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DISCONTINUOUS GROUPS AND THE EULER CLASS

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Abstract

Let M be a closed oriented surface, $\chi(M) < 0$, and let π denote its fundamental group. To every homomorphism $\phi: \pi \to PSL(2,R)$ there is associated a bundle E_ϕ of hyperbolic planes over M with a foliation transverse to the fibers; the hyperbolic structure on each fiber is invariant under the local holonomy of the foliation. Taking the Euler number of the underlying oriented disc bundle one obtains a mapping $e: Hom(\pi, PSL(2,R)) \to Z$. We prove that $|e(\phi)|$ assumes its maximum value (which by the work of Milnor and Wood equals $|\chi(M)|$) if and only if ϕ is an isomorphism of π onto a discrete subgroup of $PSL(2,R)$. The proof consists of showing there exists a hyperbolic structure on M with holonomy ϕ by constructing a section of E_ϕ transverse to the leaves of the foliation. The surface M is decomposed into smallest pieces, and the section is constructed over each piece one at a time. To construct the section over the pieces, it is necessary to consider surfaces with boundary; an appropriate relative version of the theorem is stated and proved. Applications of this result include a homological criterion for structural stability of projective actions of π on the circle and a characterization of Anosov foliations among transversely projective codimension-one foliations of circle bundles over M.

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Table of Contents

Introduction

Let M be a closed orientable surface with a Riemann metric of constant negative curvature. The metric defines a system of local isometries of M with the hyperbolic plane X; hence the metric gives M the (local) structure of hyperbolic geometry. Globally, there is an isometry of the universal covering surface $\tilde{M}$ with X. Via this correspondence, the group π of covering transformations of $\tilde{M} \to M$ acts properly discontinuously, freely, and isometrically on X.

It is well known that a necessary and sufficient condition for an isometric action of π on X, i.e. a homomorphism $\phi: \pi \to \mathrm{Isom}(X)$, to arise in this way is that ϕ be an isomorphism onto a discrete subgroup of $\mathrm{Isom}(X)$. Since this is the action of a Fuchsian group, we call such an action Fuchsian.

Let $G \cong PSL(2,\mathbb{R})$ be the group of orientation-preserving isometries of the hyperbolic plane. Corresponding to a homomorphism $\phi: \pi \to G$ we associate a geometric object, namely a hyperbolic foliated disc-bundle, or, equivalently, a projective foliated circle-bundle, over M. A hyperbolic foliated bundle E_ϕ consists of a X-bundle over M together with a foliation transverse to the fibers leaving invariant the hyperbolic structure on the fibers. A projective foliated bundle ∂E_ϕ is the boundary of the natural compactification of a hyperbolic foliated bundle; its typical fiber is the "circle-at-infinity" ∂X which is $\mathbb{RP}^1$ with the structure of projective geometry.

If $\phi \in \mathrm{Hom}(\pi,G)$ is a Fuchsian, the associated projective foliated bundle ∂E_ϕ is the unit tangent bundle $T_1(M)$ with an Anosov foliation. There are two identifications of a unit tangent space over $x \in M$ with the circle-at-infinity, obtained by flowing along a geodesic, either forwards or backwards, approaching asymptotically a unique point of ∂X. The corresponding Anosov foliations of $T_1(M)$ are just the foliations by the stable and unstable manifolds of the geodesic flow.

The hyperbolic foliated bundle E_ϕ corresponding to a Fuchsian ϕ is topologically the tangent disc-bundle. Like the canonical embedding of a smooth manifold in its tangent bundle by the zero-section, a manifold with hyperbolic structure embeds canonically as a section of a hyperbolic foliated bundle. These canonical sections are characterized by the property that they are transverse to the foliation as well as the fibers. Indeed there is natural correspondence between hyperbolic structures on M, Fuchsian actions, and sections of hyperbolic foliated bundles which are transverse to the foliation.

The main result of this paper is that if a representation $\phi: \pi \to G$ determines a hyperbolic foliated bundle E_ϕ which is topologically the tangent bundle then ϕ is Fuchsian. In other words, E_ϕ is the tangent bundle if and only if there exists a transverse (with respect to the foliation) section $M \to E_\phi$.

Giving X an orientation, a hyperbolic foliated bundle becomes an oriented disc bundle. Oriented disc bundles over M are classified up to isomorphism by their Euler class (or Euler number if M is oriented) which lies in $H^2(M;\mathbb{Z}) \cong \mathbb{Z}$. If $\phi \in \mathrm{Hom}(\pi,G)$ then the Euler class $e(\phi)$ of E_ϕ is an invariant of the representation ϕ. Hence the mapping

$$e: \mathrm{Hom}(\pi,G) \longrightarrow \mathbb{Z}$$

expresses the oriented bundle type of a hyperbolic foliated bundle, while forgetting its foliation. It amounts to enlarging the structure group from G with the discrete topology to G with the usual topology.

Theorem A. If $\phi \in \mathrm{Hom}(\pi, PSL(2,\mathbb{R}))$, then $|e(\phi)| \leq |\chi(M)|$; equality holds, $e(\phi) = \pm\chi(M)$ if and only if $\phi: \pi \to PSL(2,\mathbb{R})$ is an isomorphism onto a discrete subgroup.

A more geometrically stated version of Theorem A is:

Theorem B. (i) Let $(E_\phi, \mathcal{F})$ be a hyperbolic foliated bundle over M. Then E_ϕ is topologically the tangent disc bundle of M if and only if it admits a section transverse to $\mathcal{F}$.

(ii) Let $(\partial E_\phi, \mathcal{F})$ be a projective foliated bundle over M. Then ∂E_ϕ is topologically the unit tangent bundle of M if and only if $\mathcal{F}$ is an Anosov foliation of $T_1(M)$.

The inequality $|e(\phi)| \leq |\chi(M)|$ is due to J. Milnor [39] and J. Wood [56]. They prove this inequality is sharp; Theorem A is therefore a converse to the sharpness of this inequality.

One consequence of Theorem A is that it provides necessary and sufficient conditions for an isometric action on the hyperbolic plane to be properly discontinuous (i.e. like a group of deck transformations in a covering space) in terms of a characteristic class of the action. The Euler class is the only "obstruction" for an action to be discontinuous in the sense that $\phi \in \mathrm{Hom}(\pi, G)$ is Fuchsian provided that $|e(\phi)|$ is maximized.

In Anosov [1] it is proved that the geodesic flow on $T_1(M)$ is structurally stable. From this follows the structural stability of the corresponding Anosov foliations, and hence all Fuchsian actions of π

on the circle $S^1_\infty = \mathbb{RP}^1$. We will prove a converse: if $\phi: \pi \to SL(2,\mathbb{R})$ defines a projective action on S^1_∞ then this action is structurally stable if and only if ϕ is Fuchsian. Hence maximizing the Euler class also characterizes structural stability for a certain class of projective actions on the circle. Indeed, $|e(\phi)| = |\chi(M)|$ is equivalent just to ϕ being <u>approximated</u> by structurally stable actions.

From the continuity of $e: \mathrm{Hom}(\pi,G)/G \to \mathbb{Z}$ we derive the well-known fact that the Fuchsian actions form a closed subset. For reasons holding more generally, this set is also open. Since the orientation-preserving Fuchsian actions form a space which is a principal $PSL(2,\mathbb{R})$-bundle over a $-3\chi(M)$-cell (Teichmuller space), it follows that $e^{-1}(\chi(M))$ and $e^{-1}(-\chi(M))$ are each (3g-3)-cells and connected components of $\mathrm{Hom}(\pi,G)/G$ (g denotes the genus of M). With regard to the other components of $\mathrm{Hom}(\pi,G)$ we propose the following conjecture.

<u>Conjecture</u>. The fibers of the map $e: \mathrm{Hom}(\pi,G) \to \mathbb{Z}$ are connected. Hence the connected components are the 4g-3 sets $e^{-1}(n)$, $|n| \leq 2g-2$.

In general we develop the theme that the global (e.g. connectedness) properties of spaces like $\mathrm{Hom}(\pi,G)$ for G any Lie group and π finitely presented are reflected in the topological properties (as described by characteristic classes) of bundles with structure group G over spaces with fundamental group π (for example, $M = K(\pi,1)$). In this direction we prove that if π is the fundamental group of a closed orientable surface then

<u>Proposition C</u>. $\mathrm{Hom}(\pi,PSL(2,\mathbb{C}))$ has two connected components. $\mathrm{Hom}(\pi,SL(2,\mathbb{C}))$ is connected.

This paper is organized as follows. The first chapter contains background on geometric structures (sometimes called "integrable," "rigid," "flat," "locally flat," etc.) and their relation to foliated bundles. In particular we characterize (G,X)-structures by their developing sections, sections to their tangent G-foliated X-bundles. Early in this chapter we define geometric structures transverse to foliations and we adopt the point of view that a geometric structure on a manifold or transverse to a foliation is just a section of a foliated bundle transverse to the foliation.

The second chapter is background on hyperbolic geometry, projective geometry and $PSL(2,\mathbb{R})$. In addition to establishing notation and terminology we suggest a picture of $PSL(2,\mathbb{R})$ (which I learned from Sullivan) in which we interpret some of our later techniques. In particular, to get a model for a $PSL(2,\mathbb{R})$-structure over a curve in M, we desire a uniform way of lifting elements of $PSL(2,\mathbb{R})$ to the universal cover $\widetilde{PSL}(2,\mathbb{R})$; although we must sacrifice continuity, these lifts are still a useful tool in defining "special sections," the sort of sections we begin with over curves, before extending them over M.

In the appendix to §2 we classify projective structures on the circle (originally due to Kuiper [35]). In addition to combining the abstract theory of §1 with the discussion of $PSL(2,\mathbb{R})$ in §2, these structures appear naturally in our study of hyperbolic structures on surfaces with boundary as "ideal sections" over the boundary. Algebraically, isotopy classes of $\mathbb{R}P^1$-structures on the circle correspond to elements of the universal cover group $\widetilde{SL}(2,\mathbb{R})$.

In §3 we address the question of hyperbolic structures on surfaces. These structures, of course, have been studied extensively as the subject of Fuchsian groups. The first part (3.1-3.2) reviews well-known properties of Fuchsian actions. In 3.3 we discuss the variety of representations. In 3.5 we unveil our plan to study the global properties of representation spaces by means of the characteristic classes of the associated foliated bundles.

The Euler class of a hyperbolic foliated bundle, e: $\mathrm{Hom}(\pi,G) \to \mathbb{Z}$ is discussed in 3.6 where we state Theorem A. From it we deduce (3.7) that Fuchsian actions constitute two connected components of $\mathrm{Hom}(\pi,G)$. Its definition as an obstruction class appears in 3.9. Proposition C is proved in 3.10 where we characterize components of $\mathrm{Hom}(\pi,\mathrm{PSL}(2,\mathbb{C}))$ by W_2. This allows us to calculate the irreducible components of the (quasiprojective) algebraic variety $\mathrm{Hom}(\pi,\mathrm{PSL}(2,\mathbb{R}))$. While there are exactly two irreducible components, there are at least $4g-3$ topological components. We use Proposition C to study generic properties of representations $\phi \in \mathrm{Hom}(\pi,G)$ in the appendix to §3. One consequence (which is actually stronger than what we actually need for the proof of Theorem A is that isomorphisms $\phi: \pi \to G$ are dense in all homomorphisms $\pi \to G$, at least for the choices $G = \mathrm{SL}(2,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{C})$, $\mathrm{SU}(2)$. Finally in 3.15 this genericity result is used to characterize Fuchsian actions on the circle as the only structurally stable projective actions of π on the circle (which lift to $\mathrm{SL}(2,\mathbb{R})$).

In 3.16 we introduce another point of view concerning the Euler class: namely as an area for "singular" hyperbolic structures. This leads us to conjecture a generalization of Theorem A to other semi-simple groups G than $\mathrm{PSL}(2,\mathbb{R})$, where the Euler class is replaced by a characteristic class of the foliation derived from an invariant volume. The Euler class occupies a somewhat special position in that it is both an obstruction class and a "volume class."

The last three chapters discuss the proof of Theorem A. Basically, we shall use the assumption that $e(\phi) = \pm\chi(M)$ to construct a section f of E_ϕ transverse to the foliation. First we decompose M into "smallest" subsurfaces M_i with $\chi(M_i) = -1$, and state a relative version of the Milnor-Wood inequality over the M_i, i.e. $|e(\phi|_{M_i})| \leq 1$. Then it should follow from an additivity property of the Euler class that $|e(\phi)| \leq |\chi(M)|$ (the Milnor-Wood inequality). More importantly, equality is reached for M if and only if equality is reached for each M_i. We prove a relative version of the sharpness statement, Theorem B, "if $e(\phi) = \pm\chi(M)$ then E_ϕ admits a transverse section" over each M_i, and then glue together the sections over the M_i to produce a global transverse section $M \to E_\phi$.

However, to do this requires some technical definitions. We must define boundary data both for the relative Euler class and for the transverse sections we construct. This is carried out in §4. We call a section σ of a projective foliated bundle ∂E_ϕ an "ideal section"; then the relative Euler class $e(\phi;\sigma)$ of E_ϕ with respect to an ideal section σ over ∂M is the obstruction to extending σ to an ideal section over all of M.

Ideal sections which are transverse to the foliation carry natural real-projective structures. Real projective structures on the circle are shown (Appendix, §2) to correspond to elements of the universal covering group $\widetilde{PSL}(2,\mathbb{R})$; different choices of lifts correspond to different homotopy classes of ideal sections.

Since we desire a numerical estimate on the relative Euler class, we must be careful in our choice of ideal section; thus we define "special ideal sections" which are to depend only on the holonomy around the given boundary component. Special sections are "special" only in that the relative Milnor-Wood inequality is true using them. Since there is no continuous section $PSL(2,\mathbb{R}) \to \widetilde{PSL}(2,\mathbb{R})$, special sections cannot even be chosen to depend continuously on the holonomy.

We also need suitable boundary conditions for the sections we construct. Since we plan to glue the sections over the pieces together, ideal sections are unsatisfactory boundary conditions. Instead we realize special ideal sections as "special interior sections." For example if c is a simple closed curve in M and $\phi(c)$ is hyperbolic, a special interior section may take c to the $\phi(c)$-invariant geodesic. More generally, special interior sections take values on invariant circles, horocycles, and equidistant curves.

Chapter 5 is the heart of the proof. There we attempt a detailed analysis of hyperbolic foliated bundles over a pair-of-pants (i.e. sphere minus three discs). The main results are summarized in Theorem 5.1. We proceed to classify such bundles, or equivalently, representations $\phi \in \mathrm{Hom}(\pi,G)$, by associating to ϕ a geometric object, a triangular configuration in $\mathbb{R}P^2$. This configuration completely determines ϕ, at least as long as ϕ satisfies the generic condition that $\phi(\pi)$ is not solvable. This condition means that $\phi(\pi)$ has no fixed point in $\mathbb{R}P^2$, where the hyperbolic plane X is represented as a conic in $\mathbb{R}P^2$ (the Klein model). This triangle Δ is defined by the property that if I_{BC}, I_{CA}, and I_{AB} are projective involutions fixing the respective sides of Δ and leaving the conic ∂X invariant, then

$$\phi(A) = I_{CA}I_{AB}$$
$$\phi(B) = I_{AB}I_{BC}$$
$$\phi(C) = I_{BC}I_{CA}$$

We say that "ϕ has been factored into reflections (or symmetries)," depending on whether the involutions reverse or preserve the orientation of X. Using the triangle Δ a transverse section is constructed by truncating Δ to form a right hexagon, assuming certain geometric conditions on the triangular configuration. Finally, we compute the relative Euler class $e(\phi;\sigma)$ with respect to the special ideal section σ over ∂M by constructing deformations of ϕ to known examples. Our conclusion is that $e(\phi;\sigma)$ must be -1, 0, or +1, and if $e(\phi;\sigma) = -1$, then generically E_ϕ admits an orientation-preserving transverse section which restricts to a special interior section over ∂M.

Finally in §6 we prove Theorem A. We use Theorem 5.1 to find a section $f: M \to E_\phi$ which is transverse over $\cup \operatorname{int} M_i$. To show that this section is actually transverse over M, we use the fact that the $f|_{M_i}$ are either all orientation-preserving or orientation-reversing. Consequently f cannot suffer a "fold" along some component of ∂M_i, which implies f is transverse. However, the section is constructed for generic ϕ only, so it is now necessary to show that the generic approximation was actually unnecessary. This follows from an analysis on how the transverse sections are constructed: the approximation was necessary only if, for some component $C = \partial M_i \cap \partial M_j$ $\phi(C)$ is elliptic, but we show that unless $\phi(C)$ is hyperbolic there cannot exist transverse sections over $M_i \cup M_j$ with the correct boundary behavior. This last step, incidentally, may be avoided by the use of the "known" fact that Fuchsian actions form a closed set in $\mathrm{Hom}(\pi,G)$; however we prefer a more elementary proof and in any case deduce this fact as a corollary to our main theorem.

§1. Geometric Structures and Sections to Foliated Bundles

This chapter is a brisk introduction to the theory of (G,X)-structures on manifolds and transverse to foliations. Similar treatments can be found in Kulkarni [36], Thurston [51] and Sullivan and Thurston [54]. We develop the point of view that such (G,X)-structures are canonically identified with sections of certain "foliated bundles." These foliated bundles themselves possess transverse geometric structures, since the foliations are transverse to the fibers which carry the geometric structure.

In this chapter alone do we apply the following general notation: Let X be a connected real analytic manifold with an analytic transformation group G which acts strongly effectively on X: two transformations in G which agree on a nonempty open subset of X must be identical. Later on we will specialize to $G = PSL(2,\mathbb{R})$ and X is either the hyperbolic plane or $\mathbb{RP}^1$.

Definition 1.1. A (G,X)-structure on a manifold M is defined by an open covering $\{U_\alpha\}$ of M, and diffeomorphisms $\psi_\alpha: U_\alpha \longrightarrow \psi_\alpha(U_\alpha) \subset X$ such that whenever $U_\alpha \cap U_\beta \neq \emptyset$, the mapping $\psi_\alpha \circ \psi_\beta^{-1}$ extends from $\psi_\beta(U_\alpha \cap U_\beta)$ to a transformation $g_{\alpha\beta}: X \to X$ in G. Such a collection $\{(U_\alpha, \psi_\alpha)\}$ is called a (G,X)-atlas. A (G,X)-structure, then, is a maximal (G,X)-atlas.

A manifold with a (G,X)-structure will be called a (G,X)-manifold. If M and N are (G,X)-manifolds, then $f: M \to N$ is a (G,X)-map if and only if for every coordinate chart (U_α, ψ_α) in the maximal atlas for M, and every chart (V_β, ϕ_β) for N, with $f(U_\alpha) \cap V_\beta \neq \emptyset$, then $\phi_\beta \circ f \circ \psi_\alpha^{-1}$ extends to a transformation $X \to X$ in G.

Definition 1.2. For every classical geometry, we obtain a category of (G,X)-manifolds. The simplest examples occur when G is the group of isometries of a homogeneous Riemannian manifold X; then a (G,X)-manifold M is just a Riemannian metric locally isometric to X. Taking $X = S^n, \mathbb{R}^n$, or $\mathbb{H}^n$, we obtain spherical, Euclidean, and hyperbolic structures. For these cases (G,X)-structures are just metrics of constant curvature.

