Low-rank matrix estimation and rank-one matrix decompositions: when nonlinear analysis meets statistics

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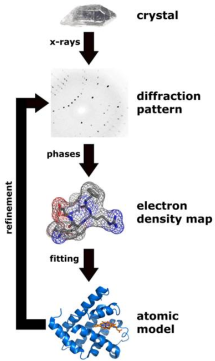
Problems to consider:

- What do we measure (model) ?
- What do we know about X (prior) ?
- How do we want to estimate X (principle) ?

Phase Retrieval

X-Ray Chrystallography



$$I(k) = C\left|\int e^{2\pi i\langle k,r\rangle}\rho(r)dr\right|^2$$

Unknown: ρ, the electron density.
Measurement: $I(k)$, difraction pattern intensity at wavevector k.

(from http://en.wikipedia.org/wiki/X-ray_crystallography)

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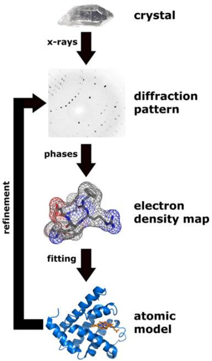

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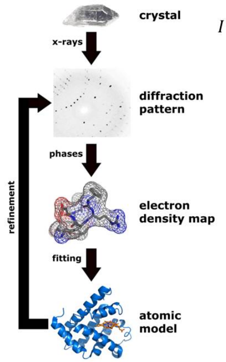

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Abstract form:

$$y_k = |\langle x, f_k\rangle|^2 \ ,\ 1 \le k \le m.$$

Quantum Tomography

Quantum Measurements

Quantum Mechanics:

- Observables are represented by self-adjoint operators, e.g. position x, momentum $P = i\hbar\frac{d}{dx}$, spin Σ, energy H, total angular momentum J, etc.

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- Quantum States: Two types: pure states, associated to a wave function Ψ. Mixed states, denoted M.

Quantum theory postulates that, in a *pure state* Ψ, an observable, say Σ, may take one of the values in its spectrum, say s, with probability $p_\Sigma(s) = |\langle\Psi, f_s\rangle|^2$, where f_s is the normalized eigenfunction $\Sigma f_s = sf_s$. In particular, the average (expected value) of Σ is

$$\mathbb{E}[\Sigma] = \sum s p_\Sigma(s) = \langle\Sigma\Psi, \Psi\rangle$$

Quantum Tomography

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$$\mathbb{E}[\Sigma] = \sum sp_\Sigma(s) = \langle\Sigma\Psi, \Psi\rangle = trace(\Sigma\Psi\Psi^*).$$

In a *mixed state* M, the expectation is replaced with $\mathbb{E}[\Sigma] = trace(M\Sigma)$.

Quantum Tomography

Problem

Given a quantum system in (mixed) quantum state M, and a set of observables $Y_1, \cdots, Y_m$ that can be measured simultaneously, assume we know

$$y_k = trace(MY_k) \quad , \quad 1 \leq k \leq m.$$

The problem is to estimate (compute) the PSD M that has to satisfy additionally $trace(M) = 1$.

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The problem is to estimate (compute) the PSD M that has to satisfy additionally $trace(M) = 1$.
To make this problem more tractable we shall assume $rank(M)$ is small. (M has low rank)

Setup

Notations

$H = \mathbb{R}^n$ or $H = \mathbb{C}^n$, finite dimensional Euclidean space.

- $Sym(\mathbb{R}^n) = \{T \in \mathbb{R}^{n \times n}\ ,\ T = T^T\}$ or $Sym(\mathbb{C}^n) = \{T \in \mathbb{C}^{n \times n}\ ,\ T = T^*\}$
- Convex cone of PSD: $Sym^+(H) = \{T \in Sym(H)\ ,\ T = T^* \geq 0\}$

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- Convex cone of PSD: $Sym^+(H) = \{T \in Sym(H)\ ,\ T = T^* \geq 0\}$
- Quantum states: $St(H) = \{T \in Sym^+(H)\ ,\ trace(T) = 1\}$
- Cone of mixed signatures matrices:

  $$\mathcal{S}^{p,q}\{T \in Sym(H)\ ,\ T \text{ has at most } p \text{ positive and } q \text{ negative eigenvalu}$$

  In particular $\mathcal{S}^{1,0} = \{xx^*\ ,\ x \in H\}$, set of rank (at most) one PSDs.
- Low-rank quantum states $St^r(H) = \{T \in Sym^+(H)\ ,\ trace(T) = 1\ ,\ rank(T) \leq r\}$

Problem Formulation

Models

Measurement maps:

$$\alpha : Sym^{+}(H) \to \mathbb{R}^m \quad , \quad (\alpha(X))_k = \sqrt{trace(XF_k)}$$

$$\beta : Sym^{+}(H) \to \mathbb{R}^m \quad , \quad (\beta(X))_k = trace(XF_k)$$

where $F_1, \cdots, F_m \in Sym^{+}(H)$ are fixed PSD matrices.

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Prior Information: Assume the unknown matrix X belongs to a class of PSD matrices $\mathcal{S}$:

- Phase Retrieval: $\mathcal{S} = \mathcal{S}^{1,0} = \{xx^* \ , \ x \in H\}$.
- Quantum Tomography:
  $\mathcal{S} = St^r(H) = \{X = X^* \geq 0 \ , \ trace(X) = 1 \ , \ rank(X) \leq r\}$.

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Matrix Estimation Problem: Estimate X given $y = \alpha(X) + \nu$ or $y = \beta(X) + \nu$ and knowing à priorly that $X \in \mathcal{S}$.

Problem Formulation

The phase retrieval problem

- Hilbert space $H = \mathbb{C}^n$, $\hat{H} = H/T^1$, frame $\mathcal{F} = \{f_1, \cdots, f_m\} \subset \mathbb{C}^n$ and

$$\alpha : \hat{H} \to \mathbb{R}^m \quad , \quad \alpha(x) = (|\langle x, f_k \rangle|)_{1 \leq k \leq m} .$$

$$\beta : \hat{H} \to \mathbb{R}^m \quad , \quad \beta(x) = \left(|\langle x, f_k \rangle|^2\right)_{1 \leq k \leq m} .$$

  The frame is said *phase retrievable* (or that it gives phase retrieval) if α (or β) is injective.
- The general *phase retrieval problem* a.k.a. *phaseless reconstruction*: Decide when a given frame is phase retrievable, and, if so, find an algorithm to recover x from $y = \alpha(x)$ (or from $y = \beta(x)$) up to a global phase factor.

Problem Formulation
Lipschitz Reconstruction

Assume $\mathcal{F}$ is phase retrievable.
Our Problems Today:

1. Are the nonliner maps α, β bi-Lipschitz with respect to appropriate metrics?
2. Do they admit left inverses that are globally Lipschitz?
3. What are the Lipschitz constants? What is the structure of local Lipschitz bounds?
4. What is the average performance of any reconstruction scheme (Cramer-Rao Lower Bounds)?

