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FLAT BUNDLES WITH SOLVABLE HOLONOMY

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ABSTRACT. Let G be a solvable linear Lie group. We show that for every flat principal G-bundle ξ over a CW-complex M, there is a finite-sheeted covering space $p: \hat{M} \to M$ such that $p^*\xi$ is trivial as a principal G-bundle. This result is used to show that every affine manifold with solvable fundamental group has a finite covering which is parallelizable.

In this note M denotes a connected manifold or CW-complex with fundamental group π and G denotes a Lie group. A bundle ξ over M with structure group G is *virtually trivial* if and only if there is a finite covering space $p: \hat{M} \to M$ such that $p^*\xi$ is a trivial bundle. We call ξ a *flat G-bundle* if the structure group has been reduced to a totally disconnected subgroup $\Gamma \subset G$, the *holonomy group* of the flat bundle. A vector bundle is flat if the associated principal bundle is flat.

The following result is known to others (e.g. D. Sullivan [6]) but there seems to be no published proof. Our purpose is to supply one.

THEOREM 1. *Let ξ be a flat vector bundle over M whose holonomy group is finitely generated and contains a solvable subgroup of finite index. Then ξ is virtually trivial.*

The proof is based on

THEOREM 2. *Suppose G is a solvable Lie group with finitely many components, which admits a faithful matrix representation. Let ξ be a flat principal G-bundle over M whose holonomy group is finitely generated. Then ξ is virtually trivial.*

PROOF OF THEOREM 1 FROM THEOREM 2. Let ξ be a flat vector bundle over M, induced by a representation $\phi: \pi \to \mathrm{GL}(n; R)$. Assuming the hypotheses of Theorem 1, we may pass to a finite covering of M and assume that $\phi(\pi)$ is actually solvable. Note that a subgroup of finite index in a finitely generated group is finitely generated.

Let G denote the Zariski closure of $\phi(\pi)$ in $\mathrm{GL}(n; R)$. Since any algebraic group has finitely many components and the Zariski closure of a solvable group is solvable, G satisfies the hypotheses of Theorem 2. Therefore the principal G-bundle induced by $\phi: \pi \to G$ is virtually trivial. Q.E.D.

We briefly recall how a homomorphism $h: \pi \to G$ induces a flat G-bundle, which we denote by h_b. Let $p: \tilde{M} \to M$ denote a universal covering and identify π as the group of deck transformations. Give π the diagonal action on $\tilde{M} \times G$ defined by

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$\gamma(\tilde{x}, g) = (\gamma\tilde{x}, h(\gamma)g)$. The orbit space $(\tilde{M} \times G)/\pi$ is the total space of a flat G-bundle over M; the projection sends the orbit of $(\tilde{x}, g)$ to $p(\tilde{x})$. The holonomy group is $h(\pi)$.

The following well-known results will be used. For convenience we indicate proofs.

LEMMA 3. *Let h_t: $\pi \to G$, $0 < t < 1$, be a continuous family of homomorphisms. Then $(h_0)_b$ and $(h_1)_b$ are isomorphic G-bundles. In particular, $(h_1)_b$ is a trivial G-bundle if h_0 is the trivial homomorphism.*

PROOF. There is a G-bundle ξ over $M \times [0, 1]$ such that $\xi|_{M\times\{t\}}$ is isomorphic to $(h_t)_b$. Apply the covering homotopy theorem. Q.E.D.

LEMMA 4. *Let ϕ: $G \to H$ be a homomorphism which is a homotopy equivalence of Lie groups. Then h_b is a trivial G-bundle provided $(\phi \circ h)_b$ is a trivial H-bundle.*

PROOF. ϕ induces a homotopy equivalence of classifying spaces $B\phi$: $BG \to BH$. (In fact let $EG \to BG$ be the universal principal G-bundle. Then H acts freely on the contractible space $EH = (EG \times H)/G$, where G acts on H via ϕ and G acts diagonally on $EG \times H$. Therefore $EH \to (EH)/H$ is a universal bundle for H; but $BH = (EH)/H$ is naturally identified with $(EG)/G = BG$. Thus we can take $BH = BG$.)

Let f: $M \to BG$ be the classifying map for h_b (as a G-bundle). Then $(B\phi) \circ f$: $M \to BH$ classifies $(\phi \circ h)_b$. Since $(\phi \circ h)_b$ is trivial, $(B\phi) \circ f$ is null homotopic, and therefore, so is f. Q.E.D.