(G,X)	Geometric Structure
$X = \mathbb{R}^n, \quad G = \mathrm{Aff}(\mathbb{R}^n)$	affine
$X = \mathbb{RP}^n, \quad G = \mathrm{PSL}(\mathbb{R}^{n+1})$	(real)-projective
$X =$ quadric in $\mathbb{RP}^{n+2}, \quad G = SO(n+1,1)$	conformal

(We shall sometimes refer to, e.g. real projective structures on a 1-manifold as an $\mathbb{RP}^1$-structure, etc.) Notice that $\mathbb{RP}^1$-structures are also (SO(2,1),quadric)- (i.e. flat conformal) structures.

By passing to the universal covering of X (and hence also a covering of G), we may as well assume, in the study of (G,X)-structures, that X is simply connected. Following Thurston [54] §3, we say that a (G,X)-manifold is complete if M is obtained as X/Γ, where Γ is a discrete subgroup of G which acts "freely and properly discontinuously" (i.e. as covering transformations) on X. The condition "properly discontinuous" is most neatly expressed as a discrete group acting properly. (An action of G on X is proper if and only if the map

$$G \times X \to X \times X$$
$$(g,x) \mapsto (gx,x)$$

is proper.)

If X is a complete metric space upon which G acts by isometries, then every (G,X)-structure on a compact manifold is complete.

1.3 We now describe an object "tangent" to a (G,X)-manifold M. Use the transition functions $\{g_{\alpha\beta}\}$ to put together a fiber bundle E with fiber X and structure group G over M in the usual manner: $E|_{U_\alpha} = U_\alpha \times X$ and E is obtained from $\bigcup_\alpha (U_\alpha \times X \times \{\alpha\})$ by identifying (u,x,α) to $(u,g_{\alpha\beta}x,\beta)$ whenever $U_\alpha \cap U_\beta$ is nonempty. Since the "change-of-coordinates" $U_\alpha \cap U_\beta \longrightarrow G$ is (locally) constant, it follows that the structure group G reduces to G^δ, the group G with the discrete topology. This is well-known to be equivalent to the sets $U_\alpha \times \{x\}$, $x \in X$ defining a foliation F of E. Clearly F is transverse to the fibration $E \to M$.

Just as the tangent bundle to a smooth manifold admits a canonical section (the zero section), which completely specifies the differentiable structure, there is also a canonical section $M \to E$ which specifies the (G,X)-structure on M. This section, which we call the <u>developing section</u>, is defined locally as the map

$$U_\alpha \longrightarrow U_\alpha \times X = E|_{U_\alpha}$$
$$u \longmapsto (u, \psi_\alpha u) \ .$$

(The relation $g_{\alpha\beta} \circ \psi_\beta = \psi_\alpha$ insures that the local sections over U_α extend over M.) The developing section is the "graph" of a (G,X)-structure; it is clearly transverse to F as well as the fibration $E \to M$. When M has a complete (G,X)-structure, the developing section is a "cross-section" to F, i.e. every leaf intersects $f(M)$ exactly once. In that case M is actually represented as the leaf space E/F.

The (G,X)-bundle E to M is "tangent" to the (G,X)-manifold in the following sense. Let N be a sufficiently small tubular neighborhood of the developing section in E. Then as a topological disc-bundle, N is isomorphic to a tubular neighborhood of the diagonal $M \subset M \times M$, i.e. N is topologically the tangent disc-bundle. In other words, the normal bundle of the developing section defines the tangent microbundle. (See Milnor and Stasheff [40].) Even when $X \approx \mathbb{R}^n$ (e.g. hyperbolic space), calculations are complicated by the absence of a linear vector-bundle structure on E (except, of course, for affine structures).

1.4 Let M be a (G,X)-manifold with tangent foliated (G,X)-bundle $(E,\mathcal{F})$. Transverse to $\mathcal{F}$ are the fibers X with the "universal" (G,X)-structure (the (G,X)-structure with one chart, the identity map $X \to X$). Since our later work will have applications to the general subject of geometric structures transverse to a foliation, we make the following definition (compare Thurston [54], Haefliger [24]).

<u>Definition</u>. Let $(M,\mathcal{F})$ be a foliated manifold. A <u>(G,X)-structure transverse</u> to $\mathcal{F}$ is a maximal atlas $\{(U_\alpha,\psi_\alpha)\}$ where $\{U_\alpha\}$ is an open cover of M, and the charts $\psi_\alpha \colon U_\alpha \to X$ are submersions such that the leaves of $\mathcal{F}|_{U_\alpha}$ are the preimages $\psi_\alpha^{-1}(x)$, $x \in X$ and such that whenever $U_\alpha \cap U_\beta \neq \emptyset$, there is some $g_{\alpha\beta} \in G$ such that $= g_{\alpha\beta}\psi_\beta$.

Note that the codimension of $\mathcal{F}$ must equal the dimension of X. If $\mathcal{F}$ is the foliation of M by points, then a (G,X)-structure transverse to $\mathcal{F}$ is just a (G,X)-structure on M.

1.5 Suppose that F is a foliation of M having a (transverse) (G,X)-structure. Suppose that $f: V \to M$ is a map transverse to F. The induced foliation on V then carries a natural (G,X)-structure too. When $\dim V = \dim X$, then f automatically defines a (G,X)-structure on V. The foliation F defines an equivalence relation on maps f as above, whereby two maps f_1 and f_2 are equivalent if and only if there is a map $F: V \times [1,2] \longrightarrow M$ such that $F|_{V\times\{i\}} = f_i$ and the sets $F(\{v\} \times [1,2])$ are contained in leaves of F. One can actually define (G,X)-structures transverse to F as assignments of (G,X)-structures on V to maps $V \to M$ transverse to F such that F-equivalent maps are assigned isomorphic (G,X)-structure.

1.6 Now we come to the key definition. For different treatments, see Hirsch-Thurston [29], Lawson [37], Kamber-Tondeur [34], and Wood [56].

<u>Definition</u>. A <u>foliated</u> (G,X)-<u>bundle</u> (or "G-foliated X-bundle") over M is a pair (E,F) where E is a fiber bundle over M with fiber X and F is a (G,X)-foliation over E transverse to the fibers. By the preceding remark, each fiber has a (G,X)-structure; therefore we require the (G,X)-structure on each fiber to be the universal (G,X)-structure on X, i.e. the one with only one chart.

Notice that this last compatibility condition forces the fiber bundle $E \to M$ to have structure group G, given the discrete topology. Indeed one may easily define a foliated (G,X)-bundle to be a fiber bundle with fiber X and structure group G, where G is given the discrete topology (compare Steenrod [48]).

Examples

(1) The product bundle $M \times X \to M$ has a foliation with leaves $M \times \{x\}$; in this case the structure group can be reduced to $G = \{1\}$.

(2) If M has a (G,X)-structure then its tangent (G,X)-bundle (defined in 1.2) is a foliated (G,X)-bundle.

(3) Suppose that $\pi = \pi_1(M)$ and $\phi\colon \pi \to G$ is any homomorphism. Then ϕ defines an action of π on $\tilde{M} \times X$ which, on the first factor, is by deck transformations of the universal covering $\tilde{M} \to M$, and, on the second factor by $\pi \xrightarrow{\phi} G \hookrightarrow \mathrm{Diff}(X)$. The quotient, which we denote by E_ϕ, fibers over M by $[(\tilde{m},x)] \mapsto [\tilde{m}]$ with fibers $\{\tilde{m}\} \times X$. On the other hand the foliation with leaves $[\tilde{M} \times \{x\}]$ do not necessarily form a fibration since ϕ need not define a free proper action of π on X. We denote this foliation by $\mathcal{F}_\phi$. Clearly $(E_\phi, \mathcal{F}_\phi)$ defines a foliated (G,X)-bundle over M. We call $(E_\phi, \mathcal{F}_\phi)$ the foliated (G,X)-bundle associated to ϕ, and ϕ the holonomy homomorphism of the foliated (G,X)-bundle $(E_\phi, \mathcal{F}_\phi)$.

As is well-known, the last case (3) includes all foliated (G,X)-bundles. To see this, let $(E,\mathcal{F})$ be a foliated (G,X)-bundle over M. Choose a base-point x_0 and a homeomorphism $\eta\colon E_{x_0} \to X$ of the fiber of E over x_0 with X. We use η and the foliation $\mathcal{F}$ to define the holonomy homomorphism $\phi\colon \pi_1(M;x_0) \to G$ as follows. First note that in E, a point is completely determined by which fiber and which leaf it lies, at least locally. Therefore if $U \subset M$ is a sufficiently small open set, there are well-defined canonical identifications $\eta_{xy}\colon E_x \to E_y$ of nearby fibers, satisfying the requirement that $u \in E_x$ and $\eta_{xy}(u)$ lie in the same leaf of $\mathcal{F}$.

Hence we may cover a closed loop γ by such open sets and obtain a sequence $\eta_{x_0x_1}, \eta_{x_1x_2}, \ldots, \eta_{x_kx_0}$, of canonical identifications, $x_0, x_1, \ldots, x_k \in \gamma$; the composite $\eta_{x_0x_1}, \ldots, \eta_{x_kx_0}$ is the holonomy $\phi(\gamma)$. It is not hard to prove that the holonomy defines a homomorphism $\pi_1(M;x_0) \longrightarrow G$.

The holonomy is the only obstruction to "trivializing" $(E,\mathcal{F})$, i.e. representing $(E,\mathcal{F})$ as the product foliated bundle. For if the holonomy is trivial, there is no ambiguity to finding a global trivialization $E \to X$ which collapses leaves to single points.

Thus when pulled back to the universal cover $\tilde{M}$ of M, the foliated bundle is a product and the fiber X is just the leaf space. To pass back down to M, one needs to consider how the group π of deck transformations identifies distinct leaves. This identification is precisely the one arising from the action ϕ, so $(E,\mathcal{F})$ is indeed $(E_\phi, \mathcal{F}_\phi)$.

As the only ambiguity in defining ϕ arises from $\eta\colon E_{x_0} \longrightarrow X$, we may compose η with any transformation g in G; the corresponding holonomy is then $\phi^g\colon \gamma \longmapsto g\phi(\gamma)g^{-1}$. A more drastic kind of change arises when we compose with a homeomorphism $h\colon M \to M$; then the new holonomy is $\phi \circ h_*$, where h_* is the induced map on $\pi_1(M;x_0)$. Let $\mathrm{Aut}_M(\pi)$ denote the subgroup of $\mathrm{Aut}(\pi)$ $(\pi = \pi_1(M;x_0))$ induced by homeomorphisms $h \in \mathrm{Homeo}(M)$; then we consider the left action of $\mathrm{Aut}_M(\pi)$ on the space $\mathrm{Hom}(\pi,G)$ of homomorphisms $\pi \to G$, and the right action of G defined by ϕ^g.

Theorem 1.7. There is a canonical bijection between isomorphism classes of foliated (G,X)-bundles over M and elements of the orbit space $(\mathrm{Aut}_M(\pi))\backslash\mathrm{Hom}(\pi,G)/G$.

The space $(\mathrm{Aut}_M(\pi))\backslash\mathrm{Hom}(\pi,G)/G$ of foliated (G,X)-bundles is often called the moduli space; the space $\mathrm{Hom}(\pi,G)/G$ is sometimes called the deformation space. Elements of the deformation space correspond to foliated (G,X)-bundles over M, together with a marking of M, i.e. a fixed isomorphism $\pi_1(M;x_0) \to \pi$. These spaces will be discussed further in §3.3.

1.8 Let M be a (G,X)-manifold with marking $\pi \cong \pi_1(M;x_0)$ and tangent foliated (G,X)-bundle $(E,\mathcal{F})$. By 1.5 there exists $\phi \in \mathrm{Hom}(\pi,G)$, well defined up to conjugacy in G, with $E = E_\phi$, $\mathcal{F} = \mathcal{F}_\phi$; we call ϕ the holonomy of the (G,X)-structure on M. A central problem is to determine which $\phi \in \mathrm{Hom}(\pi,G)$ arise as holonomy of (G,X)-structures on M.

Theorem 1.8. Let $\phi \in \mathrm{Hom}(\pi,G)$. Then there is a natural bijection between (G,X)-structures on M with holonomy ϕ and sections of E_ϕ transverse to $\mathcal{F}_\phi$.

Proof. Consider the map which assigns to a (G,X)-structure its developing section. We show that this map is bijective, in the sense that every $\mathcal{F}_\phi$-transverse section to E_ϕ is the developing section of some (G,X)-structure.

Let $f\colon M \to E_\phi$ be an $\mathcal{F}_\phi$-transverse section. By 1.5, f defines a (G,X)-structure on M. Explicitly, the charts are defined as follows. Cover $f(M)$ by sets U'_α open in E_ϕ such that $\mathcal{F}_\phi|_{U'_\alpha}$ is induced by a foliation chart $\psi'_\alpha\colon U'_\alpha \to X$. The (G,X)-structure on M has for charts $\psi_\alpha\colon U_\alpha \to X$, $\psi_\alpha = \psi'_\alpha \circ f$ where $U_\alpha = f^{-1}(U_\alpha)$. Clearly f is the developing section for the (G,X)-structure on M (see Fig. 1.1).

We shall say that two (G,X)-structures are <u>isotopic</u> if there is a diffeomorphism of M, isotopic to the identity, which carries the developing section of one (G,X)-structure to the developing section of the other. By definition isotopic (G,X)-structures have the same holonomy, which is really only well-defined up to conjugacy in G. Conversely, it is not difficult to prove ([17] or [54], 5.1) that two "nearby" (G,X)-structures are isotopic if they have the same holonomy.

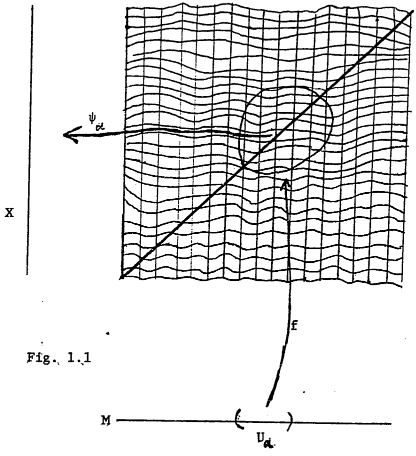


Fig. 1.1

Remark. Theorem 1.8 directly generalizes to (G,X)-structures transverse to a foliation. The "tangent foliated bundle" is topologically the normal bundle of the foliation, in a neighborhood of the developing section.

1.9 Suppose $(E_\phi, \mathcal{F}_\phi)$ is a foliated (G,X)-bundle over M with holonomy ϕ. Since $E_\phi = \tilde{M} \times_\phi X$ (as in 1.5), a section $f \colon M \to E_\phi$ is just a map $\tilde{f} \colon \tilde{M} \to X$ which is ϕ-equivariant, i.e. if $g \in \pi_1(M)$ then the following diagram commutes:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \phi(g)} \\ \tilde{M} & \xrightarrow{\tilde{f}} & X \end{array}$$

If f is the developing section of a (G,X)-structure, then the map $\tilde{f}$ is called the developing map of this structure. In some instances it is more convenient to consider the developing map as an equivariant local diffeomorphism although in the present work it is more conceptual to consider the sections themselves.

1.10 Now we state the Basic Problem in the study of (G,X)-structures:

Basic Problem. Given $\phi \in \mathrm{Hom}(\pi_1(M),G)$ classify all (G,X)-structures on M (if any) with holonomy ϕ. Equivalently, classify all sections of the (G,X)-bundle E_ϕ which are transverse to the (G,X)-foliation $\mathcal{F}_\phi$.

We shall "solve" the Basic Problem when X is the hyperbolic plane and G its group of orientation-preserving isometries. Since X is contractible all sections of an X-bundle E_ϕ are homotopic. In general one should also ask for the existence of a transverse section in a given homotopy class of sections. Furthermore since every hyperbolic structure on a closed manifold M is complete, there is only one structure (up to isotopy) with a given holonomy ϕ. We find necessary and sufficient conditions on $(E_\phi, \mathcal{F}_\phi)$ to admit a transverse section.

In 1.2 we mentioned that one necessary condition is that E_ϕ be the tangent disc bundle. This condition is independent of $\mathcal{F}_\phi$ and the main result of this paper is that this necessary condition is also sufficient.

1.11 As in 1.8, a (G,X)-structure on M is just a transverse section to a foliated (G,X)-bundle over M. An arbitrary section of a foliated (G,X)-bundle $(E_\phi, \mathcal{F}_\phi)$ may be regarded as a singular (G,X)-structure on M, where the singularities of the (G,X)-structure occur where the section fails to be transverse to $\mathcal{F}_\phi$. (Singular (G,X)-structures play the analogous role for (G,X)-structures that Haefliger structures play for foliations.)

The most singular kind of (G,X)-structure arises from a "constant" section. Namely, a stationary point $x \in X$ for an action $\phi \in \mathrm{Hom}(\pi, G)$ determines a section which is also a leaf. Such a section which lies in a leaf is called constant. The associated developing map is a constant map.

A necessary condition for a singular (G,X)-structure f to be homotopic to a nonsingular (G,X)-structure is that a tubular neighborhood of f(M) in E is the tangent microbundle. When M is open, the Smale-Hirsch-Phillips-Haefliger-Gromov theorem (see Haefliger [23]) implies that this necessary condition is also sufficient. Of course, when M is the interior of a manifold with boundary, one has very little control of the mapping of ∂M.

When M is closed, the homotopy-theoretic necessary condition is quite far from being sufficient. One may try to remedy this by allowing particularly nice singularities, such as folds. We say that a singular (G,X)-structure $f: M \to E_\phi$ <u>has folds along</u> a closed submanifold $V \subset M$ if and only if the associated developing map $D_f: \tilde{M} \to X$ has folds on $\tilde{V} \subset \tilde{M}$ (i.e. there are C^0 coordinates $(x_1,\ldots,x_n)$ in a neighborhood of $\tilde{v} \in \tilde{V}$ such that $D_f(x_1,\ldots,x_n) = (x_1^2,x_2,\ldots,x_n)$ and $\tilde{V}$ is the set $(0,x_2,\ldots,x_n)$. See Fig. 1.2 for an $\mathbb{R}$-structure on S^1 folded along S^0, with trivial holonomy.

For mappings of closed manifolds whose only singularities are folds, Èliašberg [10] has extended classical immersion theory. His methods extend to the study of sections of foliated bundles. For example, if $\dim M = \dim X = 2$ and f is a singular (G,X)-structure, then f is homotopic to one whose worst singularities are folds if and only if the normal bundle of $f(M)$ in E_ϕ has vanishing Stiefel-Whitney class w_2. Again, one has very little geometric control over the mapping of the folding locus.