1-3: Worst Case Performance
4: Average Case Performance

Metric Space Structures on $\hat{H}$

Topological Structures

Let $H = \mathbb{C}^n$. The quotient space $\hat{H} = \mathbb{C}^n / T^1$, with classes induced by $x \sim y$ if there is real φ with $x = e^{i\varphi} y$.

Topologically:

$$\hat{\mathbb{C}}^n = \{0\} \cup \left((0,\infty) \times \mathbb{CP}^{n-1}\right)$$

with

$$\mathring{\hat{\mathbb{C}}}^n = \hat{\mathbb{C}}^n \setminus \{0\} = (0,\infty) \times \mathbb{CP}^{n-1}$$

a real analytic manifold of real dimension $2n-1$.

Another embedding is into the space of symmetric matrices $Sym(\mathbb{C}^n)$. Specifically let

$$\mathcal{S}^{p,q}(H) = \{T \in Sym(H)\ ,\ T \text{ has at most } p \text{ pos.eigs. and } q \text{ neg.eigs}\}$$

Then:

$$\kappa_\beta : \hat{H} \to \mathcal{S}^{1,0}\ ,\ \hat{x} \mapsto = xx^*\ ,\ \text{is an embedding.}$$

Metric Space Structures on $\hat{H}$

The matrix norm-induced metric structure

Fix $1 \leq p \leq \infty$. The *matrix-norm induced distance*

$$d_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ \ d_p(\hat{x}, \hat{y}) = \|xx^* - yy^*\|_p$$

with the p-norm of the singular values. In the case $p = 2$ we obtain

$$d_2(x, y) = \sqrt{\|x\|^4 + \|y\|^4 - 2|\langle x, y\rangle|^2}$$

Lemma (BZ15)

1. $(d_p)_{1\leq p\leq\infty}$ *are equivalent metrics and the identity map* $i : (\hat{H}, d_p) \to (\hat{H}, d_q)$, $i(x) = x$ *has Lipschitz constant*

$$Lip^d_{p,q,n} = \max(1, 2^{\frac{1}{q} - \frac{1}{p}}).$$

2. *The metric space* $(\hat{H}, d_p)$ *is isometrically isomorphic to* $\mathcal{S}^{1,0}$ *endowed with the p-norm via* $\kappa_\beta : \hat{H} \to \mathcal{S}^{1,0}$, $x \mapsto \kappa_\beta(x) = xx^*$.

Metric Space Structures

The natural metric structure

Fix $1 \leq p \leq \infty$. The *natural metric*

$$D_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ \ D_p(\hat{x}, \hat{y}) = \min_{\varphi} \left\| x - e^{i\varphi} y \right\|_p$$

with the usual p-norm on $\mathbb{C}^n$. In the case $p = 2$ we obtain

$$D_2(\hat{x}, \hat{y}) = \sqrt{\|x\|^2 + \|y\|^2 - 2|\langle x, y \rangle|}$$

Lemma (BZ15)

1. *$(D_p)_{1 \leq p \leq \infty}$ are equivalent metrics and the identity map $i : (\hat{H}, D_p) \to (\hat{H}, D_q)$, $i(x) = x$ has Lipschitz constant*

$$Lip^D_{p,q,n} = \max(1, n^{\frac{1}{q} - \frac{1}{p}}).$$

2. *The metric space $(\hat{H}, D_2)$ is Lipschitz isomorphic to $\mathcal{S}^{1,0}$ endowed with the 2-norm via $\kappa_\alpha : \hat{H} \to \mathcal{S}^{1,0}$, $x \mapsto \kappa_\alpha(x) = \frac{1}{\|x\|} xx^*$.*

Metric Space Structures

Distinct Structures

Two different structures: topologically equivalent, BUT the metrics are NOT equivalent:

Lemma (BZ15)

The identity map $i : (\hat{H}, D_p) \to (\hat{H}, d_p)$, $i(x) = x$ is continuous but it is not Lipschitz continuous. Likewise, the identity map $i : (\hat{H}, d_p) \to (\hat{H}, D_p)$, $i(x) = x$ is continuous but it is not Lipschitz continuous. Hence the induced topologies on $(\hat{H}, D_p)$ and $(\hat{H}, d_p)$ are the same, but the corresponding metrics are not Lipschitz equivalent.

Main Results
Lipschitz inversion: α

Theorem (BZ15)

Assume $\mathcal{F}$ is a phase retrievable frame for H. Then:

1. *The map $\alpha : (\hat{H}, D_2) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz. Let $\sqrt{A_0}, \sqrt{B_0}$ denote its Lipschitz constants: for every $x, y \in H$:*

$$A_0 \min_{\varphi} \|x - e^{i\varphi}y\|_2^2 \leq \sum_{k=1}^{m} ||\langle x, f_k\rangle| - |\langle y, f_k\rangle||^2 \leq B_0 \min_{\varphi} \|x - e^{i\varphi}y\|_2^2.$$

2. *There is a Lipschitz map $\omega : (\mathbb{R}^m, \|\cdot\|_2) \to (\hat{H}, D_2)$ so that: (i) $\omega(\alpha(x)) = x$ for every $x \in \hat{H}$, and (ii) its Lipschitz constant is $Lip(\omega) \leq \frac{4+3\sqrt{2}}{\sqrt{A_0}} = \frac{8.24}{\sqrt{A_0}}$.*

Main Results

Lipschitz inversion: β

Theorem (BZ15)

Assume $\mathcal{F}$ is a phase retrievable frame for H. Then:

1. *The map $\beta : (\hat{H}, d_1) \rightarrow (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz. Let $\sqrt{a_0}, \sqrt{b_0}$ denote its Lipschitz constants: for every $x, y \in H$:*

$$a_0 \|xx^* - yy^*\|_1^2 \leq \sum_{k=1}^{m} \left| |\langle x, f_k \rangle|^2 - |\langle y, f_k \rangle|^2 \right|^2 \leq b_0 \|xx^* - yy^*\|_1^2.$$

2. *There is a Lipschitz map $\psi : (\mathbb{R}^m, \|\cdot\|_2) \rightarrow (\hat{H}, d_1)$ so that: (i) $\psi(\beta(x)) = x$ for every $x \in \hat{H}$, and (ii) its Lipschitz constant is $Lip(\psi) \leq \frac{4+3\sqrt{2}}{\sqrt{a_0}} = \frac{8.24}{\sqrt{a_0}}$.*

Main Results
Statistical models

- A general noisy measurement process is given by:

$$y_k = |\langle x, f_k \rangle + \mu_k|^p + \nu_k \quad , \quad 1 \leq k \leq m,$$

where $(\mu_k)_k, (\nu_k)_k$ are two noise processes.

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- AWGN Model: $\mu_k = 0$, $p = 2$ and $\nu_k \sim \mathbb{N}(0, \sigma^2)$ i.i.d.

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- Non-AWGN Model: $\mu_k \sim \mathbb{CN}(0, \rho^2)$, i.i.d. and $\nu_k = 0$,
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Want:

1) Fisher Information Matrix $\mathbb{I} = \mathbb{E}\left[(\nabla_x \, log\, p(y;x))(\nabla_x \, log\, p(y;x))^*\right].$

2) Cramer-Rao Lower Bounds for unbiased estimators.