PROOF OF THEOREM 2. Let ξ be induced by h: $\pi \to G$ with $h(\pi) = \Gamma$. Let $G_0 \subset G$ be the identity component. Then $h^{-1}G_0$ has finite index in π. We may replace ξ by the induced bundle ξ_0 over the covering space corresponding to $h^{-1}G_0$. The holonomy group of ξ_0 is $\Gamma \cap G_0 \subset G_0$. *Therefore we may assume G is connected.*

The assumptions on G imply that the commutator subgroup G' is closed and contractible (see Hochschild [3, Chapter XVIII, Theorem 3.2]). Therefore the natural homomorphism ϕ: $G \to G/G'$ is a homotopy equivalence. It follows from Lemma 4 that ξ is virtually trivial provided $(\phi \circ h)_b$ is virtually trivial. Therefore we may replace G by G/G'. Hence *we assume that G is a connected abelian Lie group.*

Now Γ is a finitely generated abelian group, so there is a free abelian subgroup $\Gamma_0 \subset \Gamma$ of finite index. Passing to the covering of M corresponding to $h^{-1}\Gamma_0$, *we assume Γ is free abelian.*

Let ψ: $\tilde{G} \to G$ be the universal covering group. Since Γ is free abelian and $\tilde{G}$ is abelian, there is a homomorphism θ: $\Gamma \to \tilde{G}$ with $\psi \circ \theta =$ identity of Γ.

Consider the commutative diagram

$$\begin{array}{ccc} & & \tilde{G} \\ & \overset{\theta\circ h}{\nearrow} & \downarrow\psi \\ \pi & \xrightarrow[h]{} & G \end{array}$$

Now $\tilde{G}$ is a 1-connected abelian Lie group; hence it is a vector group. For any $t \in R$ there is a homomorphism $u_t\colon \pi \to \tilde{G}$, $\gamma \mapsto t[(\theta \circ h)(\gamma)]$. Therefore $\psi \circ u_t\colon \pi \to G$, $0 \leqslant t \leqslant 1$, is a continuous family of homomorphisms connecting the trivial homomorphism to h. It follows from Lemma 3 that h_b is trivial. Q.E.D.

One particular case of interest occurs when M is an affine manifold, i.e. when M is locally modelled on an affine space so that overlapping charts are identified by affine maps. In that case the tangent bundle of M is a flat vector bundle. From Theorem 1 we deduce the following:

COROLLARY 5. *Let M be an affine manifold whose fundamental group is finitely generated and has a solvable subgroup of finite index. Then M is virtually parallelizable, i.e. M has a finite covering which is parallelizable.*

It is an open question whether every affine manifold is virtually parallelizable. In particular this would imply the famous conjecture (posed by Chern) that the Euler characteristic of a compact affine manifold must vanish. This was proved in the complete case by Kostant and Sullivan [8]. Using examples of Milnor [4], Smillie has constructed examples [5] of compact manifolds having nonzero Euler characteristic, whose tangent bundles are isomorphic to flat vector bundles. However it seems quite difficult to determine whether the *manifolds* can be made affine.

Auslander and Szczarba [7] give an example of a 5-dimensional compact flat orientable Riemannian manifold with solvable fundamental group, whose Stiefel-Whitney class w_2 is nonzero. This shows that not all orientable affine manifolds are *stably* parallelizable.

In [2] Hirsch and Thurston prove the following related result: Let $p\colon E \to M$ be a bundle with compact fiber X and structure group $\Gamma \subset \mathrm{Homeo}(X)$. Suppose that p is induced by a representation $\phi\colon \pi_1(M) \to \Gamma$ and Γ is amenable (e.g. a finite extension of a solvable group). Then the induced homomorphism $p^*\colon H^*(M; R) \to H^*(E; R)$ is injective. This is in some sense a cohomological version of Theorem 2 when the structure group is not a Lie group G as above.

Using considerably deeper methods than these used here, Deligne and Sullivan [1] prove the sharp result that if ξ is a flat *complex* vector bundle (i.e. a $\mathbf{C}^n$-bundle induced by a representation $\phi\colon \pi_1(M) \to \mathrm{GL}(n; \mathbf{C})$) over a compact polyhedron M, then ξ is virtually trivial. As noted above there are flat real vector bundles (see [4]) which have nonzero Euler class and hence are nontrivial over any finite covering.

REFERENCES

1. P. Deligne and D. Sullivan, *Fibrés vectoriels á groupe structural discret*, C. R. Acad. Sci. Paris **281** (1975), 1081–1083.

2. M. Hirsch and W. Thurston, *Foliated bundles, invariant measures and flat manifolds*, Ann. of Math. (2) **101** (1975), 369–390.

3. G. Hochschild, *The structure of Lie groups*, Holden-Day, San Francisco, Calif., 1965.

4. J. Milnor, *On the existence of a connection with curvature zero*, Comment Math. Helv. **32** (1958), 215–223.

5. J. Smillie, *Flat manifolds with nonzero Euler characteristic*, Comment Math. Helv. **52** (1977), 453–456.

6. D. Sullivan, Conversation with W. Goldman, February, 1980.

7. L Auslander and R. H. Szczarba, *Characteristic classes of compact solvmanifolds*, Ann. of Math. (2) **76** (1962), 1–8.

8. B. Kostant and D. Sullivan, *The Euler characteristic of an affine space form is zero*, Bull. Amer. Math. Soc. **81** (1975), 937–938.

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