Our immediate interest in folds, however, is that our method for constructing sections will frequently give "folded sections" (sections whose only singularities are folds) and we wish to detect these singularities. Let $f: M \to E_\phi$ be a (G,X)-structure whose only singularities are perhaps folds along components of a closed submanifold V of codimension-one. Let us assume that E_ϕ is an orientable X-bundle and that we have chosen an orientation on X. If $f: M \to E_\phi$ is $\mathcal{F}$-transverse then it makes sense to speak of whether f preserves or

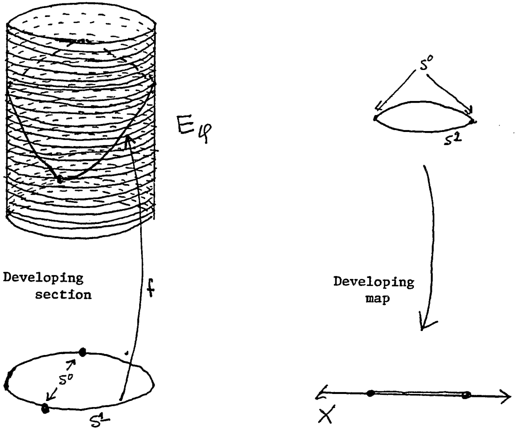


Fig. 1.2 A folded structure on the circle

reverses orientation. If f has folds in V then $f|_{M-V}$ is transverse.

Proposition 1.12. If for every component M_i of $M-V$, the section $f|_{M_i}$ is orientation-preserving (resp. orientation-reversing), then f is transverse. Otherwise, f suffers a fold along $\partial M_i \cap \partial M_j$ where $f|_{M_i}$ preserves orientation and $f|_{M_j}$ reverses orientation. In terms of a developing map for f, there is a fold along V if and only if a collar neighborhood of V in M_i and M_j get mapped to the same side of the developing image of V.

Proof. Consider a developing map F for f. It suffices to consider a tubular neighborhood of V. The theorem then follows from the elementary topological fact that if h is any map $D^{n-1} \times [-1,1] \to \mathbb{R}^n$ which is a homeomorphism on $D^{n-1} \times [-1,0)$ and on $D^{n-1} \times (0,1]$, then h is a homeomorphism if and only if $F|_{D^{n-1}\times(0,1]}$ and $F|_{D^{n-1}\times[-1,0)}$ both preserve or reverse orientation; in that case $F(D^{n-1} \times (0,1])$ and $F(D^{n-1} \times [-1,0))$ lie on opposite sides of $F(D^{n-1} \times \{0\})$; otherwise F suffers a fold on $D^{n-1} \times \{0\}$.

§2. Hyperbolic Geometry and the Projective Geometry of PSL(2,ℝ)

We shall mainly be interested in the case where X is the hyperbolic plane and G is the group of its orientation-preserving isometries. In this chapter we review some facts about hyperbolic geometry which are used later in the sequel. We do not discuss discontinuous groups acting on the hyperbolic plane in this chapter; see §3 for a brief review. Since most of this material is quite standard, we refer the reader to one of the many existing treatments for details, e.g. Harvey [25] §2, Thurston [54], Siegel [45], Coxeter [7], and Busemann and Kelly [5].

One of our goals will be a global picture of $PSL(2,\mathbb{R})$ and how it relates to hyperbolic geometry. We shall draw a picture of $PSL(2,\mathbb{R})$, and in the appendix give a geometric interpretation of elements of the universal covering group $\widetilde{PSL}(2,\mathbb{R})$ as projective structures on the circle.

2.1 There are various models for hyperbolic geometry. One model uses $X = \{z \in \mathbb{C}: \operatorname{Im} z > 0\}$ the upper half-plane with metric $y^{-2}(dx^2+dy^2)$ which has curvature (-1). Geodesics are then circular arcs orthogonal to $\partial X = \mathbb{R} \cup \{\infty\}$. The group of (orientation-preserving) isometries is equal to the group of (orientation-preserving) conformal transformations of the sphere $\mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$ leaving $\mathbb{R} \cup \{\infty\} = \mathbb{R}P^1$ invariant. Hence G is the projective linear group $PSL(2,\mathbb{R})$. The homomorphism $SL(2,\mathbb{R}) \to PSL(2,\mathbb{R})$ has kernel $\{\pm I\}$ so $SL(2,\mathbb{R})$ is a double covering of G, and every element of G can be represented in exactly two ways.

Another model, which we shall use frequently, is the projective model due to Klein. Let $\{[x,y,z] \in \mathbb{RP}^2 : -x^2+y^2+z^2 = 0\}$ be a conic in $\mathbb{RP}^2$. Its complement has two components, one of which is the disc $X = \{[x,y,z]: -x^2+y^2+z^2 < 0\}$ and the other the Moebius band $X^* = \{[x,y,z]: -x^2+y^2+z^2 > 0\}$. The original conic is $\partial X = \partial X^*$ and the group of projective transformations leaving X, X^* or ∂X invariant is $SO(2,1)$. Following Hilbert there is an $SO(2,1)$-invariant hyperbolic metric, determined intrinsically by the domain X in projective space [5]. The identity component of $SO(2,1)$ is the group of orientation-preserving isometries and is isomorphic to $PSL(2,\mathbb{R})$. In an analogous way, X^* has a canonical $SO(2,1)$-invariant Lorentz metric of constant nonzero curvature.

In all of these models, the boundary ∂X plays an important role. Intrinsically the "circle-at-infinity" is defined as equivalence classes of parallel (i.e. forward asymptotic) geodesic rays; two oriented geodesics are parallel (see Fig. 2.1) if corresponding points remain a bounded distance apart (in that case the distance necessary approaches zero). The "circle-at-infinity" ∂X is sometimes denoted by S^1_∞ and its points are often called <u>ideal points</u>.

An isometry of X defines a projective transformation of ∂X. This is easily seen in the upper half-plane model where isometries act by linear fractional transformations. In the Klein model there are various ways of identifying a conic, such as ∂X, with $\mathbb{RP}^1$ (see Fig. 2.2). (In higher dimensions projective geometry is replaced by conformal geometry as the natural geometry to use on the sphere-at-infinity).

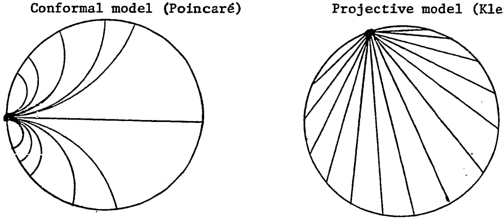


Fig. 2.1

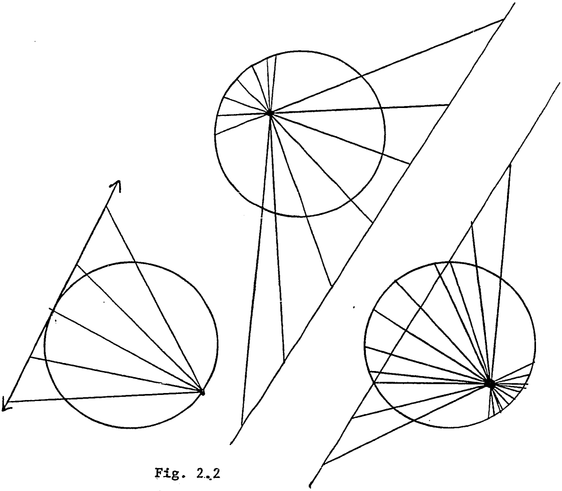


Fig. 2.2

2.2 Since the Lorentz inner product in $\mathbb{R}^3$ defines a duality on $\mathbb{R}^3$, i.e. an isomorphism of $\mathbb{R}^3$ with its dual vector space, a conic in $\mathbb{RP}^2$ defines a projective duality. That is, to every point p in $\mathbb{RP}^2$ (corresponding to a line in $\mathbb{R}^3$ containing 0) there is a dual projective line p^* (corresponding to the dual "annihilating" plane containing 0), and vice versa. Points in X correspond to lines completely contained in X^* and points in X^* correspond to lines intersecting ∂X in two points, i.e. geodesics in X (Fig. 2.3). In this way every point in $\mathbb{RP}^2$ is interpreted in the geometry of X.

2.3 Now we shall classify the elements of G. This may be done either algebraically, by using Jordan normal form for matrices in $SL(2,\mathbb{R})$, or geometrically, using the synthetic geometry of X. Either method implies that an element $T \in G$, $T \neq 1$, has either exactly one fixed point, in X (elliptic), in ∂X (parabolic), or three fixed points, one in X^* and the other two on ∂X (hyperbolic). We proceed to discuss these three cases.

Elliptic elements of G stabilize a unique point $x \in X$. Such a transformation is rotation through some angle θ, $-\pi < \theta \leq \pi$ and is represented by a matrix conjugate to

$$T_\theta = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

Note that the centralizer of such an elliptic element is the elliptic one-parameter subgroup fixing x, which is compact and conjugate to $PSO(2) \subset PSL(2,\mathbb{R})$. However, the parametrization of the one-parameter subgroup is not uniquely determined, for we may write

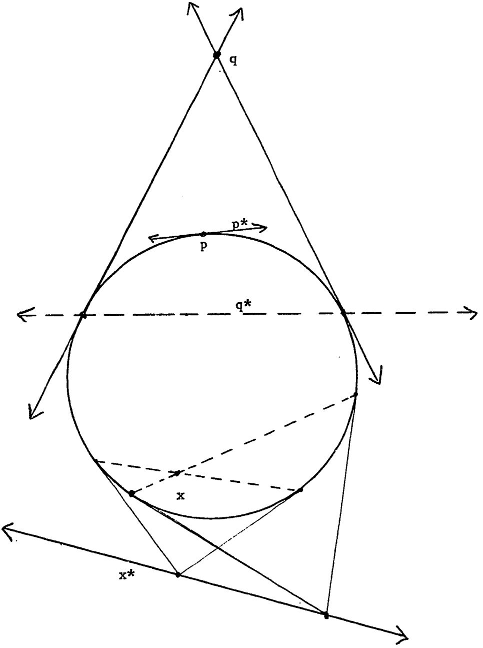


Fig. 2.3 The projective duality defined by a conic

$$T_\theta = \exp(\tau_\theta)$$

where

$$\tau_\theta = \begin{pmatrix} 0 & -\frac{\theta}{2} - \pi k \\ \frac{\theta}{2} + \pi k & 0 \end{pmatrix} \in sl(2,\mathbb{R})$$

for every $k \in \mathbb{Z}$. To remedy this situation we define the principal-value logarithm of an elliptic transformation by the formula

$$\text{Log } P^{-1}\begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} P = P^{-1}\begin{pmatrix} 0 & -\frac{\theta}{2} \\ \frac{\theta}{2} & 0 \end{pmatrix} P$$

where $P \in G$ and $-\pi < \theta \leq \pi$. Note that Log has discontinuities at elements with $\theta = \pm\pi$; such elements are rotations of angle $180^\circ = \pi$ and may be characterized as those elements of G of order two. We call such elements symmetries and we denote the subset of G consisting of symmetries by Sym. Note that the map

$$X \longrightarrow \text{Sym}$$

$$x \longmapsto (\text{symmetry about } x)$$

is a homeomorphism which naturally identifies X with a subset at G.

2.4 Parabolic elements are limiting cases of elliptic elements as their fixed points tend to $\partial X = S^1_\infty$. Parabolic elements of G fall into two conjugacy classes, depending on the direction they move points on ∂X; they are represented by matrices

$$\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$$

As with elliptic elements, the centralizer of a parabolic element is precisely the one-parameter subgroup containing it. Unlike elliptic elements the parametrization of parabolic one parameter subgroups is unique, that is, if T is parabolic, there is a unique solution $(\text{Log } T)$ to $\exp(\text{Log } T) = T$. Specifically, if T is represented by

$$T^{-1}\begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}T ,$$

$T \in G$, then $\text{Log } T$ is the unique element of $sl(2;\mathbb{R})$ given by

$$T^{-1}\begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix}T$$

The orbits of elliptic one-parameter groups are the concentric circles about the fixed point in X. As the point moves towards $y \in \partial X$, then the circles approach curves which are the orbits of the parabolic one-parameter set group fixing y. These <u>horocycles</u> are realized in the Poincaré model as circles tangent to ∂X at y and may be defined synthetically as the orthogonal trajectories to the family of geodesics asymptotic to y.

2.5 Hyperbolic elements are those elements of G which leave a geodesic (necessarily unique) invariant. Once again the centralizer of a hyperbolic is the one-parameter subgroup containing it. A hyperbolic element has three fixed points in $\mathbb{RP}^2_y$: the endpoints in ∂X of the invariant geodesic as well as the dual point in X^*. We call the fixed point in X^* the <u>preferred</u> fixed point because although it is not the unique fixed point, it nonetheless characterizes the hyperbolic one-parameter subgroup.

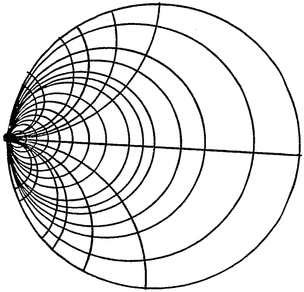


Fig. 2.4

As with the parabolic case, the parametrization of a hyperbolic one-parameter subgroup is unique. Thus we may define the logarithm of a hyperbolic element by the formula

$$\operatorname{Log} P^{-1}\begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} P = P^{-1}\begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix} P$$

The orbits of a hyperbolic one-parameter subgroup in X are all curves (called equidistant curves) which run between the two fixed points on ∂X. However, only one of these orbits is a geodesic.

Thus we have classified the elements of G other than 1 into three types. We have defined a logarithm $\operatorname{Log}: G \longrightarrow g = sl(2;\mathbb{R})$ which is uniquely determined except in the elliptic case. Log is continuous except along the set Sym of symmetries; there it must necessarily suffer a discontinuity. We say that a path $g_t \in G$ is admissible (or special) if the map $t \mapsto \operatorname{Log} g_t$ is continuous. Certainly if $g_t \notin \mathrm{Sym}$, then g_t is admissible. Admissible deformations will play an important role in our proof of the Main Theorem.

2.6 Now we wish to digress very briefly to discuss one class of orientation-reversing isometries, the reflections. Given a geodesic in X, there is a unique orientation-reversing involution fixing that line; conversely every involution which reverses orientation, reflects in a line. Indeed, the fixed point set of this map is the projective line in $\mathbb{RP}^2$ extending the geodesic and the dual point in X^*. A reflection R_L in a line L commutes with the hyperbolic one-parameter subgroup leaving L invariant; on the other hand, if T is a parabolic transformation fixing an endpoint of L, then $R_L T R_L = T^{-1}$. Thus all parabolic elements are conjugate in $SO(2,1)$. It is also interesting to note that for every non-parabolic element of G, there is some involution in $SO(2,1)$ commuting with it; in fact the only elements of G which commute with a reflection (other than the identity 1) are symmetries and hyperbolic elements.

2.7 The first picture we draw will be of $G = PSL(2,\mathbb{R})$. From the Killing form G has a bi-invariant Lorentz metric of constant curvature. G is topologically an open solid torus. Let $1 \in G$ be the identity element. Then the light cone emanating from 1 is precisely the set of all parabolic element $Par \cup \{1\}$. Inside it lies the union of elliptic one-parameter subgroups. Halfway around the pinched torus $\{1\} \cup Par \cup Ell$ lies the hyperbolic plane embedded as Sym. Finally adding the hyperbolic one-parameter subgroups fills out the whole $\mathbb{RP}^2$ field-of-vision around $1 \in G$.

The stabilizer of a point in $\mathbb{RP}^2$ is the normalizer of the corresponding one-parameter group; these also turn out to be the maximal solvable subgroups. Elliptic one-parameter subgroups are their own normalizers. On the other hand the stabilizers of ideal points are 2-dimensional and are conjugated to $Aff(\mathbb{R})$. The stabilizer of a geodesic (ultra-ideal point) contains the hyperbolic one-parameter group as a subgroup of index two; in addition to translating along the geodesic, the stabilizer contains symmetries about points on the geodesic. Such a symmetry interchanges the two ideal fixed points of the hyperbolic elements. Hence, if Γ is a subgroup of G, the following are equivalent:

Γ solvable

$\Leftrightarrow$ Γ fixes a point in $\mathbb{R}\mathbb{P}^2$

$\Leftrightarrow$ Γ is either a group of rotations about a point, a group of orientation-preserving affine transformations on $\mathbb{R}^1$ or a group of isometries on a line

To better picture the conjugacy classes in $G = PSL(2,\mathbb{R})$ we shall use the trace mapping trace: $SL(2,\mathbb{R}) \longrightarrow \mathbb{R}$. Since $\text{trace}(-A) = -\text{trace } A$, the trace mapping itself is not defined on G, but $|\text{trace}|$ and trace^2 are. However, in the following picture we draw the level sets of $t = \text{trace}: SL(2,\mathbb{R}) \longrightarrow \mathbb{R}$.

It is well-known that the trace of an element of $SL(2,\mathbb{R})$ (or $SL(2,C)$) is a conjugacy invariant; except for the case $\text{trace} = \pm 2$ the trace of a matrix in $SL(2)$ is a complete conjugacy invariant. One knows the type of element of $SL(2,\mathbb{R})$ by computing its trace:

$\lvert\text{trace } A\rvert > 2$	hyperbolic
$\lvert\text{trace } A\rvert = 2$	$\pm$(parabolic or identity)
$\lvert\text{trace } A\rvert < 2$	elliptic

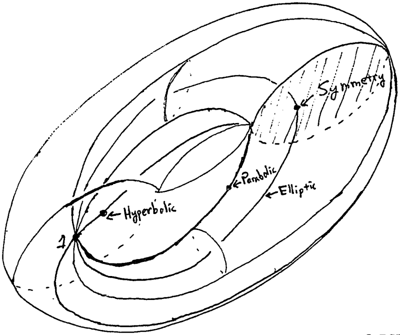


Fig. 2.5a A picture of PSL(2;ℝ)

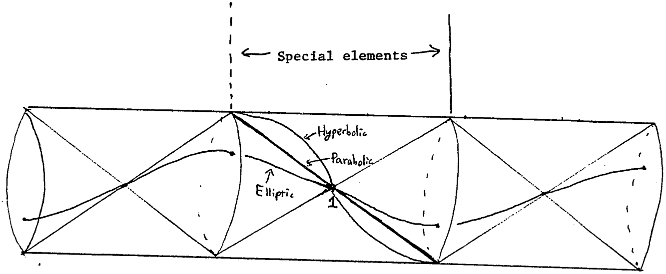


Fig. 2.5b A picture of $\widetilde{PSL}(2;\mathbb{R})$

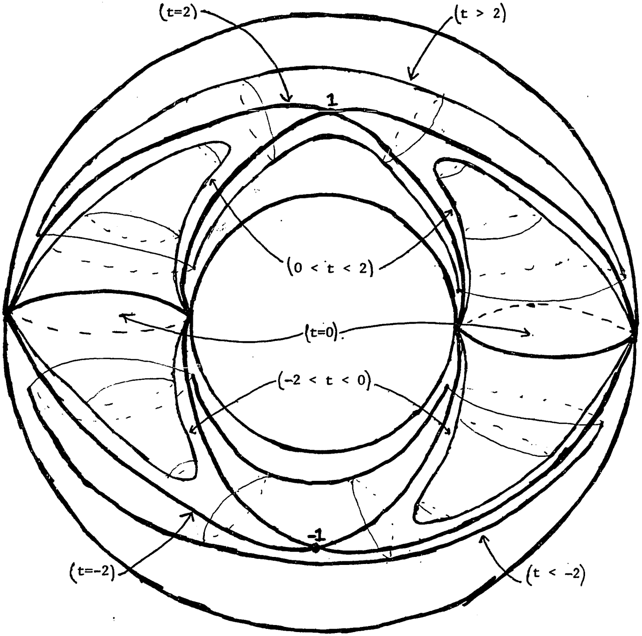


Fig. 2.5c. A picture of SL(2;R) and the level sets trace = t

Appendix to §2. Real Projective Structures on the Circle

We begin the study of PSL(2,$\mathbb{R}$)-structures by the classification of the closed $\mathbb{RP}^1$-manifolds. In addition to illustrating the theory of §1 and §2, real projective structures in dimension one play a key role in our study of PSL(2,$\mathbb{R}$)-structures on surfaces. Indeed one may see certain analogies between the situation in dimensions one and two. We shall find a natural interpretation of conjugacy classes in the universal covering group $\tilde{G} = \tilde{SL}(2,\mathbb{R})$ and real projective structures on the circle. The classification of closed $\mathbb{RP}^1$-manifolds is due to Kuiper [35] although the connection with $\tilde{SL}(2,\mathbb{R})$ is new.