Main Results

Fisher Information Matrix

$$\mathbb{I}^{AWGN,real}(x) = \frac{4}{\sigma^2}\sum_{k=1}^{m} |\langle x, f_k\rangle|^2 f_k f_k^T = \frac{4}{\sigma^2}\sum_{k=1}^{m}(f_k f_k^T)xx^T(f_k f_k^T) \quad \text{[Bal12]}.$$

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$$\mathbb{I}^{AWGN,cplx}(x) = \frac{4}{\sigma^2}\sum_{k=1}^{m} \Phi_k \xi \xi^* \Phi_k \quad \text{[Bal13, BCMN13]}.$$

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$$\mathbb{I}^{AWGN,cplx}(x) = \frac{4}{\sigma^2}\sum_{k=1}^m \Phi_k \xi \xi^* \Phi_k \quad \text{[Bal13, BCMN13]}.$$

$$\mathbb{I}^{nonAWGN,cplx}(x) = \frac{4}{\rho^4}\sum_{k=1}^m \left(G_1\left(\frac{\langle \Phi_k \xi, \xi\rangle}{\rho^2}\right) - 1\right) \Phi_k \xi \xi^* \Phi_k$$

$$= \frac{4}{\rho^2}\sum_{k=1}^m G_2\left(\frac{\langle \Phi_k \xi, \xi\rangle}{\rho^2}\right) \frac{1}{\langle \Phi_k \xi, \xi\rangle} \Phi_k \xi \xi^* \Phi_k \quad \text{[Bal15]}.$$

where

$$G_1(a) = \frac{e^{-a}}{8a^3}\int_0^\infty \frac{I_1^2(t)}{I_0(t)} t^3 e^{-\frac{t^2}{4a}} dt \quad , \quad G_2(a) = a(G_1(a) - 1).$$

Main Results

AWGN vs. non-AWGN: Comparisons and Identifiability

Let B be the frame upper bound.

Lemma

$$\frac{\sigma^2}{\rho^4}\left(G_1\left(\frac{B\|x\|^2}{\rho^2}\right)-1\right)\mathbb{I}^{AWGN,cplx}(x)\leq \mathbb{I}^{nonAWGN,cplx}(x)\leq \frac{\sigma^2}{\rho^4}\mathbb{I}^{AWGN,cplx}(x)$$

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Theorem

The following are equivalent:

1. *The frame $\mathcal{F}$ is phase retrievable;*
2. *For every $0\neq x\in\mathbb{C}^n$, $rank(\mathbb{I}^{nonAWGN,cplx}(x))=2n-1$;*
3. *For every $0\neq x\in\mathbb{C}^n$, $rank(\mathbb{I}^{AWGN,cplx}(x))=2n-1$;*

Main Results

The Cramer-Rao Lower Bound

Fix $z_0 \in \mathbb{C}^n$, $\|z_0\| = 1$, let $\zeta_0 = [real(z_0)\ imag(z_0)]^T$ and set

$$\Omega_{z_0} = \{\xi \in \mathbb{R}^{2n}\ ,\ \langle \xi, \zeta_0 \rangle) \geq 0, \langle \xi, J\zeta_0 \rangle) = 0\}.$$

Let $\Pi_{z_0} = 1 - J\zeta_0\zeta_0^* J^*$ with J the symplectic form matrix.

Theorem

Assume a measurement model $y_k = |\langle x, f_k \rangle + \mu_k|^p + \nu_k$ *with* $\xi = [real(x)\ imag(x)]^T \in \mathring{\Omega}_{z_0}$. *Then the covariance of any unbiased estimator* $\omega : \mathbb{R}^m \to \mathbb{C}^n$ *is bounded below by*

$$Cov[\omega(y); \xi] \geq (\Pi_{z_0} \mathbb{I}(\xi) \Pi_{z_0})^\dagger\ .$$

If one chooses the global phase so that $\langle \omega(y), x \rangle \geq 0$ *(*$z_0 = x$*) then:*

$$Cov[\omega(y); \xi] \geq (\mathbb{I}(\xi))^\dagger\ .$$

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If one chooses the global phase so that $\langle \omega(y), x \rangle \geq 0$ $(z_0 = x)$ *then:*

$$Cov[\omega(y); \xi] \geq (\mathbb{I}(\xi))^\dagger .$$

Is this the optimal bound?

Main Results
Prior Works

Prior literature:

- 2012: **B.**: Cramer-Rao lower bound in the real case; **Eldar&Mendelson** : map α in the real case

$$\|\alpha(x)-\alpha(y)\| \geq C\|x-y\|\|x+y\|.$$

- 2013: **Bandeira,Cahill,Mixon,Nelson**: improved the estimate of C. **B.**: β bi-Lipschitz in real and complex case.
- 2014: **B.&Yang**: Find the exact Lipschitz constant for α in the real case - the constants A_0, B_0; **B.&Z.**:constructed a Lipschitz left inverse for β.
- 2015: **B.&Z.**: Proved α is bi-Lipschitz in the complex case; constructed a Lipschitz left inverse. **B.**: lower Lipschitz constant A_0 connected to CRLB of a non-AWGN model.

Main Results

Key relationship between deterministic and stochastic bounds

The central object: $\mathcal{R}(\xi) = \sum_{k=1}^{m} \Phi_k \xi \xi^T \Phi_k$.

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Best inversion scheme ψ that is lossless in the absence of noise achieves:

$$d_1(\psi(c), \psi(d))^2 \leq \frac{68}{a_0}\|c-d\|_2^2.$$

An efficient estimator (i.e. unbiased that achieves CRLB) ω^0 achieves:

$$\mathbb{E}\left[\|\omega^0(y) - x\|_2^2; x\right] \leq \frac{(2n-1)\sigma^2}{4a_0\|x\|^2} = \frac{2n-1}{4a_0\, SNR}.$$

Proofs
Overview

Deterministic bounds: The proofs involve several steps (details in [BZ15]).

1. Part 1: Injectivity $\longrightarrow$ bi-Lipschitz: Upper bounds are not too hard; lower bounds: relatively easy for β (the "square" map), but relatively hard for α.
2. Part 2: Left inverse construction is done in three steps:
   1. The left inverse is first extended to $\mathbb{R}^m$ into $Sym(H)$ using Kirszbraun's theorem;
   2. Then we show that $\mathcal{S}^{1,0}(H)$ is a Lipschitz retract in $Sym(H)$;
   3. The proof is concluded by composing the two maps.

The stochastic bounds: Direct computations and a bit of luck! [Bal15]

Proofs

Part 1a: Bi-Lipschitzianity of α

$$\alpha : \hat{H} \to \mathbb{R}^m \quad , \quad \alpha(x) = (|\langle x, f_k \rangle|)_{1 \leq k \leq m}$$

The homogeneity of α shows that

$$L(x,y) = \frac{\|\alpha(x) - \alpha(y)\|}{D_p(x,y)}$$

is homogeneous of degree 0: $L(tx,ty) = L(x,y)$, for every $t > 0$.
This reduces the problem to the unit ball: $1 = \|x\| \geq \|y\|$.
The upper bound was computed in [BCMN13]:

$$\sup_{x \neq y} \frac{\|\alpha(x) - \alpha(y)\|^2}{D_2(x,y)^2} = B \ \ \text{(upper frame bound)}.$$

Proofs

Part 1a: Bi-Lipschitzianity of α - cont'd

A compactness argument shows the lower bound is positive if and only if the local lower bound is positive:

$$\inf_{\|z\|=1} \lim_{r\to 0} \inf_{\substack{x,y\in\hat{H} \\ D_2(x,z)<r \\ D_2(y,z)<r}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2} > 0.$$

This bound is computed explicitely and shown positive: Computations involve the realification framework and other delicate nonlinear expansions.

