ 

If $A = 0$ then $\gamma_+(A) = 0$ and

$$0 \leq \gamma_+(B_j) \leq n \text{trace}(B_j) \leq n^2 \|B_j\|_{Op}.$$ 

Hence $\lim_j \gamma_+(B_j) = 0$.

Assume $\text{rank}(A) = r > 0$ and let $\lambda_r > 0$ denote the smallest strictly positive eigenvalue of $A$. Let $\varepsilon \in (0, \frac{1}{2})$ be arbitrary. Let $J = J(\varepsilon)$ be so that

$$\|A - B_j\|_{Op} < \varepsilon \lambda_r$$

for all $j > J$. Let $B_j = \sum_{k=1}^{n^2} y_{j,k} y_{j,k}^*$ be the optimal decomposition of $B_j$ such that $\gamma_+(B_j) = \sum_{k=1}^{n^2} \|y_{j,k}\|_1^2$.

Let $\Delta_j = A - (1 - \varepsilon) P_{A,r} B_j P_{A,r}$. By Lemma L1, for any $j > J$,

$$\gamma_+(A) \leq (1 - \varepsilon) \gamma_+(P_{A,r} B_j P_{A,r}) + \gamma_+(\Delta_j) \leq (1 - \varepsilon) \sum_{k=1}^{n^2} \|P_{A,r} y_{j,k}\|_1^2 + n \text{trace}(\Delta_j).$$
Proof of Continuity of $\gamma_+$ (cont)

Pass to a subsequence $j'$ of $j$ so that $y_{j',k} \to y_k$, for every $k \in [n^2]$, and 
$\gamma_+(B_{j'}) \to \lim \inf_j \gamma_+(B_j)$. Then 
\[ \lim_{j'} \sum_{k=1}^{n^2} \| P_{A,r} y_{j',k} \|^2_1 = \lim_{j'} \sum_{k=1}^{n^2} \| y_{j',k} \|^2_1 = \lim \inf_j \gamma_+(B_j) \]

On the other hand, $\lim_j \text{trace}(\Delta_j) = \varepsilon \text{trace}(A)$. Hence:
\[ \gamma_+(A) \leq (1 - \varepsilon) \lim \inf_j \gamma_+(B_j) + \varepsilon \text{trace}(A) \]

Since $\varepsilon > 0$ is arbitrary, it follows $\gamma_+(A) \leq \lim \inf_j \gamma_+(B_j)$.

The inequality $\lim \sup_j \gamma_+(B_j) \leq \gamma_+(A)$ follows from Lemma L2 similarly: with 
$\Delta_j = B_j - (1 - \varepsilon)P_{B_j,r} A P_{B_j,r}$ and $A = \sum_{k=1}^{n^2} x_k x_k^*$ optimal,
\[ \gamma_+(B_j) \leq (1 - \varepsilon) \gamma_+(P_{B_j,r} A P_{B_j,r}) + n \text{trace}(\Delta_j) = (1 - \varepsilon) \sum_{k=1}^{n^2} \| P_{B_j,r} x_k \|^2_1 + n \text{trace}(\Delta_j). \]

Next take limsup of lhs by noticing $P_{B_j,r} \to P_{A,r}$ and $\lim \sup_j \| \Delta_j \|_{O_p} = \varepsilon \| A \|_{O_p}$:
\[ \lim \sup_j \gamma_+(B_j) \leq (1 - \varepsilon) \gamma_+(A) + n^2 \varepsilon \| A \|_{O_p} \]. Take $\varepsilon \to 0$ and result follows. □
Proof of Lemmas

Proof of Lemma L1

Let $P = P_{A,r}$ and $\Delta = A - (1 - \varepsilon)P_{A,r}BP_{A,r}$. For any $x \in \mathbb{C}^n$:

$$\langle \Delta x, x \rangle = \langle APx, Px \rangle - (1 - \varepsilon)\langle BPx, Px \rangle = \langle (A - (1 - \varepsilon)B)Px, Px \rangle =$$

$$= \varepsilon \langle APx, Px \rangle + (1 - \varepsilon)\langle (A - B)Px, Px \rangle \geq \varepsilon \lambda_r \|Px\|^2 - (1 - \varepsilon)\|A - B\|_{Op} \|Px\|^2 \geq 0$$

because $\|A - B\|_{Op} \leq \frac{\varepsilon \lambda_r}{1 - \varepsilon}$.

Proof of Lemma L2

Let $P = P_{B,r}$ and $\Delta = B - (1 - \varepsilon)P_{B,r}AP_{B,r}$. Let $C = B - P_{B,r}BP_{B,r} \geq 0$. Let $\mu_r$ be the $r^{th}$ eigenvalue of $B$. Note $|\mu_r - \lambda_r| \leq \|A - B\|_{Op} \leq \varepsilon \lambda_r$. Thus $\mu_r \geq (1 - \varepsilon)\lambda_r$. For any $x \in \mathbb{C}^n$:

$$\langle \Delta x, x \rangle = \langle Cx, x \rangle + \langle BPx, Px \rangle - (1 - \varepsilon)\langle APx, Px \rangle = \langle Cx, x \rangle + \varepsilon \langle BPx, Px \rangle +$$

$$+ (1 - \varepsilon)\langle (B - A)Px, Px \rangle \geq \langle Cx, x \rangle + (\varepsilon \mu_r - (1 - \varepsilon)\|A - B\|_{Op})\|Px\|^2 \geq 0$$

because $\|A - B\|_{Op} \leq \varepsilon \lambda_r \leq \frac{\varepsilon \mu_r}{1 - \varepsilon}$. 