2.8 First we discuss those projective structures on the circle M^1 which are <u>affine structures</u>. Recall that the affine group $Aff_+(\mathbb{R})$ of the real line $\mathbb{R}$ is the isotropy group of PSL(2,$\mathbb{R}$) acting by projective transformations on $\mathbb{RP}^1$. Labeling the fixed point on $\mathbb{RP}^1$ as ∞, the complement $\mathbb{RP}^1 - \{\infty\}$ is $\mathbb{R}$ with the group $Aff_+(\mathbb{R})$ acting by (orientation-preserving) affine transformations $x \mapsto ax+b$, $a > 0$.

Since the developing image of a compact affine manifold contains no fixed points of the affine holonomy [14] and $\pi_1(M)$ is cyclic, the holonomy group leaves invariant either the 1-form dx or (translating any fixed points to 0) $x^{-1}dx$, depending on whether $a \neq 1$ or $a = 1$. In the first case the holonomy acts by translations and the affine manifold $M = \mathbb{R}/\mathbb{Z}$; in the second case M is covered by one of the components of $\mathbb{R}-\{0\}$, say the positive part $\mathbb{R}_+$, and as an affine manifold M is a "Hopf circle" $\mathbb{R}_+/\{a^n: n\in\mathbb{Z}\}$. It is easy to see that these are the only possibilities. Thus the space of all isotopy classes of affine structures on a closed 1-manifold M is naturally identified with the half-closed infinite interval $[1,\infty)$.

When generalizing to the case of projective structures there are two new complications which arise. First, the holonomy may have no fixed points at all on $\mathbb{RP}^1$ and hence not be conjugate to a subgroup of $\mathrm{Aff}(\mathbb{R})$. Secondly, the developing map $\tilde{M} \to \mathbb{R}$ may fail to be injective. Indeed we have the following characterization:

Theorem 2.9. Let M be a closed $\mathbb{RP}^1$-manifold. Then M is projectively equivalent to an affine manifold if and only if its developing map $\tilde{M} \to \mathbb{RP}^1$ is injective.

In further contrast to affine structures on the circle, note that $\mathbb{RP}^1$ itself has its "universal" projective structure. The developing map is the universal covering map $\tilde{M} \to M = \mathbb{RP}^1$ and the holonomy is trivial. Unlike the affine case the developing image consists entirely of stationary points of the holonomy. One obtains infinitely many similar examples by passing to an n-sheeted covering $(\mathbb{RP}^1)^{(n)}$ of this universal $\mathbb{RP}^1$-manifold.

To understand these and other examples we introduce a further invariant of projective manifolds, which Nagano and Yagi term modified holonomy. Let $\widetilde{\mathbb{RP}}^1 \longrightarrow \mathbb{RP}^1$ be the universal covering and choose a lift of a base point in $\mathbb{RP}^1$ to $\widetilde{\mathbb{RP}}^1$ as initial data.

Since $\tilde{M}$ is simply connected, there is a unique lift $\tilde{D}$ of the developing map to $\widetilde{\mathbb{RP}}^1$ which extends the initial data

$$\begin{array}{ccc} & & \widetilde{\mathbb{RP}}^1 \\ & {}^{\tilde{D}} \nearrow & \downarrow \\ \tilde{M} & \xrightarrow{D} & \mathbb{RP}^1 \end{array}$$

As the action of $PSL(2,\mathbb{R})$ on $\mathbb{RP}^1$ lifts to an action of the covering group $\widetilde{SL(2,\mathbb{R})}$ on $\widetilde{\mathbb{RP}}^1$, the <u>modified development</u> $\tilde{D}$ is equivariant with respect to a lift of the holonomy homomorphism to $\widetilde{SL(2,\mathbb{R})}$, which we call the <u>modified holonomy</u>.

$$\begin{array}{ccc} & & \widetilde{SL(2,\mathbb{R})} \\ & {}^{\tilde{\phi}} \nearrow & \downarrow \\ \pi_1 M & \xrightarrow{\phi} & PSL(2,\mathbb{R}) \end{array}$$

<u>Theorem 2.10</u>. There is a natural bijective correspondence between the set of isotopy classes of marked $\mathbb{RP}^1$-structures on S^1 and conjugacy classes in $\widetilde{SL(2,\mathbb{R})} - \{1\}$, defined by assigning the modified holonomy of a projective structure.

<u>Proof</u>. First we show that (marked) $\mathbb{RP}^1$-structures on $M \approx S^1$ with the same modified holonomy are isotopic. Let $\tilde{m} \in \tilde{M}$ be a base point and let $\tilde{D}_1$ and $\tilde{D}_2$ denote the modified developing maps of two projective structures with the same modified holonomy $\phi\colon \pi_1 M = \mathbb{Z} \to \widetilde{SL(2,\mathbb{R})}$. By a trivial deformation we may assume that $\tilde{D}_1(\tilde{m}) = \tilde{D}_2(\tilde{m})$. Now let g_1 denote the preferred (i.e. coming from the marking) generator of $\pi_1(M)$, and let $\tilde{I}$ denote the interval in $\tilde{M}$ with endpoints $\tilde{m}$ and $g_1\tilde{m}$. Then $\tilde{D}_i$ is completely determined by its restriction to the fundamental interval $\tilde{I} \subset \tilde{M}$, which is an immersion

(and hence a homeomorphism) of $\tilde{I}$ onto an interval in $\widetilde{\mathbb{RP}}^1 \approx \mathbb{R}$. Since any two homeomorphisms from an interval into $\mathbb{R}$ which agree on the endpoints are isotopic (relative to the endpoints), there is an isotopy $h_t: \tilde{I} \to \tilde{I}$, $h_t|_{\partial\tilde{I}} = id_{\partial\tilde{I}}$ and $h_0 = id_{\tilde{I}}$ such that $\tilde{D}_1 \circ h_1 = \tilde{D}_2$. By extending h_t to $\tilde{M} \to \tilde{M}$ by $h_t \circ (g_1^n) = (g_1^n) \circ h_t$ we have constructed an isotopy between $\tilde{D}_1$ and $\tilde{D}_2$ which defines an isotropy between D_1 and D_2. Hence two projective structures are isotopic if their modified holonomy are identical.

Now we show that every element $\phi \in \widetilde{SL(2,\mathbb{R})}$, $\phi \neq 1$ occurs as the modified holonomy of some projective structure. Since $\phi \neq 1$, there exists a point $p \in \widetilde{\mathbb{RP}}^1$ which is not stationary under ϕ. Define the modified development $\tilde{D}: \tilde{M} \longrightarrow \widetilde{\mathbb{RP}}^1$ by taking $\tilde{D}$ to be a homeomorphism from a fundamental interval $\tilde{I}$ onto the interval with endpoints p and ϕp and extending equivariantly to $\tilde{M} \to \widetilde{\mathbb{RP}}^1$. This concludes the proof of Theorem 2.10.

Proof of Theorem 2.9. The fact that the developing map of an affine 1-manifold is injective follows from the fact that every immersion $\mathbb{R} \to \mathbb{R}$ is injective. Conversely suppose M^1 is a compact $\mathbb{RP}^1$-manifold whose developing map $D: \tilde{M} \longrightarrow \widetilde{\mathbb{RP}}^1$ is injective; we show that M is actually affine. If the holonomy is trivial or elliptic, then the image of a fundamental interval $\tilde{I}$ in $\tilde{M}$ under D will wrap around $\mathbb{RP}^1$ at a constant angular rate. Then there will be some $n > 0$ such that $\phi(g_1)^n D(\tilde{I})$ intersects $D(\tilde{I})$. Consequently in this case D is not injective. Hence $\phi(g_1)$ is either hyperbolic or parabolic; in particular $\phi(g_1)$ has a stationary point.

A projective structure is affine if and only if the holonomy fixes a projective hyperplane disjoint from the developing image. Therefore it suffices for Theorem 2.9 to prove that any fixed point p lies outside $D(\tilde{M})$. Suppose otherwise; then p lies in the image of some fundamental interval $\tilde{I} \subset \tilde{M}$. Since $\phi(g_1)p = p$, it lies in the image $D(g_1^n\tilde{I})$ of every fundamental interval, contradicting the injectivity of D. This completes the proof of Theorem 2.9, and together with Theorem 2.10, the classification of closed affine 1-manifolds.

2.11 Note that when $\phi = 1$, we obtain a "developing map" for a singular projective structure which is a constant map, obtaining the <u>constant projective structure</u>. In this way we have a concrete geometric object (either an honest projective structure or a constant projective structure) which exactly corresponds to an arbitrary element of the covering group $\widetilde{SL(2,\mathbb{R})}$.

We shall like to find some natural way of associating to an element of $PSL(2,\mathbb{R})$ some projective structure having that as holonomy. It is impossible to do this in a continuous fashion since there is no continuous cross-section to the covering projection $\widetilde{SL(2,\mathbb{R})} \longrightarrow PSL(2,\mathbb{R})$.

One procedure begins as follows. Write $M = \mathbb{R}/\mathbb{Z}$ so the preferred generator g_1 of the group $\pi_1(M)$ of deck transformations of $\tilde{M}$ is $t \mapsto t+1$. Then a developing map will be a map $\mathbb{R} \xrightarrow{D} \mathbb{RP}^1$ which is equivariant in the sense that $D(t+1) = \phi(g_1)D(t)$ where ϕ is the holonomy. Notice that if $t \mapsto \exp t \log \phi(g_1)$ is a one-parameter subgroup which has $\phi(g_1)$ as its time-one map, then such a D may be defined by

$$D(t) = (\exp -t(\log \phi(g_1)))y .$$

If y is not a stationary point for the holonomy then D is nonsingular. Moreover specifying $\log \phi(g_1)$ to be the principal-value $\operatorname{Log} \phi(g_1)$ (defined in 2.3-2.5) we obtain, for each $\phi(g_1) \in PSL(2,\mathbb{R})$, a projective structure with that as holonomy (which is constant if $\phi(g_1) = 1$). We call this singular projective structure <u>special</u>. We emphasize that the assignment of special structures is discontinuous; it fails to be continuous precisely at the set Sym where Log fails to be continuous.

We examine these special structures as well as the other projective structures by picturing their foliated bundles. We shall represent the foliated $\mathbb{RP}^1$-bundle over M as a square, with opposite sides identified, making it into a torus. The $\mathbb{RP}^1$-fibers will be vertical lines which we do not draw; the leaves however will be drawn.

<u>Case 1</u>: $\phi(g_1) = 1$

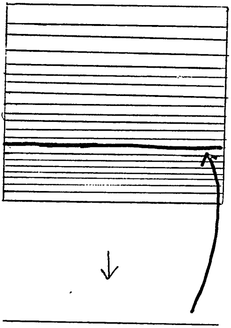
In this case all the leaves are closed and we have the product foliation of $M \times \mathbb{RP}^1$. The special section is just a leaf, that is, it is the constant projective structure. On the other hand, elements of $\pi_1 PSL(2,\mathbb{R}) = \operatorname{Ker}(\widetilde{SL(2,\mathbb{R})} \to PSL(2,\mathbb{R}))$ which are n times a generator of $\pi_1 PSL(2,\mathbb{R})$ correspond to n-fold covers of the universal $\mathbb{RP}^1$-manifold $\mathbb{RP}^1$. If f is the developing section and $p: M \times \mathbb{RP}^1 \to \mathbb{RP}^1$ denotes projection, then $p \circ f$ is a covering of degree n. Pictured in Fig. 2.6 are the cases $n = 0, 1, 2$.

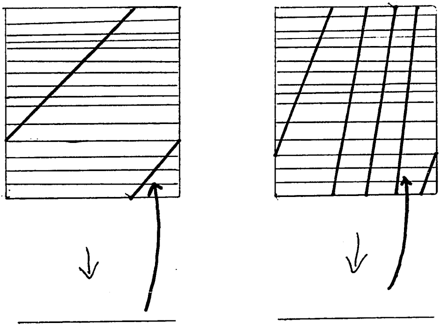
<u>Case 2</u>: $\phi(g_1)$ is parabolic.

In this case all the projective structures are nonsingular. The special one is just the Euclidean structure on $\mathbb{R}/\mathbb{Z}$. The various

Fig. 2.6 Developing sections for RP^1-structures with trivial holonomy

To the right is constant section, which is the developing section for the "special" projective structure with trivial holonomy. Below are pictured the developing sections for the "universal projective structure" RP^1 (left) and its four-fold cover.





projective structures with holonomy ϕ are distinguished by the intersection number of the developing section with the unique closed leaf. In Fig. 2.7 we draw developing sections, one for the special structure and one which intersects the compact leaf twice.

<u>Case 3</u>: $\phi(g_1)$ is hyperbolic.

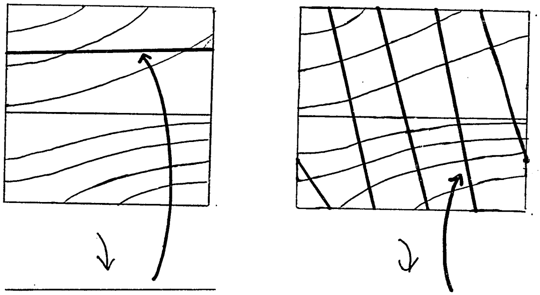
This case is almost exactly the same as the previous one, except there are two closed leaves. Fig. 2.8 shows a special developing section and one which intersects each of the compact leaves once.

<u>Case 4</u>: $\phi(g_1)$ is elliptic.

In this case the foliation is a linear foliation of the torus. Suppose first that $\phi(g_1)$ is a rotation of angle $\theta \neq 0$, $-\pi < \theta < \pi$. Then in every homotopy class of sections there is a transverse section. Any fiber serves as a cross-section to the foliation, and as the first-return map is rotation by angle θ the leaves of the foliation can be taken to have slope $\tan \frac{\theta}{4}$. (This forces the slope to be less than one in absolute value. For a linear foliation of $|\text{slope}| > 1$ one can perform Dehn twists about the fiber taking the original foliation to one with $|\text{slope}| < 1$, preserving the foliated bundle structure.) Then the "horizontal" curve is a special section. The first picture in Fig. 2.9 is of a special section. Clearly as $\theta \to 0$ the special sections converge to the constant section, as we would expect since Log is continuous near the identity.

Now we come to the more delicate case when $\phi(g_1) \in \text{Sym}$. Recall that Log is continuous as $\theta \nearrow \pi$. Then we may take $\theta = \pi$, slope $= 1$ and the special section looks horizontal. But taking $\theta = -\pi$ the horizontal section is no longer special and we must twist one about a

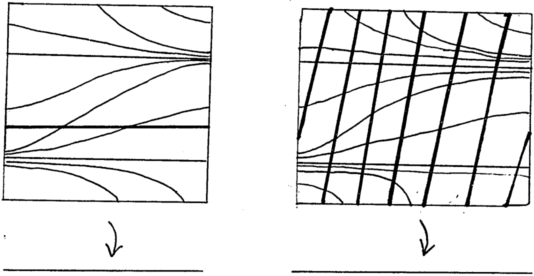
Fig. 2.7 Projective structures with parabolic holonomy



A special section is disjoint from any compact leaf and defines an affine structure

This section is not special since it intersects a compact leaf four times

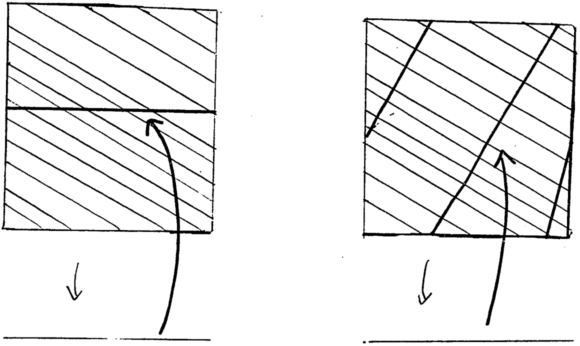
Fig. 2.8 Projective structures with hyperbolic holonomy



A special section, defining an affine structure

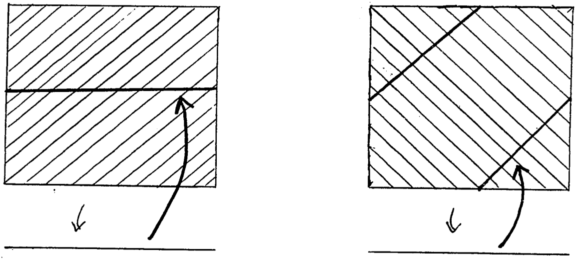
A "non-special" section

Fig. 2.9. Projective structures with elliptic holonomy



A special section

Fig. 2.10. Special sections with holonomy in Sym



$\theta = \pi$ $\qquad$ $\theta = -\pi$

fiber. In Fig. 2.10 are pictures of special sections in this case.

For the classification of the affine structures on the 2-torus, see Arrowsmith and Furness [2] or Nagano and Yagi [41]. In [16] the real projective structures on T^2 are classified. One of the consequences of [16] (which is unfortunately not stated) is:

Theorem. Let M be a 2-torus or a Klein bottle with an $\mathbb{RP}^2$-structure. Then either the projective structure is an affine structure or the developing map is not a covering onto its image.

The first examples of non-affine projective structures are due to Smillie [57] and to Sullivan and Thurston [51].

§3. Fuchsian Actions and the Milnor-Wood Inequality

Now we turn to the subject of this paper: hyperbolic structures on surfaces. Let M be a closed orientable surface, $\chi(M) < 0$ and let $\pi = \pi_1(M;x_0)$ for some base-point x_0. Let $\tilde{M} \to M$ be a universal covering space with π as its group of deck transformations. Let X denote the hyperbolic plane and G the group of orientation-preserving isometries of X. A (G,X)-structure on M will be called a hyperbolic (-geometry) structure on M; a foliated (G,X)-bundle will be called a hyperbolic foliated bundle.

Let $\partial X = S^1_\infty$ denote the circle-at-infinity. A foliated $(G,\partial X)$-bundle will be called a projective foliated bundle. Note that every hyperbolic foliated bundle E_ϕ determines a projective foliated bundle ∂E_ϕ such that $E_\phi \cup \partial E_\phi$ is a G-foliated closed-disc bundle over M; the bundle ∂E_ϕ is the ideal bundle of E_ϕ.

It follows from 1.7 that the correspondence $\phi \mapsto E_\phi$ defines a bijection between the deformation space $\mathrm{Hom}(\pi,G)/G$ and isomorphism classes of hyperbolic (resp. projective) foliated bundles over the marked surface M. The space $\mathrm{Hom}(\pi,G)/G$ will be one of the main objects of study.