Proofs

Part 1b: Bi-Lipschitzianity of β

Key Remark (B.Bodmann,Casazza,Edidin - 2007): The nonlinear map β is the restriction of the linear map

$$\mathbb{A} : Sym(H) \to \mathbb{R}^m \quad , \quad \mathbb{A}(T) = (\langle Tf_k, f_k \rangle)_{1 \leq k \leq m}$$

Specifically: $\beta(x) = \mathbb{A}(xx^*) = (|\langle x, f_k \rangle|^2)_{1 \leq k \leq m}$.

$$\begin{aligned} \|\beta(x) - \beta(y)\| = \|\mathbb{A}(xx^*) - \mathbb{A}(yy^*)\| &= \|\mathbb{A}(xx^* - yy^*)\| \\ &= \|xx^* - yy^*\| \|\mathbb{A}\left(\frac{xx^* - yy^*}{\|xx^* - yy^*\|}\right)\| \end{aligned}$$

$$a_0 = \min_{T \in \mathcal{S}^{1,1}, \|T\|_1 = 1} \|\mathbb{A}(T)\| > 0 \quad , \quad b_0 = \max_{T \in \mathcal{S}^{1,1}, \|T\|_1 = 1} \|\mathbb{A}(T)\|$$

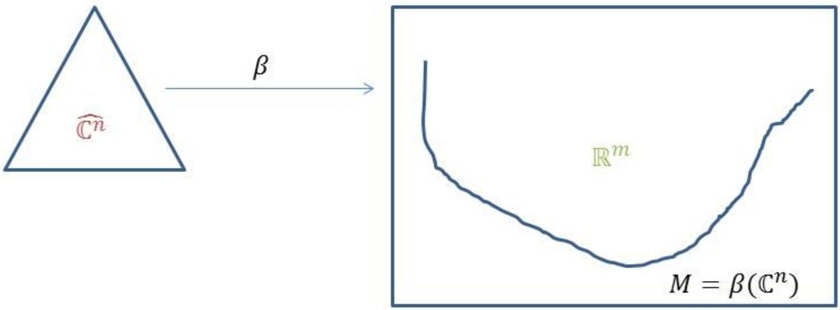
Proofs

Part 2a: Extension of the inverse for α

We know $\alpha : (\hat{H}, D_2) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz:

$$A_0 D_1(x,y)^2 \le \|\alpha(x) - \alpha(y)\|^2 \le b_0 D_2(x,y)^2$$

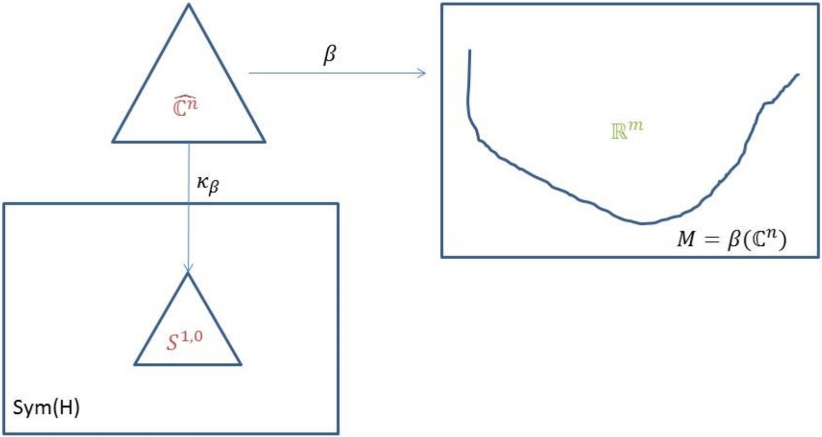
Let $M = \alpha(\hat{H}) \subset \mathbb{R}^m$.



Proofs

Part 2a: Extension of the inverse for α

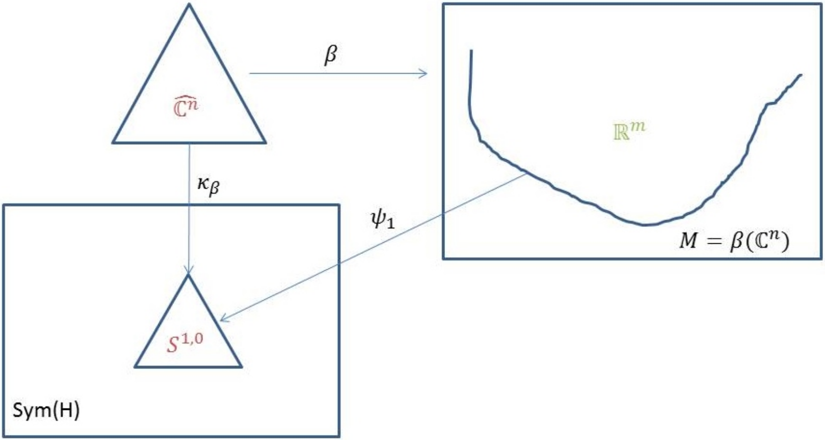
First identify $\hat{H}$ with $\mathcal{S}^{1,0}(H)$.



Proofs

Part 2a: Extension of the inverse for α

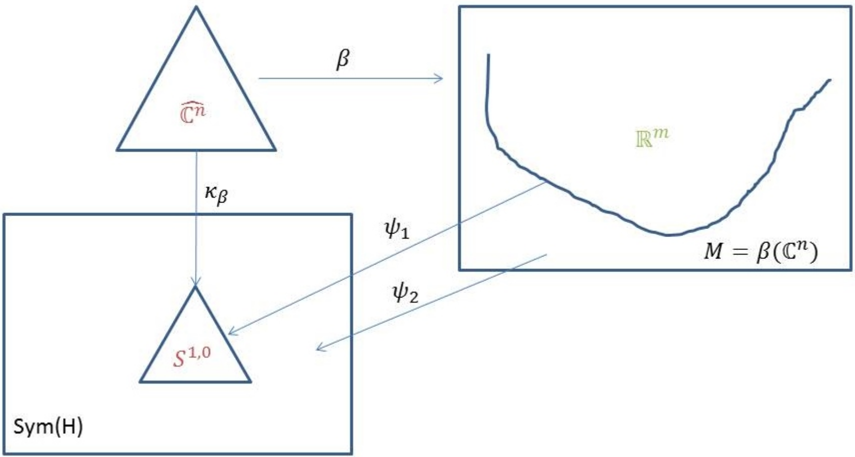
Then construct the local left inverse $\omega_1 : M \to \hat{H}$ with $Lip(\omega_1) = \frac{1}{\sqrt{A_0}}$.