Suppose now that M has been given a hyperbolic structure. Then by 1.8 there is a hyperbolic foliated bundle E_ϕ over M and a section $f\colon M \longrightarrow E_\phi$ transverse to the foliation. (The holonomy $\{\phi\}$ is well-defined as an element of $\mathrm{Hom}(\pi,G)/G$.) In general it is difficult to characterize which representations ϕ occur as holonomy of (G,X)-structures, but because of the special nature of our choice of (G,X), a great deal is known. Because a hyperbolic structure is a Riemannian metric, it follows from the Hopf-Rinow theorem that if M is closed then every hyperbolic structure is complete as a Riemannian metric. It is not hard to show (see, e.g. Thurston [54], 3.6-3.7) that this is equivalent to completeness of the (G,X)-structure (see 1.2), that is, a developing map for this hyperbolic structure is a homeomorphism $\tilde{M} \to X$. Hence the holonomy action of π is a faithful, free, proper, isometric action on X, i.e. an action by deck transformations. This, in turn, is equivalent to the homomorphism $\phi: \pi \to G$ being an isomorphism onto a discrete subgroup. Such an action ϕ of π (either on X or ∂X) will be called a Fuchsian action.

We are interested in characterizing Fuchsian ϕ among all $\phi \in \mathrm{Hom}(\pi,G)$. We list several classical equivalent conditions (for the proofs, the reader is referred to, e.g. Harvey [25] or Siegel [45]).

Proposition 3.1. For $\phi \in \mathrm{Hom}(\pi,G)$ the following conditions are equivalent:

(i) $\phi: \pi \to G$ is an isomorphism onto a discrete subgroup.

(ii) The action of π on X defined by ϕ is a faithful, free, proper action.

(iii) ϕ is injective and $\phi(\pi)$ acts properly discontinuously on X.

(iv) ϕ is the holonomy of a hyperbolic structure on M.

It follows from 1.8 that we may add the following condition:

(v) The hyperbolic foliated bundle E_ϕ admits a transverse section.

Later on in this chapter we shall add the next two conditions:

(vi) ϕ lifts to $\pi \to SL(2,\mathbb{R})$ and the projective action of π on $\partial X \approx \mathbb{RP}^1$ defined by ϕ is structurally stable.

(vii) ϕ lifts to $SL(2,\mathbb{R})$ and the action on ∂X may be approximated by structurally stable actions.

In terms of the projective bundles we have the following condition, its equivalence with (i)-(v) being a tautology:

(viii) ∂E_ϕ is the unit tangent bundle of M and $\mathcal{F}_\phi$ is an Anosov foliation.

The basic result of this whole paper is that we can adjoin another condition:

(ix) ∂E_ϕ is the unit tangent bundle of M.

Chapter 3 discusses the equivalence of these conditions. Proofs of the nonstandard material needed for the equivalence of (i) through (viii) are given in the appendix. The equivalence with condition (ix) is given in the last three chapters.

3.2 The subject of discrete subgroups of $G = PSL(2,\mathbb{R})$ is a well-developed subject with roots dating back a century. We will not discuss either the history or the details but rather refer to the reference [25]. We will only discuss those aspects which are relevant to our purposes; in particular we wish to characterize discrete subgroups as "subgroups consisting practically entirely of hyperbolic elements."

In general discrete subgroups Γ contain parabolic and elliptic elements as well as hyperbolic elements; however such elements correspond to various special properties of the quotient surface X/Γ. If Γ has no elliptic elements, the quotient X/Γ is diffeomorphic to a smooth surface; then conjugacy classes of cyclic subgroups of Γ containing parabolic elements correspond to noncompact ends of X/Γ of finite area (see [25] or Thurston [54]). Conjugacy classes of cyclic elliptic subgroups of Γ (such subgroups are all finite in discrete Γ) correspond to branch points on X/Γ, where the projection $X \to X/\Gamma$ is a ramified covering. Since every discrete subgroup Γ contains a normal subgroup Γ' of finite index without elliptic elements (Selberg [44]), we can realize X/Γ as the quotient of the unbranched surface X/Γ' by the finite group Γ/Γ'.

Now suppose the hyperbolic surface X/Γ has finite area. Then all the ends of X/Γ have finite area and correspond to conjugacy classes of parabolic subgroups. By removing finitely many horodiscs we are left with a compact hyperbolic surface with horocycle boundary.

Finally suppose X/Γ is compact. In all other cases Γ is a free group; in this case it is the fundamental group of a closed surface. Then Γ acts properly with compact fundamental domain. By passing to a normal subgroup of finite index we may assume Γ has no elliptic elements and X/Γ is a closed unbranched hyperbolic surface. Every element of Γ except the identity, then, is hyperbolic; conversely, we have the following fact:

<u>Proposition 3.2</u>. If $\Gamma \subset G$ consists entirely of hyperbolic elements (except for the identity) and is nonabelian, then Γ is discrete.

This theorem is originally due to J. Nielsen and was generalized by Siegel [45] as well as by Fenchel and Nielsen. More recently, Jørgensen [32] has found a further generalization which deals with the corresponding question for subgroups of $SL(2,\mathbb{C})$ (this is much harder). We do not give a proof of 3.2, although probably a proof follows from our techniques. We later use this fact in characterizing Fuchsian actions by their structural stability.

3.3 In this section we discuss the local properties of the variety of representations $\mathrm{Hom}(\pi,G)$ and the corresponding deformation spaces $\mathrm{Hom}(\pi,G)/G$ and moduli spaces $\mathrm{Aut}(\pi)\backslash\mathrm{Hom}(\pi,G)/G$. The space of representations $\mathrm{Hom}(\pi,G)$ is in a natural way an "algebraic variety" in $\underbrace{G\times\cdots\times G}_{k}$: let $\pi = \langle\gamma_1,\ldots,\gamma_k | R_1(\gamma_1,\ldots,\gamma_k) = \cdots = R_\ell(\gamma_1,\ldots,\gamma_k) = 1\rangle$ be a finite presentation; then $\mathrm{Hom}(\pi,G)$ is the set $\{(g_1,\ldots,g_k)\in G\times\cdots\times G \mid R_1(g_1,\ldots,g_k) = \cdots = R_\ell(g_1,\ldots,g_k) = 1\}$. The variety $\mathrm{Hom}(\pi,G)$ admits a natural $(\mathrm{Aut}(\pi)\times G)$-action, defined by composition with $\mathrm{Aut}(\pi)$ and conjugation by G.

In general one cannot expect these spaces to be well-behaved. There are basically three kinds of pathology which occur. First the variety $\mathrm{Hom}(\pi,G)$ can be singular. Secondly G may not act properly, even on the set of simple points $\mathrm{Hom}(\pi,G)^{-}$. The G-orbits define a singular "foliation" of $\mathrm{Hom}(\pi,G)$, whose "normal bundle" is the "bundle" of cohomology groups, $H^1(\pi;\mathcal{G}_{\mathrm{Ad}\,\phi})$ where $\mathcal{G}$ is the Lie algebra of G. When, for example, some leaves are not closed, the deformation space $\mathrm{Hom}(\pi,G)/G$ may fail to be Hausdorff. Finally there is a third kind of pathology which can occur if the action of $\mathrm{Aut}(\pi)$ on $\mathrm{Hom}(\pi,G)/G$ is not required to be proper or free. In our case ($G = PSL(2,\mathbb{R})$, X = the hyperbolic plane, $\pi = \pi_1(M^2)$, $\chi(M^2) < 0$) $\mathrm{Aut}(\pi)$ acts properly but not freely (see [25], §2, §7) and the moduli space has (cone-like) singularities which locally look like orbit spaces of finite groups acting (i.e. "orbifold" singularities). This comes from representations $\phi\colon \pi \to G$ which are invariant under nontrivial subgroups of $\mathrm{Aut}(\pi)$. For projective structures on the (oriented) circle, only the second type of pathology occurs, i.e. G doesn't act properly and freely. The deformation space is the quotient $(\widetilde{SL(2,\mathbb{R})}-\{1\})/\mathrm{Ad}$ (see Fig. 2.5). For real affine structures on the 2-torus all three types of pathology occur. Curiously, the deformation space for the real projective structures is a disjoint union of the space of affine structures and countably many disjoint $\mathbb{R}^4$'s.

However for $\pi = \pi_1(M^2)$ and $G = PSL(2,\mathbb{R})$, the space $\mathrm{Hom}(\pi,G)$ is a real quasiprojective variety (because $PSL(2,\mathbb{R})$ is the Zariski open subset $\{[a,b,c,d] \mid ad-bc \neq 0\}$ of $\mathbb{RP}^3$). Indeed there is a rather precise description of the local singular behavior of the deformation space. This follows from the following fact, which is taken from Gunning [20], [21], [22]:

<u>Proposition 3.3</u> (Gunning). Let $G = SL(2,\mathbb{C})$, and $\pi = \pi_1(M^2)$, $\chi(M^2) < 0$.

(1) The singular variety of $\mathrm{Hom}(\pi,G)$ is the collection of all reducible representations.

(2) The group G acts properly on the set $\mathrm{Hom}(\pi,G)^-$ of simple points (i.e. irreducible representations, by (1)) and the projection $\mathrm{Hom}(\pi,G)^- \longrightarrow \mathrm{Hom}(\pi,G)^-/G$ is a principal G-bundle.

The same proof applies to $G = PSL(2,\mathbb{C})$ instead of $SL(2,\mathbb{C})$. Since by Whitney [55], complexifying a real algebraic variety preserves simple points and singular points, 3.3(1) carries over now to $G = PSL(2,\mathbb{R})$. Since $PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$, it follows directly from 3.3(2) that the action of $PSL(2,\mathbb{R})$ on $\mathrm{Hom}(\pi,PSL(2,\mathbb{R}))^-$ is proper.

Consequently $Hom(\pi, PSL(2,\mathbb{R})^{-}/PSL(2,\mathbb{R}))$ <u>is a real analytic manifold of dimension</u> $-3\chi(M) = \dim H^1(\pi; G_{Ad\ \phi})$ for $\phi \in Hom(\pi, PSL(2,\mathbb{R}))^{-}$ (for more details on the computation of the tangent space of $Hom(\pi,G)/G$, see Gunning [20] or Goldman [17]).

3.4 Our goal is to characterize Fuchsian actions among all actions in $Hom(\pi,G)$. Therefore it is of interest to know how the space consisting of Fuchsian ϕ sits inside $Hom(\pi,G)$. More to the point we consider the set of G-conjugacy classes of Fuchsian ϕ as a subset of $Hom(\pi,G)/G$. This space is precisely the set of all isotopy classes of hyperbolic structures on M, and is a disjoint union of <u>two</u> copies of the Teichmuller space of M.

Suppose that $\phi: \pi \to G$ is Fuchsian. Then there is a homeomorphism $M \longrightarrow X/\phi(\pi)$ which defines a hyperbolic structure on M. Since G preserves orientation we are entitled to orient X; since M is orientable we choose an orientation for M also. Then the homeomorphism $M \longrightarrow X/\phi(\pi)$ may either preserve or reverse orientation. By simply changing the orientation on M we establish a bijection between these two collections of Fuchsian actions. Hence the space consisting of Fuchsian actions $R^{\pm}(\pi,G) \subset Hom(\pi,G)$ consists of two copies of the space $R(\pi,G)$ of orientation-preserving Fuchsian actions. The space $R(\pi,G)$ is well known to be a principal G-bundle over the Teichmuller space of M.

We shall not go into the theory of Teichmuller space any deeper than the well-known fact that the Teichmuller space for M is a $3|\chi(M)|$-dimensional cell. This fact was known to Fricke and Klein, where the present point of view--a set of isomorphisms $\pi \to G$ --was adopted. There are a number of excellent modern accounts of this, using coordinates derived by Fenchel and Nielsen; for this and much more see Harvey [25], Thurston [54], Fathi-Laudenbach-Poénaru [11].

Since Teichmuller space is a deformation space for (G,X)-structures, viewing it as the set of holonomy representations, it is an open subset of $\mathrm{Hom}(\pi,G)$ (see Goldman [17] or Thurston [54], 5.1). Since M is compact, every hyperbolic structure on M is complete, and the map

$$\left\{\begin{matrix}\text{isotopy classes}\\ \text{of } (G,X)\text{-structures on } M\end{matrix}\right\} \longrightarrow \mathrm{Hom}(\pi,G)$$

which assigns to a (G,X)-structure its holonomy is injective. Hence we may regard Teichmuller space as an open subset of the deformation space $\mathrm{Hom}(\pi,G)/G$. (This fact is originally due to A. Weil [60], [61].)

It is also true that Teichmuller space is a <u>closed</u> subset of $\mathrm{Hom}(\pi,G)$. This fact seems to have been known classically, although we shall give another proof, based on Theorem A. One proof, by Helling [27], describes representations by means of their characters, and exhibits the isomorphisms onto discrete subgroups as a closed subset. Another proof comes out of a more general theorem concerning Kleinian groups. This result (due, in various parts, to Chuckrow, Marden, Yamamoto, and Jørgensen) states the following: if $\phi_n \in \mathrm{Hom}(\Gamma,SL(2,\mathbb{C}))$ is a sequence of isomorphisms onto discrete groups, which converges to a representation ϕ, then ϕ is itself an isomorphism onto a discrete subgroup. See Jørgensen [31], Thurston [54], §9, and Harvey [25].

Theorem A is a quantitative elucidation of the fact that Teichmuller space is closed in the deformation space. We exhibit it as the fiber $e^{-1}(\chi(M))$ of a continuous map $e\colon \mathrm{Hom}(\pi,G)/G \longrightarrow \mathbb{Z}$.

Since it is open and connected, we deduce that Teichmuller space is precisely one connected component of $\mathrm{Hom}(\pi,G)/G$. Counting the other component with reversed orientation, we see that two of the connected components of $\mathrm{Hom}(\pi,G)/G$ are (6g-6)-dimensional.

3.5 For the moment let us consider the following general situation: G is a real algebraic group (i.e. quasiprojective variety over $\mathbb{R}$ such that $G \times G \to G$, $(a,b) \mapsto ab^{-1}$ is an algebraic morphism), X is a G-space, and π is the fundamental group of a compact polyhedra M. Then conjugacy classes $\{\phi\} \in \mathrm{Hom}(\pi,G)/G$ of representations ϕ are identified by isomorphism classes of foliated (G,X)-bundles E_ϕ over M. Thus the canonical map which forgets the foliation but expresses the topological type of the bundle is a map

$$(*) \qquad \mathrm{Hom}(\pi,G)/G \to \left\{\begin{matrix}\text{X-bundles over M} \\ \text{with structure group G}\end{matrix}\right\}$$

Now the space $\mathrm{Hom}(\pi,G)$ is a real algebraic variety, and it follows from Whitney [55] that $\mathrm{Hom}(\pi,G)$ has finitely many connected components. Hence there are only finitely many X-bundles over M which admit G-foliations (Sullivan [50]).

If ϕ_t is a deformation in $\mathrm{Hom}(\pi,G)$, then $\bigcup_{t\in I} E_{\phi_t} \times \{t\}$ is an X-bundle over $M \times I$. By the covering homotopy property the isomorphism type of the X-bundle E_{ϕ_t} does not vary. Since an algebraic variety is locally path-connected, the map (*) is continuous. The classification of all X-bundles over M is accomplished using obstruction theory (Steenrod [48]) which assigns to a given bundle, characteristic classes, which are generally elements of some cohomology group of M. (Compare Chapter I of Hirzebruch [30].) We develop the theme that global properties of spaces of representations are detected by the associated characteristic classes.

The first of these characteristic classes arise as follows. Let G_0 be the identity component of G and $G/G_0 = \pi_0(G)$ be the group of components of G. Taking $X = G$ the bundle E_ϕ is a principal G-bundle and the first obstruction to its triviality is the class $\sigma_1(\phi) \in H^1(M;\pi_0(G))$ which via the Hurewicz isomorphism $H^1(M;\pi_0(G)) \cong \mathrm{Hom}(\pi, G/G_0)$ is just the composite

$$\pi \xrightarrow{\phi} G \longrightarrow G/G_0 \ .$$

For example, if $G = SO(2,1)$, then $\sigma_1(\phi)$ is just the first Stiefel-Whitney class $w_1(\phi) \in H^1(M;\mathbb{Z}/2)$. Elements of $\mathrm{Hom}(\pi,SO(2,1))/SO(2,1)$ correspond to <u>not necessarily oriented</u> hyperbolic foliated bundles over M; thus it follows from the exact sequence

$$PSL(2,\mathbb{R}) \longrightarrow SO(2,1) \longrightarrow \mathbb{Z}/2$$

that $\mathrm{Hom}(\pi,PSL(2,\mathbb{R}))$ is the preimage of 0 under $w_1\colon \mathrm{Hom}(\pi,SO(2,1) \longrightarrow H^1(M;\mathbb{Z}/2)$.

The next obstruction takes values in $H^2(M;\pi_1(G))$. If G is connected, so $\sigma_1(\phi) \in H^1(M;\pi_0(G)) = 0$, then $\sigma_2(\phi)$ is defined. By definition $\sigma_1(\phi)$ vanishes if and only if ϕ lifts to $\pi \to \tilde{G}$. For example, if $G = GL(n,\mathbb{R})$, $SO(n)$, $SO(n,\mathbb{C})$, $n > 2$, then $\pi_1(G) = \mathbb{Z}/2$ and $\sigma_2(\phi)$ is just the second Stiefel-Whitney class $w_2(\phi)$ of the associated n-plane bundle. For $G = PSL(2,\mathbb{R})$ and $X =$ the hyperbolic plane, then $w_2(\phi)$ is just the mod 2 reduction of the Euler class $e(\phi)$ of the oriented G-foliated X-bundle E_ϕ over M with holonomy ϕ.

3.6 Now let M be a closed oriented surface, $\pi = \pi_1(M)$, X is the hyperbolic plane and G is the group of orientation-preserving isometries. Orient X; then the Euler class of a (G,X)-bundle is defined. Thus we obtain a continuous map

$$\mathrm{Hom}(\pi,G) \xrightarrow{e} H^2(M;\mathbb{Z}) \cong \mathbb{Z}$$

(where $H^2(M;\mathbb{Z}) \cong \mathbb{Z}$ is essentially the orientation) where $e(\phi)$ is to be the Euler number of the hyperbolic foliated bundle E_ϕ. This map expresses precisely the topological type of E_ϕ as an oriented disc-bundle but forgets the foliation with its transverse hyperbolic structure.

<u>Theorem 3.6</u>. If $\phi \in \mathrm{Hom}(\pi,G)$, then $|e(\phi)| \leq |\chi(M)|$; equality $e(\phi) = \chi(M)$ (resp. $-\chi(M)$) holds if and only if E_ϕ admits an orientation-preserving (resp. orientation-reversing) transverse section.

The inequality $|e(\phi)| \leq |\chi(M)|$ is due to Milnor [39] in the special case that ϕ lifts to $SL(2,\mathbb{R})$. Wood [56] proved this inequality for the case of an arbitrary foliated circle bundle (i.e. $\phi\colon \pi \to \mathrm{Homeo}(S^1)$). They also proved that the image of e is precisely the interval $\{n \in \mathbb{Z}: |n| \leq |\chi(M)|\}$. Alternate proofs and generalizations to higher dimensions can be found in Benzecri [3], Dupont [8], Sullivan [50], and Smillie [46]. We also give a complete proof, since it is necessary for the proof of the sharpness statement in 3.6.

The "if" assertion is an immediate consequence of the theory developed in §1. Since X is contractible any two sections of E_ϕ

are homotopic, so a tubular neighborhood of a section depends only on the bundle. If E_ϕ admits a transverse section, then $e(\phi)$ is the Euler class of a tubular neighborhood of this section which is $\pm e(TM) = \pm\chi(M)$, the sign depending on orientation.

Corollary 3.7. If M has genus g, then $e^{-1}(2-2g)$ is a connected component of $\mathrm{Hom}(\pi,G)/G$ and corresponds to the Teichmuller space of M.

There is an immediate generalization of these theorems to arbitrary discrete cocompact subgroups Γ of $G = PSL(2,\mathbb{R})$. By Selberg's theorem [44] Γ contains a normal torsionfree subgroup π of finite index d in Γ; then X/π is a closed surface M. To every $\phi \in \mathrm{Hom}(\Gamma,G)$, the restriction $\phi|_\pi \in \mathrm{Hom}(\pi,G)$ satisfies the hypotheses of 3.6. Define a rational Euler class $e(\phi) = d^{-1}e(\phi|_{\pi_1})$; then the Milnor-Wood inequality $|e(\phi)| \leq d^{-1}\chi(M) = \frac{1}{2\pi}\,\mathrm{area}(X/\Gamma)$ follows from 3.6; if equality holds, then $|e(\phi|_\pi)| = |\chi(M)|$ so $\phi|_\pi$ is Fuchsian. It follows that ϕ itself is Fuchsian: clearly $\phi(\Gamma)$ is discrete; and if ϕ is not an isomorphism then two distinct elements in π which are conjugate by an element in $\mathrm{Ker}\,\phi$ have the same ϕ-image. For an intrinsic approach to the rational Euler class, see Thurston [54], §13.3.

Concerning the other components of $\mathrm{Hom}(\pi,G)$, we propose the following:

Conjecture 3.8. The map $e\colon \mathrm{Hom}(\pi,G)/G \to \mathbb{Z}$ has connected fibers.

3.9 In general for a discrete group π and Lie group G, there is a general method for constructing characteristic maps, which reduce to the maps w_1, e, and w_2 as special cases. These are "obstruction classes" defined as follows:

Let M be a $K(\pi,1)$ and construct the foliated principal G-bundle E_ϕ over M with holonomy $\phi \in \mathrm{Hom}(\pi,G)$. Then the first obstruction to a trivialization, i.e. a section of E_ϕ, is the class $\sigma_1(\phi) \in H^1(M;\pi_0 G)$ defined by $\pi \xrightarrow{\phi} G \longrightarrow \pi_0 G$ where $G \to \pi_0 G$ projects G onto its group of connected components. An example is w^1: $\mathrm{Hom}(\pi,SO(2,1)) \to H^1(M;\mathbb{Z}/2)$.

If G is connected, then this first obstruction vanishes. The second obstruction $\sigma_2(\phi) \in H^2(M;\pi_1 G)$ then starts the classification of ϕ by their associated bundle types. This class is the obstruction to lifting $\phi: \pi \to G$ to $\pi \to \tilde{G}$. Now 3.8 implies, together with the Milnor-Wood inequality, when $\pi = \pi_1(M^2)$ that $\mathrm{Hom}(\pi,PSL(2,\mathbb{R}))$ has $4g-3$ connected components. The subset of those $\phi: \pi \to G$ which lift to $\pi \to SL(2,\mathbb{R})$ then has $2g-1$ components, since the obstruction to lifting ϕ is precisely $w_2(\phi) = e(\phi) \pmod 2$.

More generally, we propose the following conjecture concerning fundamental groups of closed surfaces:

<u>Conjecture 3.9</u>. Let π be the fundamental group of a closed surface of genus $g > 1$ and G a connected Lie group. Then the map

$$\sigma_2 \colon \mathrm{Hom}(\pi, G) \longrightarrow \pi_1(G)$$

has connected fibers.

In the appendix to §3 we shall verify this conjecture for $G = PSL(2,\mathbb{C})$ and several other Lie groups. Namely, we shall prove

Theorem 3.10. The inverse images of $w_2 \colon \mathrm{Hom}(\pi, PSL(2,\mathbb{C})) \longrightarrow \mathbb{Z}/2$ are the connected components of $\mathrm{Hom}(\pi, PSL(2,\mathbb{C}))$. (According to Gunning [22], part 3, a proof of this can be worked out following Helling [27].)

We present a proof of Theorem 3.10 in the appendix (3.18) to §3. Using similar methods, we may prove 3.9 for $G = SL(2,\mathbb{C})$, $SU(2)$ and $SO(3)$ as well. The case $G = SU(2)$ has been treated by Newstead [42]; Newstead actually computes some of the higher homotopy groups of the spaces $\mathrm{Hom}(\pi, SU(2))$.

It follows from Gunning's result (see 3.3) that each connected component of $\mathrm{Hom}(\pi, PSL(2,\mathbb{C}))$ is irreducible; thus $\mathrm{Hom}(\pi, PSL(2,\mathbb{C}))$ has two irreducible components. Since $\mathrm{Hom}(\pi, PSL(2,\mathbb{C}))$ is the complexification of the real algebraic variety $\mathrm{Hom}(\pi, PSL(2,\mathbb{R}))$, it follows (Whitney [55]) that the variety $\mathrm{Hom}(\pi, PSL(2,\mathbb{R}))$ has exactly two irreducible components, namely, the preimages of

$$w_2 \colon \mathrm{Hom}(\pi, PSL(2,\mathbb{R})) \longrightarrow \mathbb{Z}/2 \, .$$

Hence, as Hyman Bass pointed out, the Euler map

$$\mathrm{Hom}(\pi, PSL(2,\mathbb{R})) \longrightarrow \mathbb{Z}$$

cannot be algebraic on the real algebraic variety $\mathrm{Hom}(\pi, PSL(2,\mathbb{R}))$.

However there is a recipe for computing $e(\phi)$, or more generally $\sigma_2(\phi)$. A standard presentation of π, the fundamental group of a surface of genus g, is $\langle A_1,\ldots,A_g,B_1,\ldots,B_g \mid [A_1,B_1]\cdots[A_g,B_g]=1\rangle$ where we employ the notation $[X,Y] = X^{-1}Y^{-1}XY$. For $\phi \in \mathrm{Hom}(\pi,G)$, choose lifts $\tilde{\phi}(A_i)$, $\tilde{\phi}(B_i)$ of $\phi(A_i)$ and $\phi(B_i)$ to the universal covering group $\tilde{G}$. Since different lifts $\tilde{A}$, $\tilde{B}$ of $A, B \in G$ to $\tilde{G}$ differ by elements of $\pi_1(G) \subset$ center $\tilde{G}$, the value of the commutator $[\tilde{A},\tilde{B}] \in \tilde{G}$ is independent of the choice of lifts. We denote the commutator $[\tilde{A},\tilde{B}]$ by $\tilde{[}A,B]$; thus we define a map $\tilde{[}\ ,\ ]: G\times G \longrightarrow \tilde{G}$. Applying the relation $[A_1,B_1]\cdots[A_g,B_g]$ to $\tilde{\phi}(A_i)$ and $\tilde{\phi}(B_i)$ we obtain $\tilde{[}\phi(A_1),\phi(B_1)]\cdots\tilde{[}\phi(A_g),\phi(B_g)] \in \mathrm{Ker}(\tilde{G}\to G) = \pi_1(G)$. Clearly the left-hand side is the obstruction to lifting ϕ to $\pi \to \tilde{G}$. For a more geometric description of $\sigma_2(\phi)$ see Proposition 4.9, which follows Milnor [39] (or Wood [56]).

For general π_1 one can define $\sigma_2(\phi)\cdot u$, where $u \in H_2(\pi)$, by representing the homology class u by a "surface" $M^2 \longrightarrow K(\pi,1)$ and using the above formula for $\pi = \pi_1(M^2)$.

Before turning to the Euler class, we shall discuss the Stiefel-Whitney class $w_2(\phi)$. If $\phi \in \mathrm{Hom}(\pi,PSL(2,\mathbb{R}))$, then composing ϕ with the natural inclusion $PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$ we obtain a map

$$i: \mathrm{Hom}(\pi,PSL(2,\mathbb{R})) \longrightarrow \mathrm{Hom}(\pi,PSL(2,\mathbb{C}))$$

which makes the following diagram commute:

$$\begin{array}{ccc} \mathrm{Hom}(\pi,PSL(2,\mathbb{R})) & \longrightarrow & \mathrm{Hom}(\pi,PSL(2,\mathbb{C})) \\ \downarrow e & & \downarrow w_2 \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 \end{array}$$

For $\phi \in \mathrm{Hom}(\pi, PSL(2,\mathbb{C}))$, the characteristic class $w_2(\phi)$ vanishes whenever the associated foliated $\mathbb{C}P^1$-bundle over M with holonomy is a product bundle; otherwise this bundle is the twisted S^2-bundle over M.

Geometrically, $w_2(\phi)$ is the obstruction to finding a section of the $\mathbb{C}P^1$-bundle with holonomy ϕ which has at worst fold singularities with respect to the foliation transverse to fibers (see 1.11). We do not discuss the intriguing question of existence of transverse sections to the foliated $\mathbb{C}P^1$-bundle with holonomy ϕ, i.e. complex-projective structures on Riemann surfaces, but see Hejhal [26], Gunning [22], and Sullivan and Thurston [51].

3.11 The Euler class of a hyperbolic (or projective) foliated bundle is a topological invariant of its bundle. It is important to note that global properties of the foliation can be used to determine $e(\phi)$. Viewing $e(\phi)$ as the Godbillon-Vey invariant of $\mathcal{F}_\phi$ involves an interpretation of $e(\phi)$ as the area of a singular hyperbolic structure.

Let $(E_\phi, \mathcal{F}_\phi)$ be a hyperbolic foliated bundle over M. Let $f: M \to E_\phi$ be a section, i.e. a "singular hyperbolic structure" with holonomy ϕ. There is an exterior differential form ω on E_ϕ which vanishes on each leaf and represents the hyperbolic volume form on each fiber. Using ω we define the area of f; then there is a "Chern-Gauss-Bonnet" theorem:

$$e(\phi) = \frac{1}{2\pi} \int_M f^*\omega$$

This can be proved much the same way as the usual Gauss-Bonnet theorem.

This interpretation of $e(\phi)$ as an area leads to the following generalization. Let G be a semisimple Lie group with maximal compact subgroup K. Let π be a discrete cocompact subgroup of G. Let $X = G/K$ and $M = \pi\backslash G/K$. Let ω be a G-invariant volume on X.

To every homomorphism $\phi\colon \pi \to G$ there is a foliated (G,X)-bundle E_ϕ over M. If $f\colon M \to E_\phi$ define

$$\mathrm{vol}(f) = \int_M f^*\omega$$

It is not hard to see that $\mathrm{vol}(f)$ depends only on the homotopy class of f, and hence depends only on ϕ. We write $\mathrm{vol}(\phi) = \mathrm{vol}(f)$.

Conjecture 3.12. (i) $|\mathrm{vol}(\phi)| \leq |\mathrm{vol}(M)|$

(ii) Equality holds, i.e. $|\mathrm{vol}(\phi)| = \mathrm{vol}(M)$ if and only if ϕ is an isomorphism onto a discrete subgroup.

When X is n-dimensional hyperbolic space and $G = \mathrm{Isom}_+(X) = SO_+(n,1)$, this conjecture may be proved along the lines of Gromov's proof of Mostow's rigidity theorem, as given by Thurston [54], §6 (see also Munkholm [59] for another exposition, as well as Gromov [19] for related ideas). It is now known (by Haagerup and Munkholm [58]) that Gromov's proof is valid in all dimensions $n \geq 3$. The proof of 3.12(i) follows immediately from the identification of $\mathrm{vol}(M)$ as $v_n \|M\|$ (see [54], 6.2) and a simple modification of the monotonicity assertion 6.2.1 of [54]. (Here v_n denotes the maximal volume of a simplex in X and $\|\ \|$ denotes Gromov's L^1-norm on homology.)

The verification of 3.12(ii) in this case is more difficult, although it follows exactly the same lines as Thurston's argument (6.4). The hypothesis of a map $F: M \to N$ between closed hyperbolic n-manifolds satisfying $\mathrm{vol}(M) = |\deg F|\mathrm{vol}(N)$ is replaced by a section $f: M \to E_\phi$ of a hyperbolic foliated bundle with $\mathrm{vol}(M) = \mathrm{vol}(f)$. Indeed the case of a map F is just the special case of a section when $\phi(\pi)$ is discrete and cocompact in G. One uses the assumption about $\mathrm{vol}(f)$ to extend a developing map $\tilde{f}: \tilde{M} \to X$ to a measurable map $\partial\tilde{f}: S^{n-1}_\infty \to S^{n-1}_\infty$ (where we have used the hyperbolic structure on M to identify $\tilde{M}$ with X), just as in [54], 6.4.4. The rest of the proof is just as in [54], showing that $\partial\tilde{f}$ takes the vertices of almost every regular ideal simplex to the vertices of a regular ideal simplex, and hence is conformal. This proves that $\pi_1(M)$ (as a subgroup of G) is mapped by ϕ to a conjugate subgroup of G, and hence that ϕ is an isomorphism onto a discrete subgroup. In this way, Conjecture 3.12 may be regarded as a generalization of Mostow's theorem; in particular this proof is invalid in the case $n = 2$. (I am grateful to W. Thurston for pointing out this generalization.)

3.13 Let $\phi \in \mathrm{Hom}(\pi,G)$. If ϕ is Fuchsian, then the Anosov foliation $\mathcal{F}_\phi$ of ∂E_ϕ is structurally stable (see Hirsch [28], Anosov [1]). To see this, give M a hyperbolic structure determined by ϕ. Then there are two realizations of $\mathcal{F}_\phi$ with foliations of the unit tangent bundle $T_1(M)$ (corresponding to the two natural identifications of $T_1(M)_x$ with ∂X), namely the stable-manifold and unstable-manifold for the geodesic flow on $T_1(M)$. Let $\mathcal{F}^s$ and $\mathcal{F}^u$ be these two invariant foliations of $T_1(M)$; then $\mathcal{F}^s$ and $\mathcal{F}^u$ intersect transversely in the trajectories of the geodesic flow. If $\mathcal{F}$ is a small perturbation of $\mathcal{F}^u$, then $\mathcal{F}$ and $\mathcal{F}^s$ intersect transversely in a new flow close to the geodesic flow. Anosov [1] proved that the geodesic flow is structurally stable. Therefore it follows from Anosov's theorem that $\mathcal{F}$ and $\mathcal{F}^s$ intersect in a flow ξ topologically conjugate to the geodesic flow. Since $\mathcal{F}^u$ and $\mathcal{F}^s$ are the only invariant foliations under the geodesic flow, it follows that $\mathcal{F}$ is the unstable-manifold foliation for ξ. Since ξ is topologically conjugate to the geodesic flow, $\mathcal{F}$ is conjugate to $\mathcal{F}^u$.

It follows from the fact that an Anosov foliation $(\partial E_\phi, \mathcal{F}_\phi)$ is structurally stable that the action $\phi: \pi \to G \subset \mathrm{Top}(S^1)$ on $\mathbb{RP}^1$ is structurally stable (see also Floyd [12]). Conversely we have the following:

<u>Theorem 3.14</u>. Let $\phi \in \mathrm{Hom}(\pi,G)$ lift to $\phi: \pi \to SL(2,\mathbb{R})$ (i.e. $w_2(\phi) = 0$). Then the projective action determined by ϕ of π on the circle $\mathbb{RP}^1$ is structurally stable if and only if ϕ is Fuchsian.

This theorem is proved in the appendix (3.21) to this chapter.

3.15 We prove (3.21) that Anosov foliations are characterized as the only structurally stable projectively foliated circle bundles over M (under a slightly annoying additional assumption--see 3.20). Theorem A gives another characterization, namely, every projective foliated-bundle structure on the unit tangent bundle $T_1(M)$ must be an Anosov foliation with one of its canonical transverse projective structures. Together with some results from the theory of foliations, we may obtain further characterizations.

In his thesis [52], Thurston gives sufficient conditions (e.g. no leaves are compact) for isotoping a foliation of a circle bundle over M to be transverse to the fibers (see also Levitt [38]). If we assume that F is a foliation with a transverse projective structure, then isotoping F transverse to the fibers makes F a projective foliated bundle ∂E_ϕ over M, for some $\phi \in \mathrm{Hom}(\pi, PSL(2,\mathbb{R}))$ (§1.6). The condition "$\partial E_\phi \approx T_1(M)$" is equivalent to asserting $e(\phi) = \pm\chi(M)$. Theorem A implies that ϕ is Fuchsian and F must be an Anosov foliation.

<u>Theorem 3.15</u>. If F is a transversely projective foliation of $T_1(M)$, and F has no compact leaves, then F must be an Anosov foliation.

3.16 It is also worth pointing out that the Euler class $e(\phi)$ arises again as the Godbillon-Vey invariant of F_ϕ on ∂E_ϕ, integrated over the fibers of ∂E_ϕ. This follows from the interpretation of $e(\phi)$ as a "volume" class (3.11), and the identification of the volume characteristic class as the Godbillon-Vey invariant of F_ϕ, following

Dupont [8] and Bott [4].

It is interesting to note that in this transversely projective setting, it is hard to vary the Godbillon-Vey invariant. If W^3 is a circle bundle over a surface M, and F is an analytic foliation, then by the results of Thurston F can be reduced to foliation without compact leaves (hence transverse to the fibers) by "erasing the compact leaves." This means that there is a standard procedure for introducing compact leaves and this procedure is the only way that a compact leaf may arise, if F cannot already be isotoped transverse to the fibers. It can be proved that this procedure does not change the Godbillon-Vey invariant, and hence for circle bundles over surfaces, the <u>Godbillon-Vey invariant of a transversely projective foliation depends only on the underlying manifold</u>.

Another related question is whether the Godbillon-Vey invariant of an <u>analytic</u> codimension-one foliation of the <u>unit tangent bundle</u> can be continuously varied. Thurston gives examples (Thurston [53], Rosenberg and Thurston [43], Bott [4]) of analytic foliations of circle bundles over surfaces transverse to the fibers for which the Godbillon-Vey class <u>does</u> vary; however the construction only yields bundles whose Euler number is <u>less</u> than the Euler characteristic of M.

3.17 Finally we close with one more conjecture which is an immediate generalization of Theorem A.

<u>Conjecture</u>. Let $\phi: \pi \to \mathrm{Diff}^r(S^1)$ be a C^r $(2 \le r \le \omega)$ action of the fundamental group π of a closed surface M. If $e(\phi) = \pm\chi(M)$, then the action is topologically conjugate to a Fuchsian action.

If this conjecture is true, then we may replace "transversely projective" in 3.15 by "C^r."