Proofs

Part 2a: Extension of the inverse for α

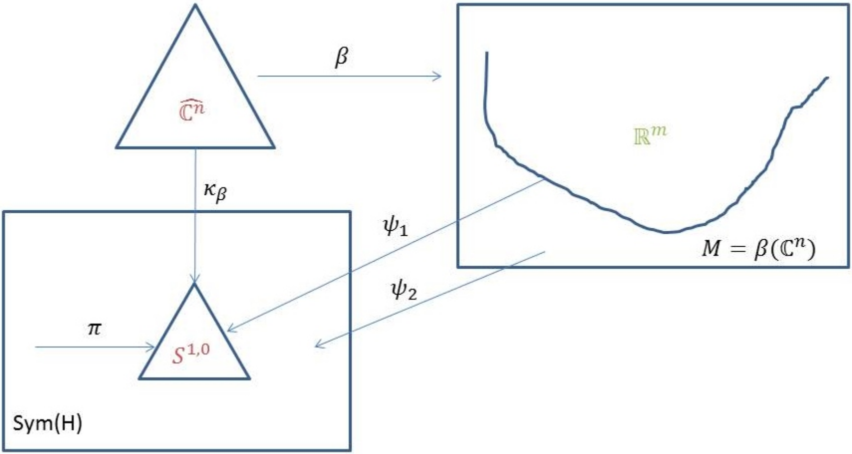
Use Kirszbraun's theorem to extend isometrically $\omega_2 : \mathbb{R}^m \to Sym(H)$.



Proofs

Part 2a: Extension of the inverse for α

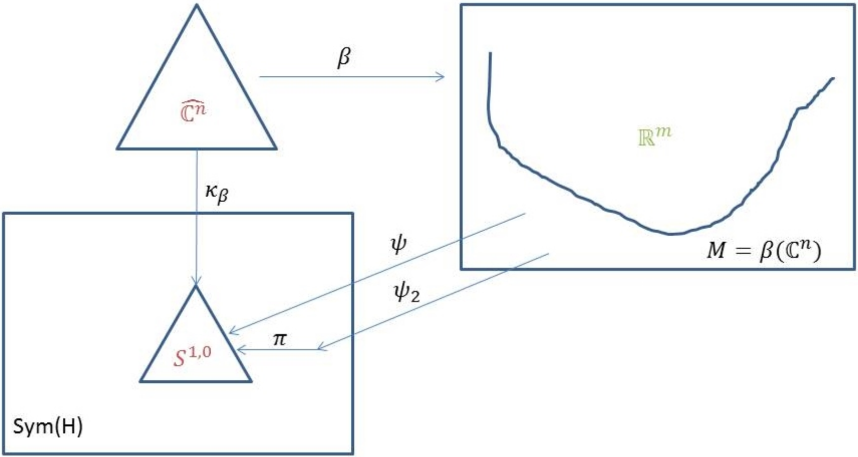
Construct a Lipschitz "projection" $\pi : Sym(H) \to \mathcal{S}^{1,0}(H)$.



Proofs

Part 2a: Extension of the inverse for α

Compose the two maps to get $\omega : \mathbb{R}^m \to \mathcal{S}^{1,0}$, $\omega = \pi \circ \omega_2$.



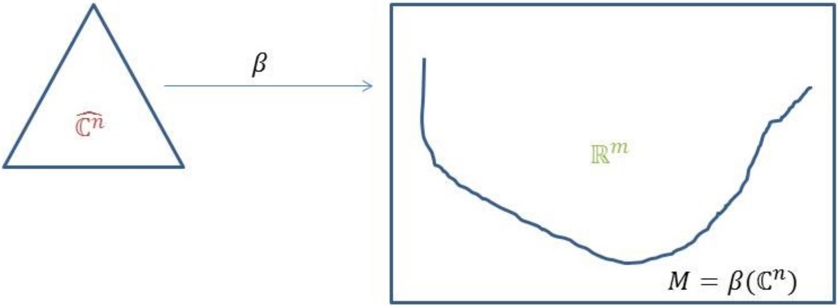
Proofs

Part 2b: Extension of the inverse for β

We know $\beta : (\hat{H}, d_1) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz:

$$a_0 d_1(x,y)^2 \leq \|\beta(x) - \beta(y)\|^2 \leq b_0 d_1(x,y)^2.$$

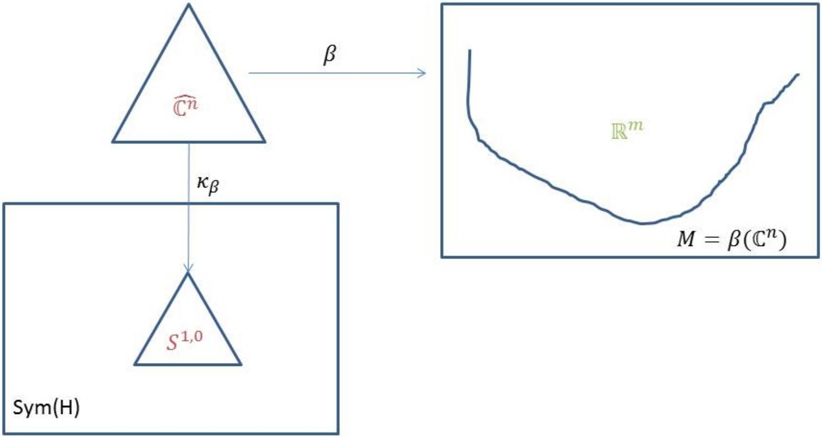
Let $M = \beta(\hat{H}) \subset \mathbb{R}^m$.



Proofs

Part 2b: Extension of the inverse for β

First identify $\hat{H}$ with $\mathcal{S}^{1,0}(H)$.



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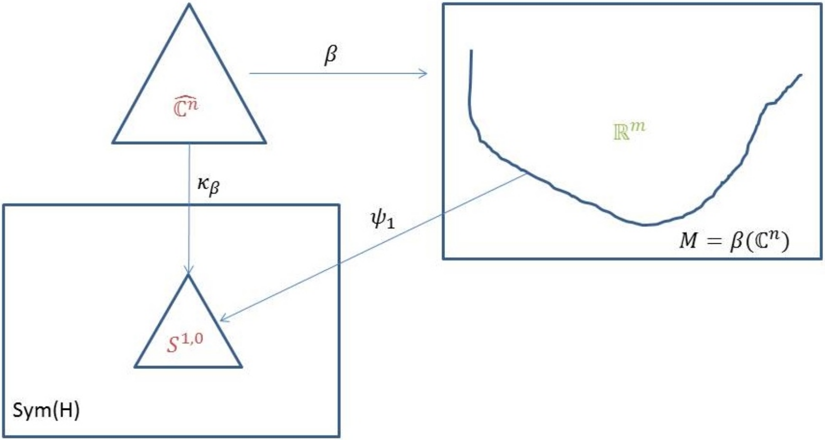
Then construct the local left inverse $\psi_1 : M \to \hat{H}$ with $Lip(\psi_1) = \frac{1}{\sqrt{a_0}}$.



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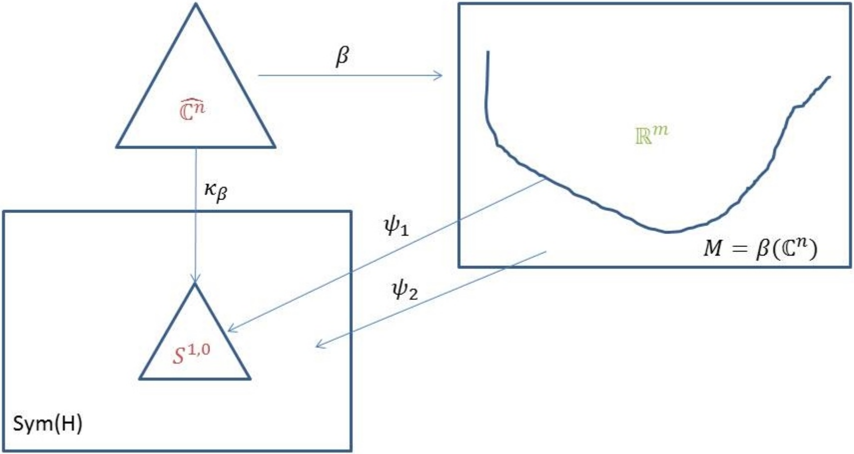

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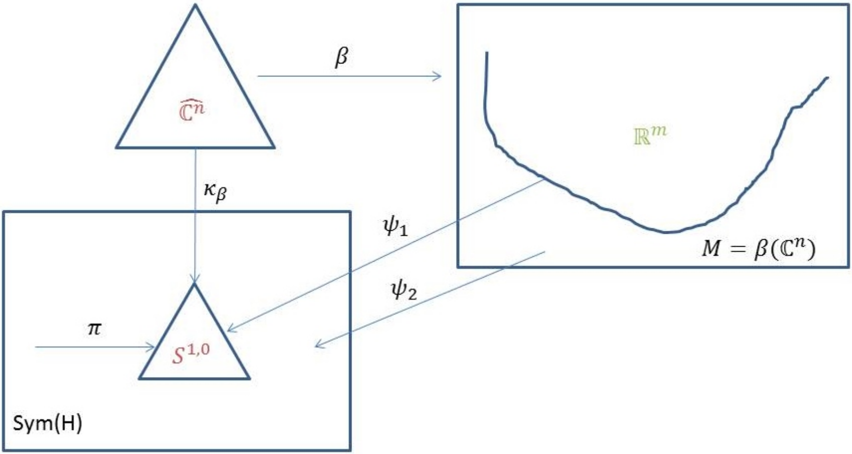
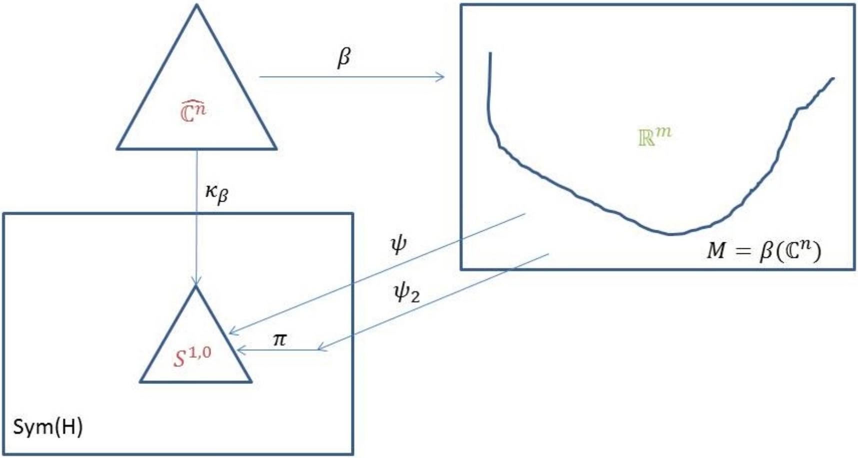
Construct a Lipschitz "projection" $\pi : Sym(H) \to \mathcal{S}^{1,0}(H)$.



Proofs

Part 2b: Extension of the inverse for β

Compose the two maps to get $\psi : \mathbb{R}^m \to \mathcal{S}^{1,0}$, $\psi = \pi \circ \psi_2$.



Proofs

Part 2: $\mathcal{S}^{1,0}(H)$ as Lipschitz retract in $Sym(H)$

Lemma

Consider the spectral decomposition of the self-adjoint operator A in $Sym(H)$, $A = \sum_{k=1}^{d} \lambda_{m(k)} P_k$. Then the map

$$\pi : Sym(H) \to \mathcal{S}^{1,0}(H) \quad , \quad \pi(A) = (\lambda_1 - \lambda_2) P_1$$

satisfies the following two properties:

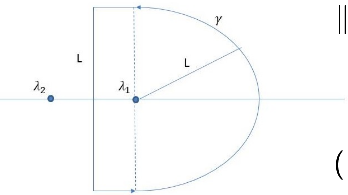
1. *for $1 \leq p \leq \infty$, it is Lipschitz continuous from $(Sym(H), \|\cdot\|_p)$ to $(\mathcal{S}^{1,0}(H), \|\cdot\|_p)$ with Lipschitz constant less than or equal to $3 + 2^{1+\frac{1}{p}}$;*
2. *$\pi(A) = A$ for all $A \in \mathcal{S}^{1,0}(H)$.*

Proof uses Weyl's inequality and spectral formula on a complex integration contour by Zwald & Blanchard (2006).

Last week: Wenbo Li [AMSC/UMD] proved that $Lip(\pi) = 2$, for $p = \infty$.

Assume simple top eigenvalues (otherwise the bound is immediate): $\pi(A) = (\lambda_1 - \lambda_2)P_1$, $\pi(B) = (\mu_1 - \mu_2)Q_1$. Then:

$$\begin{aligned} \|\pi(A) - \pi(B)\|_p &\leq (\lambda_1 - \lambda_2)\|P_1 - Q_1\|_p + |\lambda_1 - \mu_1| + |\lambda_2 - \mu_2| \\ &\leq (\lambda_1 - \lambda_2)\|P_1 - Q_1\|_p + 2\|A - B\|_p. \end{aligned}$$



$$\|P_1 - Q_1\|_p \leq \frac{1}{2\pi}\int_I \|(R_A - R_B)(\gamma(t))\|_p |\gamma'(t)| dt$$

$$R_A(z) = (A - zI)^{-1},\ R_B(z) = (B - zI)^{-1}.$$

$$(R_A - R_B)(z) = \sum_{n \geq 1} (-1)^n (R_A(z)(B - A))^n R_A(z).$$

$$\begin{aligned} \|(R_A - R_B)(\gamma(t))\|_p &\leq \sum_{n \geq 1} \|R_A(\gamma(t))\|_\infty^{n+1} \|A - B\|_p^n \\ &= \frac{\|R_A(\gamma(t))\|_\infty^2 \|A - B\|_p}{1 - \|R_A(\gamma(t))\|_\infty \|A - B\|_p} < \frac{\|A - B\|_p}{dist^2(\gamma(t), Spec(A))}. \end{aligned}$$

Proofs

Part 1: Bi-Lipschitzianity of α-cont'd

The analysis requires a deeper understanding of local behavior.