Appendix to §3: The Space of Foliated $\mathbb{C}P^1$-bundles over a Surface

3.18 The goal of this section is to prove:

Theorem 3.10. Let $G = PSL(2,\mathbb{C})$ and let π be the fundamental group of a closed orientable surface of genus $n > 1$. Then the fibers of the map

$$w_2: \mathrm{Hom}(\pi,G) \to \mathbb{Z}/2$$

are the connected components of $\mathrm{Hom}(\pi,G)$.

Proof. Let $A = (A_1,\ldots,A_n)$, $B = (B_1,\ldots,B_n)$ and let $R_n(A,B)$ be the word $[A_1,B_1]\cdots[A_n,B_n]$. Then π has the presentation $\langle A_1,\ldots,A_n,B_1,\ldots,B_n \mid R_n(A,B)=1\rangle$. Let $\tilde{[\ ,\ ]}: G\times G \to \tilde{G} = SL(2,\mathbb{C})$ be the canonical lift of commutators in G. Let $R_n^{\sim}: G^n\times G^n \to \tilde{G}$ be the corresponding product of lifts.

Then the preimages of w_2 are exactly the preimages of $\pi_1(G) = \{\pm 1\}$ under $R_n^{\sim}: G^n\times G^n \to \tilde{G}$ (where $\pi_1(G)$ is the center of $\tilde{G}$). Hence it is clearly sufficient to prove:

(A) For every $C \in \tilde{G} = SL(2,\mathbb{C})$ the set $X_n(C) = \{(A,B)\in G^n\times G^n: R_n^{\sim}(A,B) = C\}$ is nonempty and connected.

We shall reduce the proof of (A) to the case $n = 1$. When $n = 1$ (A) says that every element $P \in \tilde{G}$ is a commutator $P = \tilde{[X,Y]}$ and that the ways of expressing P as $\tilde{[X,Y]}$ form a connected set. Suppose inductively that (A) has been proved for $n = 1$ and $n = k-1$, where $k \geq 2$. Then we may write

$$X_k(C) = \bigcup_{P\in\tilde{G}} (X_{k-1}(P)\times X_1(P^{-1}C))$$

exhibiting $X_k(C)$ as the union of nonempty connected sets (using the induction hypothesis), parametrized by the set of all $P = \tilde{[}X,Y] \in \tilde{G}$, $(X,Y \in G)$, which is all of $\tilde{G}$ (by the case $n = 1$). Hence $X_k(C)$ is a union of nonempty connected sets along a connected set, and so it is connected.

Hence all that remains to prove is:

(B) For every $C \in \tilde{G} = SL(2,\mathbb{C})$ the set $X_1(C) \subset G \times G$ is connected and nonempty.

We shall break the proof of (B) into various special cases, depending on C. If $C = 1$, then $(A,B) \in X_1(C)$ if and only if A and B have lifts which commute. This occurs precisely when A and B lie on the same one-parameter subgroup. Clearly we may deform such a pair (A,B) along a one-parameter subgroup to $(1,1)$, proving that $X_1(C)$ is connected (it is obviously nonempty).

Similarly, for $C = -1$, a direct computation shows the $(A,B) \in X_1(C)$ if and only if there exists $P \in G$ such that $P^{-1}AP$ and $P^{-1}BP$ are represented by the matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ respectively. Since G is connected and nonempty so is $X_1(C)$.

In the remaining cases for C it is useful to have a formula. Let $A = A(\rho;\lambda)$ be represented by the matrix $\begin{pmatrix} e^{\rho} & \lambda \\ 0 & e^{-\rho} \end{pmatrix}$ and $B = B(a,b,c,d)$ be represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$. We have the following formula for $t = \text{trace}\tilde{[}A,B]$:

$$\text{(C)} \quad t = (2 - 4\sinh^2\rho) + 4\, ad \sinh^2\rho - 2(a-d)(\sinh\rho)c\lambda + c^2\lambda^2$$

Now if $C \in \tilde{G}$ satisfies $t = \text{trace } C \neq \pm 2$ then every $C' \in \tilde{G}$ with $\text{trace } C' = t$ is conjugate to C. Hence the set

$$X(t) = \{(A,B) \in G\times G \mid \operatorname{trace}\tilde{[A,B]} = t\}$$

may be written as

$$\bigcup_{\operatorname{trace} C = t} X_1(C)$$

which expresses $X(t)$ as a fiber bundle over the connected set $\{C \in \tilde{G} \mid \operatorname{trace} C = t\}$ with fiber $X_1(C)$. <u>Hence</u> $X_1(C)$ <u>is connected and nonempty if and only if</u> $X(t)$ <u>is connected and nonempty</u>.

Clearly for any value of t the equation (C) may be solved, so $X(t)$ is nonempty. We must show that the set of solutions of (C) has at most two connected components, corresponding to the two-fold ambiguity of $B = B(a,b,c,d) = B(-a,-b,-c,-d)$. Since λ may be chosen to be any nonzero complex number without affecting the conjugacy class of $A(\rho;\lambda)$, we may vary (A,B) so as to keep $c\lambda$ constant. By further holding $\sinh \rho$ constant we see that a and d (and hence b) are allowed to vary in a connected set. A pair (A,B) with $\rho = 0$ may be approximated by a nearby pair with $\rho \neq 0$; in the former case we may take $\lambda = 0$ and we see that the solutions of (C) holding fixed $\sinh \rho$ and $\lambda = 0$ as well as t $(t \neq \pm 2)$ form a connected set. Hence $X(t)$ is connected for $t \neq \pm 2$.

There are four conjugacy classes of $C \in \tilde{G}$ with $|\operatorname{trace} C| = 2$, represented by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} .$$

We shall exhibit $X(t) - X_1(\frac{t}{2}I) = \{(A,B) \in G\times G\colon \tilde{[A,B]} \notin \pi_1(G)$ and $\tilde{[A,B]}$ has trace $t\}$ as a fibration over the two conjugacy classes of C with trace ± 2. The fiber is $X_1(C)$ (or $X_1(C^{-1})$ over the other component); the involution $(A,B) \mapsto (B,A)$ must interchange these two components.

Once again we examine the solutions of (C), this time for $t = \pm 2$. Only when $A = 1$ can we not assume $\lambda \neq 0$; this case may be discarded for then $[A,B] = 1$. As before the case $\rho = 0$ may be approximated by $\rho \neq 0$ by taking $|\lambda|$ large. When now $\rho \neq 0$ we may conjugate A in order to take $\lambda = 0$. This simplifies (C) to:

$$\text{(D)} \quad t = 2 - 4bc \sinh^2\rho$$

For $t = 2$, then clearly $X_1(1)$ consists of the solutions of (D) with $b = c = 0$. Then clearly $X(2) - X_1(1)$ has four components (two from the ambiguity in $\pm(a,b,c,d)$ and two from the choice $C^{\pm 1}$), as desired. Finally, for $t = -2$, we identify $X_1(-1)$ with the solutions of (D) with $\sinh^2\rho = bc = -1$. Once again we see that the set of remaining solutions form four connected components.

This concludes the proof of 3.10.

Thus $\mathrm{Hom}(\pi, PSL(2,\mathbb{C}))$ has two connected components. The component $w_2 = 0$ consists of all $\phi \in \mathrm{Hom}(\pi, PSL(2,\mathbb{C}))$ which lift to $\tilde{\phi} \in \mathrm{Hom}(\pi, SL(2,\mathbb{C}))$. Although there are $2g$ lifts $\tilde{\phi}$ of a given ϕ, they are all deformable one to another. The proof of this fact is routine, uses similar sorts of deformation arguments, and is omitted. We may conclude that $\mathrm{Hom}(\pi, SL(2,\mathbb{C}))$ is connected.

A precisely analogous theorem may be proved for $G = SO(3)$ or $G = PSU(2) \subset PSL(2,\mathbb{C})$. In this case we obtain a result of Newstead [42] that $\mathrm{Hom}(\pi, SU(2))$ is connected.

By Gunning [20] the singular variety of $\mathrm{Hom}(\pi,SL(2,\mathbb{C}))$ consists of all reducible representations. That same proof adapts to show that its image under $\mathrm{Hom}(\pi,SL(2,\mathbb{C})) \to \mathrm{Hom}(\pi,PSL(2,\mathbb{C}))$ is the singular variety of $\mathrm{Hom}(\pi,PSL(2,\mathbb{C}))$. Hence the component $w_2 \neq 0$ of $\mathrm{Hom}(\pi,PSL(2,\mathbb{C}))$ is a connected smooth variety. It is easily proved that the set of all irreducible representations in $\mathrm{Hom}(\pi,SL(2,\mathbb{C}))$ form a connected manifold; hence the connected components of $\mathrm{Hom}(\pi,PSL(2,\mathbb{C}))$ are precisely its irreducible components as an algebraic variety.

Hence the irreducible components of the real algebraic variety $\mathrm{Hom}(\pi,PSL(2,\mathbb{R}))$ are the sets $w_2^{-1}(0) = e^{-1}(2g-2) \cup \cdots \cup e^{-1}(0) \cup \cdots \cup e^{-1}(2-2g)$ and $w_2^{-1}(1) = e^{-1}(2g-1) \cup \cdots \cup e^{-1}(1-2g)$.

Generic Properties of Representations

Now we shall prove some genericity results concerning hyperbolic and projective foliated bundles over a surface M. These results will be applied to characterize Fuchsian actions as the only structurally stable projective actions of π on $\mathbb{RP}^1$ (or, rather, the double covering $\hat{\mathbb{RP}}^1$). These results are also used in §6 as a preliminary step in the construction of a transverse section of a hyperbolic foliated bundle which is topologically the tangent bundle.

The basic result we shall prove is:

Theorem 3.19. Let π be the fundamental group of a closed surface M $(\chi(M) < 0)$. The set of all $\phi \in \mathrm{Hom}(\pi,SL(2,\mathbb{R}))$ which are injective is dense in $\mathrm{Hom}(\pi,SL(2,\mathbb{R}))$.

The proof we give applies to $SL(2,\mathbb{C})$ and $SU(2)$, and indeed any connected Lie group G containing $SU(2)$ or any covering group of $SL(2,\mathbb{R})$ such that $Hom(\pi,G)$ is connected. (The condition that $Hom(\pi,G)$ is connected is conjectured to be equivalent to the Levi subgroup of G being simply connected.)

Proposition. Suppose G is a Lie group and let π be a countable group. Then in any irreducible component C of $Hom(\pi,G)$ either:

(i) for all $\phi: \pi \to G$, $\phi \in C$, ϕ is not injective;

or (ii) injective homomorphisms are dense in C.

Proof. Suppose there exists an injective $\phi_1 \in C$. We must show that such injective ϕ are dense in C. Since $Hom(\pi,G)$ is an analytic variety and thus a Baire space, it suffices to show that for all $w \in \pi$, the set of all $\phi \in C$ with $\phi(w) = 1$ is a nowhere dense closed subset. Suppose not; then there is a nonempty open set $U \subset C$ such that $\phi(w) = 1$ for $\phi \in U$. The mapping

$$w: Hom(\pi,G) \longrightarrow G$$
$$\phi \longmapsto \phi(w)$$

is then an analytic map which is constant on an open set. Hence it is constant on the whole irreducible component C containing U, contradicting $\phi_1 \in C$ being injective. Q.E.D.

Proof of Theorem 3.19. By Gunning [20] and Whitney [55] as in 3.18, $Hom(\pi,SL(2,\mathbb{R}))$ is an irreducible variety. A Fuchsian ϕ is clearly injective; by the proposition, injections are dense in $Hom(\pi,SL(2,\mathbb{R}))$. Q.E.D.

The same proof works for $\mathrm{Hom}(\pi,SL(2,\mathbb{C}))$. By modifying the proof of the proposition, the proof can be made to work for $\mathrm{Hom}(\pi,SU(2))$. For if $w \in \pi$ is constant on a nonempty open set U in $\mathrm{Hom}(\pi,G)$, it must also be constant on a neighborhood of U in the complexification $\mathrm{Hom}(\pi,G_{\mathbb{C}})$ (this follows from the Weierstrass preparation theorem). Hence either (i) there is no isomorphism $\phi\colon \pi \to G_{\mathbb{C}}$ in a given component C, or (ii) isomorphisms $\phi\colon \pi \to G$ are dense in C. If $G = SU(2)$, then a Fuchsian $\phi\colon \pi \to SL(2,\mathbb{R}) \subset SL(2,\mathbb{C})$ is an isomorphism, so (ii) applies to show that isomorphisms are dense in $\mathrm{Hom}(\pi,SU(2))$. (As a corollary, there is an embedding of π in $SU(2)$; the proof of this, however, does not give an explicit embedding.)

<u>Remark 3.20</u>. Unfortunately we cannot prove that isomorphisms are dense in $\mathrm{Hom}(\pi,PSL(2,\mathbb{R}))$, $\mathrm{Hom}(\pi,PSL(2,\mathbb{C}))$ or $\mathrm{Hom}(\pi,SO(3))$ since we do not know whether in the components of $\mathrm{Hom}(\pi,PSL(2,\mathbb{C}))$ with $w_2 \neq 0$ there are any isomorphisms. This is equivalent to:

<u>Conjecture 3.21</u>. There exists an embedding of the group $\langle A_1,\ldots,A_g,B_1,\ldots,B_g\colon ([A_1,B_1]\cdots[A_g,B_g])^2 = 1\rangle$ in $SL(2,\mathbb{C})$.

<u>Structural Stability of Fuchsian Actions</u>

Our goal is to prove the following theorem. Let π be the fundamental group of a closed surface of genus $n > 1$.

<u>Theorem 3.14</u>. A projective action of π on the circle $\mathbb{RP}^1$ which comes from $\phi \in \mathrm{Hom}(\pi,SL(2,\mathbb{R}))$ is structurally stable if and only if it is Fuchsian.

Proof. For the structural stability of Fuchsian actions (equivalently Anosov foliations) see 3.13 and the references given there. Conversely, suppose that ϕ defines a structurally stable projective action on $\mathbb{RP}^1$. ϕ is an isomorphism, since isomorphisms are dense in $\mathrm{Hom}(\pi,SL(2,\mathbb{R}))$. It only remains to prove that $\phi(\pi)$ is discrete. Otherwise there exists some $w \in \pi$ such that $\phi(w)$ is elliptic or parabolic, by 3.2. Now the proof of Proposition 3.19 shows that for any $t \in \mathbb{R}$, the set of all $\psi \in \mathrm{Hom}(\pi,SL(2,\mathbb{R}))$ such that trace $\psi(w) \neq t$ is either a dense open set or empty. Since ϕ lies in the same irreducible component as a Fuchsian action (or the trivial representation), this set is dense. Consequently in any neighborhood U of ϕ there will exist $\psi \in U$ such that $\psi(w)$ is an elliptic element whose rotation angle is different than that of $\phi(w)$, contradicting structural stability. Q.E.D.

§4. Ideal Sections and Relative Euler Class

In this chapter our aim is to state a relative version of Theorem 3.6. Because the usual Euler class of a 2-disc bundle E_ϕ over a surface with $\partial M \neq \emptyset$ always vanishes ($H^2(M) = 0$), it is necessary to include some sort of boundary data and define the relative Euler class with respect to this boundary data.

This boundary data will be called ideal sections. Consider a projective or conformal foliated bundle ∂E_ϕ; then the ideal bundle to E_ϕ, ∂E_ϕ is the associated (n-1)-sphere bundle to the disc bundle E_ϕ. When $n = 2$ a section of ∂E_ϕ (i.e. an ideal section) is equivalent to a trivialization of ∂E_ϕ and hence of E_ϕ. The Euler class of E_ϕ is then the primary obstruction for the existence of an ideal section. Hence we make the following definition:

Definition 4.1. Let E_ϕ be a closed n-disc bundle over an oriented n-manifold M. Let σ denote an ideal section over ∂M. The relative Euler class $e(E;\sigma) \in H^n(M,\partial M) \cong \mathbb{Z}$ is the obstruction for extending $\sigma\colon \partial M \to \partial E|_{\partial M}$ to an ideal section $M \to \partial E$.

The main formal property of the relative Euler class is the following property of additivity:

Additivity Lemma 4.2. Let E denote a closed n-disc bundle over an n-manifold M and suppose V is a closed (n-1)-submanifold such that (i) $\partial M \subset V$;

(ii) V separates M into $M - V = (\text{int } M_1) \cup \cdots \cup (\text{int } M_k)$.

Let $\sigma\colon V \to \partial E|_V$ be an ideal section over V. Then

$$e(E;\sigma|_{\partial M}) = \sum_{i=1}^{k} e(E|_{M_i};\sigma|_{\partial M_i}) .$$

This formula is just the addition formula in elementary obstruction theory (see Steenrod [48], §36) so we omit the proof.

4.3 As an example of the relative Euler class, let f_i $(i = 1,2)$ be two sections of an S^1-bundle ∂E over S^1. The difference $d(f_1,f_2)$, which is the obstruction to finding a homotopy between f_1 and f_2, is the relative Euler class $e(p^*E;f)$ where $p: M\times[1,2] \to M$ is projection and $f: M\times\{1,2\} \to \partial E$ is f_i on $M\times\{i\}$, $i = 1,2$. Since sections of ∂E are classified up to homotopy by $H^2(S^1\times[1,2],S^1\times\{1,2\}) \cong \mathbb{Z}$, for any fixed ideal section f_1, there is a one-to-one correspondence

$$\{\text{homotopy classes of ideal sections}\} \leftrightarrow \mathbb{Z}$$

$$\{f_2\} \to d(f_1,f_2)$$

4.4 If E_ϕ is a hyperbolic foliated bundle then ∂E_ϕ is a conformal foliated bundle. If $\dim X = 2$, the "conformal" geometry of $\partial X = S^1_\infty$ is the projective geometry of $\mathbb{RP}^1$. Hence an ideal section over a 1-manifold is a singular projective structure. Since every section of ∂E is homotopic to either a transverse or constant section (2.11), we shall always assume that our ideal sections are either transverse or constant.

We have defined special projective structures over closed curves in 2.11. These sections of ∂E depend only on the holonomy of these curves (although they do not quite depend continuously on the holonomy). It is this property which forces us to sacrifice a continuous assignment of ideal sections. Since projective structures correspond to elements of $\widetilde{SL(2,\mathbb{R})}$, we are in effect asking for a cross-section to $\tilde{SL}(2,\mathbb{R}) \to PSL(2,\mathbb{R})$. Since no continuous cross-section exists, we must be content with the discontinuous "special" sections introduced in 2.11.

We will actually need a slight variant of these special ideal sections for the following reason. The idea of the proof is to cut the closed surface M along a closed curve V into $M-V = M_1 \cup \cdots \cup M_k$. Using additivity, one of the results of §5 and the assumption $e(\phi) = \chi(M)$, we get $e(\phi_i;\text{boundary data on } M_i) = \chi(M_i)$ for each M_i. Using more results from §5 we construct a section over M_i with prescribed boundary values. Hence we must begin the proof by choosing a preassigned section over V. This section must be a section of E_ϕ and not ∂E_ϕ since $V \subset \text{int } M$ and we plan to extend the section to one which is transverse over int(M). Hence we need an "interior realization" of the special ideal sections, the "special interior sections".