1. The *global lower* and *upper Lipschitz bounds*:

$$A_0 = \inf_{x,y\in\hat{H}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2} \ , \ B_0 = \sup_{x,y\in\hat{H}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2}$$

2. The *type I local lower* and *upper Lipschitz bounds* at $z \in \hat{H}$:

$$A(z) = \lim_{r\to 0} \inf_{\substack{x,y\in\hat{H} \\ D_2(x,z)<r \\ D_2(y,z)<r}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2} \ , \ B(z) = \lim_{r\to 0} \sup_{\substack{x,y\in\hat{H} \\ D_2(x,z)<r \\ D_2(y,z)<r}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2}$$

3. The *type II local lower* and *upper Lipschitz bounds* at $z \in \hat{H}$:

$$\tilde{A}(z) = \lim_{r\to 0} \inf_{\substack{x\in\hat{H} \\ D_2(x,z)<r}} \frac{\|\alpha(x)-\alpha(z)\|_2^2}{D_2(x,z)^2} \ , \ \tilde{B}(z) = \lim_{r\to 0} \sup_{\substack{x\in\hat{H} \\ D_2(x,z)<r}} \frac{\|\alpha(x)-\alpha(z)\|_2^2}{D_2(x,y)^2}$$

Proofs

Part 1: Bi-Lipschitzianity of α -cont'd

We need to analyze the real structure of $\hat{H}$.
Let $\varphi_1, \cdots, \varphi_m, \zeta \in \mathbb{R}^{2n}$, $\Phi_1, \cdots, \Phi_m \in Sym(\mathbb{R}^{2n})$, $J \in \mathbb{R}^{2n \times 2n}$ defined by:

$$\Phi_k = \varphi_k\varphi_k^T + J\varphi_k\varphi_k^T J^T, \varphi_k = \begin{bmatrix} real(f_k) \\ imag(f_k) \end{bmatrix}, J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \zeta = \begin{bmatrix} real(z) \\ imag(z) \end{bmatrix}$$

Key relations: $\langle z, f_k \rangle = \langle \zeta, \varphi_k \rangle + i\langle \zeta, J\varphi_k \rangle$, $|\langle z, f_k \rangle| = \sqrt{\langle \Phi_k \zeta, \zeta \rangle}$.
Consider the following objects:

$$\mathcal{R} : \mathbb{R}^{2n} \to Sym(\mathbb{R}^{2n}) \quad , \quad \mathcal{R}(\xi) = \sum_{k=1}^{m} \Phi_k \xi \xi^T \Phi_k \ , \ \xi \in \mathbb{R}^{2n}$$

$$\mathcal{S} : \mathbb{R}^{2n} \to Sym(\mathbb{R}^{2n}) \quad , \quad \mathcal{S}(\xi) = \sum_{k:\Phi_k\xi \neq 0} \frac{1}{\langle \Phi_k \xi, \xi \rangle} \Phi_k \xi \xi^T \Phi_k \ , \ \xi \in \mathbb{R}^{2n}$$

Proofs
Lipschitz bounds for α

Theorem (BZ15)

Assume $\mathcal{F}$ is phase retrievable for $H = \mathbb{C}^n$ and A, B are its optimal frame bounds. Then:

1. *For every $0 \neq z \in \mathbb{C}^n$, $A(z) = \lambda_{2n-1}(\mathcal{S}(\zeta))$ (the next to the smallest eigenvalue);*
2. $A_0 = A(0) > 0$;
3. *For every $z \in \mathbb{C}^n$, $\tilde{A}(z) = \lambda_{2n-1}\left(\mathcal{S}(\zeta) + \sum_{k:\langle z,f_k\rangle=0} \Phi_k\right)$ (the next to the smallest eigenvalue);*
4. *$\tilde{A}(0) = A$, the optimal lower frame bound;*
5. *For every $z \in \mathbb{C}^n$, $B(z) = \tilde{B}(z) = \lambda_1\left(\mathcal{S}(\zeta) + \sum_{k:\langle z,f_k\rangle=0} \Phi_k\right)$ (the largest eigenvalue);*
6. *$B_0 = B(0) = \tilde{B}(0) = B$, the optimal upper frame bound;*

Proofs

Lipschitz bounds for β

Theorem (cont'd)

7. *For every* $0 \neq z \in \mathbb{C}^n$, $a(z) = \tilde{a}(z) = \lambda_{2n-1}(\mathcal{R}(\zeta))/\|z\|^2$ *(the next to the smallest eigenvalue);*
8. *For every* $0 \neq z \in \mathbb{C}^n$, $b(z) = \tilde{b}(z) = \lambda_1(\mathcal{R}(\zeta))/\|z\|^2$ *(the largest eigenvalue);*
9. $a_0 = \min_{\|\xi\|=1} \lambda_{2n-1}(\mathcal{R}(\xi))$ *is also the largest constant to that* $\mathcal{R}(\xi) \geq a_0(\|\xi\|^2 I - J\xi\xi^T J^T)$;
10. $b(0) = \tilde{b}(0) = b_0 = \max_{\|\xi\|=1} \lambda_1(\mathcal{R}(\xi))$ *is also the* 4^{th} *power of the frame analysis operator norm* $T : (\mathbb{C}^n, \|\cdot\|_2) \to (\mathbb{R}^m, \|\cdot\|_4)$: $b_0 = \|T\|^4_{B(l^2,l^4)} = \max_{\|x\|_2=1} \sum_{k=1}^m |\langle x, f_k\rangle|^4$;
11. $\tilde{a}(0)$ *is given by* $\tilde{a}(0) = \min_{\|z\|=1} \sum_{k=1}^m |\langle z, f_k\rangle|^4$.

Quantum Tomography

Let's return to the Quantum Tomography problem:
Measurement maps:

$$\alpha : Sym^+(H) \rightarrow \mathbb{R}^m \quad , \quad (\alpha(X))_k = \sqrt{trace(XF_k)}$$

$$\beta : Sym^+(H) \rightarrow \mathbb{R}^m \quad , \quad (\beta(X))_k = trace(XF_k)$$

where $F_1, \cdots, F_m \in Sym^+(H)$ are fixed PSD matrices.

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where $F_1, \cdots, F_m \in Sym^+(H)$ are fixed PSD matrices.
Prior Information: Assume the unknown matrix X belongs to a class of PSD matrices $\mathcal{S}$:

- Phase Retrieval: $\mathcal{S} = \mathcal{S}^{1,0} = \{xx^* \ , \ x \in H\}$.
- Quantum Tomography:
  $\mathcal{S} = St^r(H) = \{X = X^* \geq 0 \ , \ trace(X) = 1 \ , \ rank(X) \leq r\}$.

Metric Structures on $\hat{H}$ and $Sym(H)$

Norm Induced Metric

Fix $1 \leq p \leq \infty$. The *matrix-norm induced distance* on $Sym(H)$:

$$d_p : Sym(H) \times Sym(H) \to \mathbb{R} \ , \ d_p(X,Y) = \|X - Y\|_p,$$

the p-norm of the singular values.
On $\hat{H}$ it induces the metric

$$d_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ d_p(\hat{x},\hat{y}) = \|xx^* - yy^*\|_p$$

In the case $p = 2$ we obtain

$$d_2(X,Y) = \|X - Y\|_F^2 \quad , \quad d_2(x,y) = \sqrt{\|x\|^4 + \|y\|^4 - 2|\langle x,y\rangle|^2}$$

Metric Structures on $\hat{H}$ and $Sym(H)$

Natural Metric

The *natural metric*

$$D_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ \ D_p(\hat{x}, \hat{y}) = \min_{\varphi} \|x - e^{i\varphi} y\|_p$$

with the usual p-norm on $\mathbb{C}^n$. In the case $p = 2$ we obtain

$$D_2(\hat{x}, \hat{y}) = \sqrt{\|x\|^2 + \|y\|^2 - 2|\langle x, y\rangle|}$$

On $Sym^+(H)$, the "natural" metric lifts to

$$D_p : Sym^+(H) \times Sym^+(H) \to \mathbb{R} \ , \ \ D_p(X, Y) = \min_{\substack{VV^* = X \\ WW^* = Y}} \|V - WU\|_p.$$

Metric Structures on $Sym(H)$

Natural metric vs. Bures/Helinger

Let $X, Y \in Sym^+(H)$. For the natural distance we choose $p = 2$:

$$D_{natural}(X,Y) = \min_{\substack{VV^* = X \\ WW^* = Y}} \|V - W\|_F$$

Fact:

$$D_{natural}(X,Y) = \min_{U \in U(n)} \|X^{1/2} - Y^{1/2}U\|_F = \sqrt{\mathrm{tr}(X) + tr(Y) - 2\|X^{1/2}Y^{1/2}\|_1}$$

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Another distance: Bures/Helinger distance:

$$D_{Bures}(X,Y) = \|X^{1/2} - Y^{1/2}\|_F = d_2(X^{1/2}, Y^{1/2})$$

Metric Structures on $Sym(H)$

Natural metric vs. Bures/Helinger

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Another distance: Bures/Helinger distance:

$$D_{Bures}(X,Y) = \|X^{1/2} - Y^{1/2}\|_F = d_2(X^{1/2}, Y^{1/2})$$

A consequence of the Arithmetic-Geometric Mean Inequality [BK00]:

$$\frac{1}{2} D^2_{Bures}(X,Y) \le D^2_{natural}(X,Y) \le D^2_{Bures}(X,Y).$$

Stability Results in Quantum Tomography

Bi-Lipschitz properties of α and β on Quantum States

Fix a closed subset $S \subset Sym^+(H)$. For instance $S = St(H)$, or $St^r(H)$, or $\mathcal{S}^{r,0}$.

Theorem

Assume $\mathcal{F} = \{F_1, \cdots, F_m\} \subset Sym^+(H)$ so that $\alpha|_S$ and $\beta|_S$ are injective. Then there are constants $a_0, A_0, b_0, B_0 > 0$ so that for every $X, Y \in S$,

$$A_0 D^2_{natural}(X, Y) \leq \sum_{k=1}^{m} \left| \sqrt{\langle X, F_k \rangle} - \sqrt{\langle Y, F_k \rangle} \right|^2 \leq B_0 D^2_{natural}(X, Y)$$

$$a_0 \|X - Y\|_F^2 \leq \sum_{k=1}^{m} |\langle X, F_k \rangle - \langle Y, F_k \rangle|^2 \leq b_0 \|X - Y\|_F^2.$$

Next Results

Lipschitz inversion of α and β on Quantum States

Consider the measurement map

$$\beta : (St^r(H), d_1) \to (\mathbb{R}^m, \|\cdot\|_2) \ \ , \ \ \beta(T) = (tr(TF_k))_{1 \leq k \leq m}$$

where $St^r(H) = \{T = T^* \geq 0 \ , \ tr(T) = 1 \ , \ rank(T) \leq r\}$.
If $r = n := dim(H)$ then $St^n(H) = St(H)$ is a compact convex set, hence a Lipschitz retract.
Conjecture: If $r < n$ then $St^r(H)$ is not contractible hence not a Lipschitz retract.

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If $r = n := dim(H)$ then $St^n(H) = St(H)$ is a compact convex set, hence a Lipschitz retract.
Conjecture: If $r < n$ then $St^r(H)$ is not contractible hence not a Lipschitz retract.
If conjecture is true, it follows that even if β is injective on rank r quantum states, it cannot admit a Lipschitz (or even continuous) left inverse defined globally on $\mathbb{R}^m$.

Next Results

Lipschitz inversion of α and β on Quantum States

Consider the measurement map

$$\beta : (St^r(H), d_1) \to (\mathbb{R}^m, \|\cdot\|_2) \quad , \quad \beta(T) = (tr(TF_k))_{1 \le k \le m}$$

where $St^r(H) = \{T = T^* \ge 0 \ , \ tr(T) = 1 \ , \ rank(T) \le r\}$.
If $r = n := dim(H)$ then $St^n(H) = St(H)$ is a compact convex set, hence a Lipschitz retract.
Conjecture: If $r < n$ then $St^r(H)$ is not contractible hence not a Lipschitz retract.
If conjecture is true, it follows that even if β is injective on rank r quantum states, it cannot admit a Lipschitz (or even continuous) left inverse defined globally on $\mathbb{R}^m$.
A similar result should hold true for

$$\alpha : (St^r(H), D_2) \to (\mathbb{R}^m, \|\cdot\|_2) \quad , \quad \alpha(T) = (\sqrt{tr(TF_k)})_{1 \le k \le m}.$$

Tensor Products

Consider $A \in \mathbb{C}^{n \times n}$. We seek "optimal" decompositions of A into a sum of rank-1 operators: $A = \sum_k u_k v_k^*$.
Assume A to be positive semi-definite: $A = A^* \geq 0$ ("covariance").
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Criterion 3:

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What we know

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For every $A \in Sym^+(\mathbb{C}^n)$,

$$\sum_{i,j} |A_{i,j}| =: \|A\|_\wedge = J_\wedge(A) \leq J_0(A) \leq J(A) \leq n\|A\|_\wedge$$

An Open Problem

A remaining open problem:Is there a universal constant $C_0 > 1$ so that for any $n \geq 1$ and every positive semidefinite $A \in \mathbb{C}^{n \times n}$,

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Why we care?

If the answer is positive, it follows that, given a trace-class positive semidefinite operator $T : f \mapsto Tf(x) = \int K(x,y)f(y)dy$ the following two statements are equivalent:

1. $K \in M^1(\mathbb{R}^2)$.
2. There are functions $g_k \in M^1(\mathbb{R})$ so that

$$T = \sum_{k\geq 0} \langle \cdot, g_k \rangle g_k$$

and $\sum_{k\geq 0} \|g_k\|_{M^1}^2 < \infty$.

Source Separation Problem: Finding a linear mixing model with minimal "blinding: spots.

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