Recall that if $C \subset M$ is a single closed curve, the special ideal section is defined by the formula

$$\begin{aligned} \mathbb{R} &\mapsto \mathbb{R} \times \partial X \\ t &\mapsto (t, \exp(-t \log \phi(c))y) \end{aligned} \tag{4.5}$$

where we have chosen a "parametrization" $C \xrightarrow{\approx} \mathbb{R}/\mathbb{Z}$ and some point $y \in \partial X$ not fixed under $\phi(C)$ (if such a point exists).

To define the special interior section, we simply allow y to be some point in X, not fixed under $\phi(C)$. (If $\phi(C) = 1$, then y can be arbitrary.)

4.6 Since the orbits of elliptic (resp. parabolic) one-parameter subgroups are circles (resp. horocycles) centered at the stationary point, we see that special interior sections map onto invariant circles (resp. horocycles). By projection from this point we obtain a natural correspondence between special ideal sections and special interior sections. When the holonomy is hyperbolic this projection occurs along lines orthogonal to the invariant geodesic and equidistant curves.

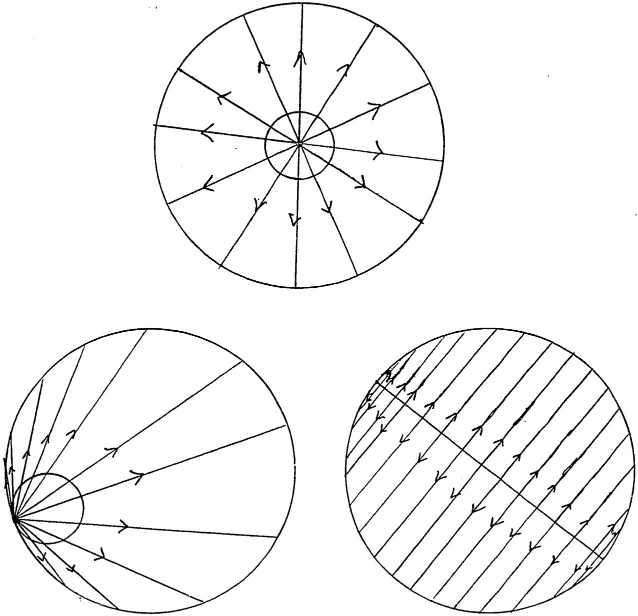
Definition 4.7. A continuous deformation $\phi_t \in \mathrm{Hom}(\pi, G)$ is admissible if and only if for each $\gamma \in \pi$ homotopic to a curve in ∂M $\phi_t(\gamma) \notin \mathrm{Sym}$. (In terms of $|\mathrm{trace}|$ this means $|\mathrm{trace}\ \phi_t(\gamma)|$ is never zero.)

If ϕ_t is admissible, then $\log \phi_t(\gamma)$ is continuous in t for γ homotopic to ∂M. Consequently the special ideal sections σ_t of E_{ϕ_t} over ∂M vary continuously in t. By the covering homotopy property the bundles E_{ϕ_t} are all equivalent and the ideal sections over ∂M are homotopic; hence $e(\phi_t;\sigma_t) = e(\phi;\sigma)$. Hence $e(\phi;\sigma)$ is continuous in ϕ as long as ϕ varies admissibly.

4.8 Now suppose that M is a surface-with-boundary and E_ϕ is a hyperbolic foliated bundle over M. We are interested in knowing which special interior sections $\partial M \to E_\phi$ might possibly extend to transverse sections over M.

The necessary conditions that a special interior section $s\colon \partial M \to E_\phi$ extend to a transverse section over M can be seen by considering a developing map. Say that a component c of ∂M is hyperbolic, parabolic, or elliptic depending on whether $\phi(c)$ is hyperbolic, parabolic, or elliptic respectively. If $f\colon \tilde{M} \to X$ is a nonsingular developing map corresponding to a transverse section of E_ϕ, then it is natural to hope that f is an embedding on some fundamental

Fig. 4.1. Projecting an ideal section to an interior section



domain Π for π acting on $\tilde{M}$. Indeed, the sections we construct will all have this property.

Corresponding to the boundary components $B_1,\ldots,B_k$ of M there will be lifts $\tilde{B}_1,\ldots,\tilde{B}_k$ to boundary segments on Π. If $f|_{\Pi}$ is an embedding then all $f(\tilde{B}_i)$ are disjoint.

Now suppose $f|_{\partial M}$ is a special interior section. That is, for each $B_i \subset \partial M$, there is $\phi(B_i)$-invariant curve to which B_i develops. If $\phi(B_i)$ is elliptic or parabolic, then we shall want f to map a collar of $\tilde{B}_i$ to the outside of a $\phi(B_i)$-invariant circle or horocycle. If $\phi(B_i)$ is hyperbolic, then f maps a collar of $\tilde{B}_i$ to one of the two regions bounded by the equidistant curve $f(\tilde{B}_i)$. Therefore in addition to the special interior section, we specify one of the regions bounded by the $\phi(B)$-invariant equidistant curve, if $\phi(B)$ is hyperbolic.

Suppose now that we are given this information: for each $B_i \subset \partial M$, a $\phi(B_i)$-invariant circle, horocycle, or equidistant curve, and if $\phi(B_i)$ is hyperbolic, also one of the regions R_i bounded by the invariant equidistant curves. For each elliptic or parabolic $\phi(B_i)$ R_i will denote the region outside the $\phi(B_i)$-invariant disc or horodisc. We want to choose the R_i sufficiently large so that for any distinct R_i, R_j, the union $R_i \cup R_j = X$. This will have the effect of making the developing map of the boundary "sufficiently small".

Definition 4.9. A special interior section $s\colon \partial M \longrightarrow E_\phi$ is sufficiently small if and only if there is a developing map f for s, and for each $\phi(B_i)$, regions R_i bounded by $f(\tilde{B}_i)$ as above, such that for every distinct B_i, $B_j \subset \partial M$, the union $R_i \cup R_j = X$.

The reason for the terminology in the above is that if s is any special interior section over ∂M, then s can always be made smaller around the elliptic and parabolic $\phi(B_i)$ in the sense that there exists another special interior section s' with a developing map which maps the invariant circles and horocycles to smaller ones in the hyperbolic plane. If the fixed points of the elliptic and parabolic $\phi(B_i)$ are all distinct, then there exists a sufficiently small interior section s. Compare Fig. 4.2.

Proposition 4.10. Let $\phi \in \mathrm{Hom}(\pi,G)$ and let σ be an ideal section of E_ϕ over ∂M. Then $e(\phi;\sigma) \in \mathbb{Z}$ may be computed by the following algorithm:

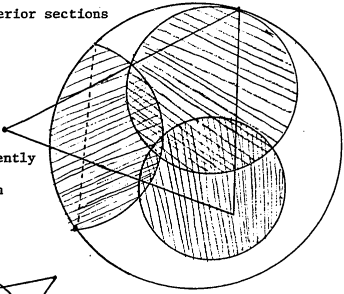
Represent π as $\langle A_1,B_1,\ldots,A_g,B_g,C_1,\ldots,C_k \mid [A_1,B_1]\cdots[A_g,B_g] C_1\cdots C_k = 1\rangle$ where ∂M is the disjoint union $C_1 \cup \cdots \cup C_k$. Each commutator $\phi([A_i,B_i])$ has a well-defined lift $\tilde{\phi}([A_i,B_i])$ to $\tilde{G}$; associated to the ideal sections σ is their modified holonomy $\tilde{C}_i \in \tilde{G}$. Then $\hat{\phi}([A_1,B_1])\cdots\tilde{C}_1\cdots\tilde{C}_k$ lies in $\mathrm{Ker}\ \tilde{G} \to G = \pi_1(G) \cong \mathbb{Z}$ and is the relative Euler number $e(\phi;\sigma)$.

This may seen directly by cutting M along curves obtaining a polygon with some of its sides identified. Going around the side of the polygon gives the word $W = [A_1,B_1]\cdots[A_g,B_g]C_1\cdots C_k$.

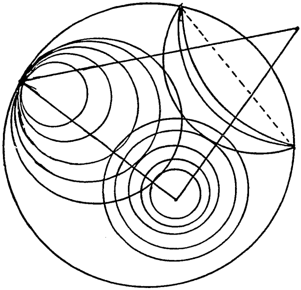
Begin to extend the ideal section σ from ∂M to int M. In fact extend σ to an ideal section over $M-\{y\}$ for some $y \in \mathrm{int}\ M$. Now the word W applied to the lifts of $\phi([A_i,B_i])$ and $\phi(C_i)$ represents the total winding number of σ over a small circle centered at y. This is clearly the obstruction to a section of ∂E.

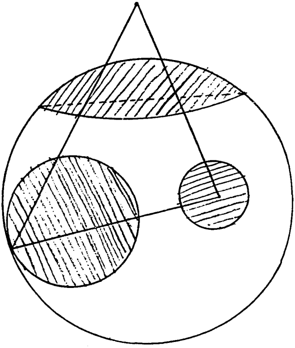
Fig. 4.2 Special interior sections

The shaded regions are the complements of the R_i.



This section is not sufficiently small, but (as below), it is easy to find one which is sufficiently small.





Here is one which is sufficiently small.

§5. Hyperbolic Structures on a Pair of Pants

In this chapter M will denote a compact oriented surface (with boundary) diffeomorphic to a sphere minus three discs, i.e. a pair of pants. Denote the three components of ∂M by A, B, C and give them orientations compatible with those of M. Let x_0 denote a base point in the interior of M and let the corresponding elements of $\pi_1(M,x_0)$ also be denoted by A, B, C. The fundamental group $\pi = \pi_1(M,x_0)$ is generated by A, B, C subject to the relation $ABC = 1$.

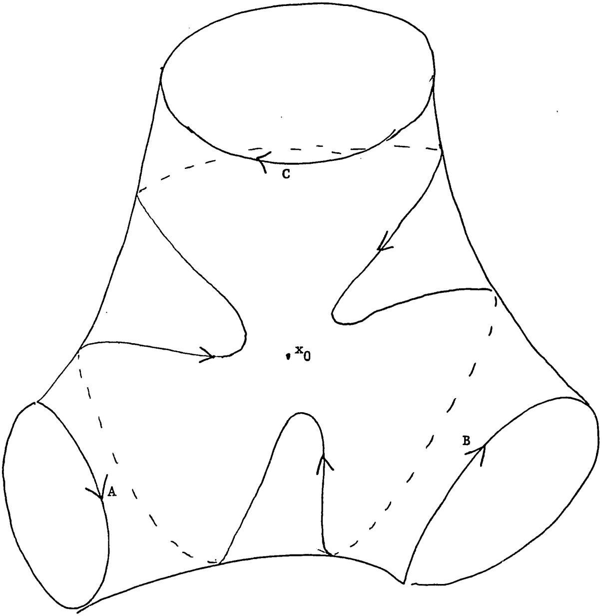
We shall be interested in the classification of hyperbolic foliated bundles E_ϕ over M where $\phi \in \mathrm{Hom}(\pi,G)$. Let $\sigma = \sigma(\phi)$ denote a special ideal section over ∂M. The goal of this chapter is to prove the following theorem.

Theorem 5.1. (i) $e(\phi;\sigma) = -1$, 0, or +1.

(ii) $e(\phi;\sigma) = 0$ provided that one of the following occurs: (a) one of $\phi(A)$, $\phi(B)$, and $\phi(C)$ equals 1 and none are symmetries; (b) $\phi(\pi)$ has a fixed point in $\mathbb{RP}^2 - X$; (c) two of $\phi(A)$, $\phi(B)$, $\phi(C)$ are hyperbolic and their axes intersect in $X \cup \partial X$.

(iii) $e(\phi;\sigma) = -1$ (resp. +1) implies that either (a) $\phi(\pi)$ is solvable, and there are two components A, B of ∂M such that $\phi(A)$, $\phi(B)$ are elliptic, or (b) E_ϕ admits an orientation-preserving (resp. orientation-reversing) transverse section $f: M \to E_\phi$ which restricts to a special interior section over ∂M. If, in case (b), $\phi(A)$ is elliptic or parabolic, then f maps a collar neighborhood of A in M to the outside of a $\phi(A)$-invariant disc or horodisc, in local foliation charts. Otherwise, in case (b), f maps A to a $\phi(A)$-invariant geodesic.

Fig. 5.1 A pair of pants



(iv) Suppose that $\phi(A)$, $\phi(B)$, and $\phi(C)$ are all hyperbolic. Then $e(\phi;\sigma) = \pm 1$ if and only if E_ϕ has a transverse section f which restricts to a special interior section over ∂M. For each hyperbolic $\phi(A)$, $f|_A$ is geodesic.

5.2 There is a simple method for constructing representations $\phi \in \mathrm{Hom}(\pi, G)$ geometrically. Using this method we obtain a natural picture of $\mathrm{Hom}(\pi,G)$ as a kind of "configuration space". Let ℓ^A, ℓ^B, and ℓ^C be any three lines in $\mathbb{RP}^2$ none of which are tangent to ∂X. Then there are unique projection involutions $I_1(\ell^A)$, etc. which fix ℓ^A, etc., and leave X invariant. Therefore $I(\ell(A))$, $I(\ell(B))$, $I(\ell(C)) \in SO(2,1)$; since the I^A are involutions, it follows that

$$(I^B I^C)(I^C I^A)(I^A I^B) = 1$$

that by taking $\phi(A) = I^B I^C$, $\phi(B) = I^C I^A$, $\phi(C) = I^A I^B$ we obtain a representation $\phi\colon \pi \to SO(2,1)$. The image $\phi(\pi)$ is contained in G if and only if I^A, I^B, and I^C either all preserve the orientation of X or reverse it; i.e. when ℓ^A, ℓ^B, ℓ^C are either all reflections or all symmetries. We shall say that ϕ has been <u>factored</u> into the involutions I^A, I^B and I^C.

Thus a representation $\phi \in \mathrm{Hom}(\pi,G)$ factors into involutions if and only if ϕ extends to $\hat{\phi} \in \mathrm{Hom}(\hat{\pi},G)$ where $\hat{\pi} \cong \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$ contains π as an index-2 subgroup in the following way: If $\hat{\pi}$ is freely generated by involutions I_{CA}, I_{AB}, I_{BC}, then

$$A \mapsto I_{CA} I_{AB}$$
$$B \mapsto I_{AB} I_{BC}$$
$$C \mapsto I_{BC} I_{CA}$$

gives the desired embedding $\pi \hookrightarrow \hat{\pi}$.

This process is described more neatly and geometrically by Thurston's notion of an "orbifold" (see [54], §13). There is a 2-orbifold $\hat{M}$ which M double covers (Fig. 5.2), and the map induced on fundamental groups by the covering $M \to \hat{M}$ is the inclusion $\pi \hookrightarrow \hat{\pi}$. Our goal in this chapter, essentially, is to use the assumption $|e(\phi;\sigma)| = 1$ to construct a hyperbolic structure on M by first constructing a hyperbolic structure on $\hat{M}$.

First we prove the following theorem:

Theorem 5.2. Suppose $\phi(\pi)$ is not solvable. Then at least one of the following must hold:

(R) ϕ can be factored into reflections, i.e. ϕ extends to $\hat{\phi}: \hat{\pi} \to SO(2,1)$ so that $\hat{\phi}(I_{AB})$, $\hat{\phi}(I_{BC})$, and $\hat{\phi}(I_{CA})$ are all reflections in lines of X.

(S) ϕ can be factored into symmetries, i.e. ϕ extends to $\hat{\phi}: \hat{\pi} \to G$.

In 5.8 we show how to pass from a hyperbolic structure on $\hat{M}$ to one on M; section 5.9 gives the basic picture of a hyperbolic structure when ϕ is Fuchsian. In 5.10-5.11 we construct hyperbolic structures on $\hat{M}$. Finally, in 5.12-5.16 we compute the relative Euler class and prove Theorem 5.1.

The proof of 5.2 involves the geometric investigation of a certain triangle associated to ϕ as follows. Every nontrivial element of G has a preferred fixed point in $\mathbb{RP}^2$ (see 2.5); thus if $\phi(\pi)$ is not solvable, the preferred fixed points p_A, p_B, p_C of $\phi(A)$, $\phi(B)$, $\phi(C)$ respectively are the vertices of a triangle (in fact four triangles)

Δ in $\mathbb{RP}^2$.

Let ℓ_{CA}, etc. be the <u>extended sides</u> of Δ, i.e. the projective lines containing p_C and p_A, etc. That $\phi(\pi)$ is nonabelian implies Δ cannot consist of a single point. Later on (5.6) we prove that Δ is actually nondegenerate, but our first step is to understand how the lines ℓ_{CA}, etc. intersect $\partial X = S^1_\infty$.

Lemma 5.3. If $\phi(\pi)$ is solvable, no line ℓ_{CA}, etc. can be tangent to ∂X.

Proof. By symmetry it suffices to show ℓ_{CA} is not tangent to ∂X. If ℓ_{CA} is tangent to ∂X at $p \in \partial X$, then $\phi(C)$ and $\phi(A)$ are both hyperbolic elements fixing p. Then $\phi(B) = \phi(A)^{-1}\phi(C)^{-1}$ also fixes p which implies $\phi(\pi)$ is solvable, a contradiction.

Lemma 5.4. Suppose $\phi(\pi)$ is not solvable and ℓ_{CA} does not intersect ∂X. Then neither ℓ_{AB} nor ℓ_{BC} intersects ∂X.

Proof. The condition "ℓ_{CA} does not intersect ∂X" is equivalent to $\phi(C)$ and $\phi(A)$ being hyperbolic with their invariant axes intersecting in X (indeed, the point of intersection is dual to the line ℓ_{CA}). Hence we must prove that if $\phi(C)$ and $\phi(A)$ are hyperbolic with intersecting axes, then $\phi(B)$ is hyperbolic and its invariant axis intersects each of the invariant axes of $\phi(C)$ and $\phi(A)$.

We may choose coordinates on $\partial X \approx \mathbb{RP}^1$ so that the fixed points of $\phi(C)$ and $\phi(A)$ are $\{1,-1\}$ and $\{r,\infty\}$ respectively. Then $\phi(C)$ and $\phi(A)$ may be represented by the matrices

$$\begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}, \quad \begin{pmatrix} e^{\rho} & -2r\sinh\rho \\ 0 & e^{-\rho} \end{pmatrix}$$

respectively. The condition that their invariant axes intersect is that $-1 < r < 1$. Now

$$\begin{aligned} \text{trace } \phi(B) &= \text{trace } \phi(B)^{-1} = \text{trace } \phi(C)\phi(A) \\ &= 2\cosh\rho\cosh\theta - 2r\sinh\rho\sinh\theta \\ &= (1+r)\cosh(\rho+\theta) + (1-r)\cosh(\rho-\theta) \\ &> 2 \end{aligned}$$

proving $\phi(B)$ is hyperbolic.

Now suppose that the invariant axes of $\phi(B)$ and $\phi(A)$ do not intersect in X. Define a deformation ϕ_t by deforming $\phi(C)$ and $\phi(A)$ along one-parameter subgroups:

$$\begin{aligned} \phi_t(C) &= \exp t \log \phi(C) \\ \phi_t(A) &= \exp t \log \phi(A) \\ \phi_t(B) &= \phi_t(A)^{-1}\phi_t(C)^{-1}, \quad 1 \geq t \geq 0 \end{aligned}$$

By the above argument $\phi_t(B)$ is hyperbolic. Moreover as $t \to 0$ the preferred fixed $p_B(t)$ point of $\phi_t(B)$ approaches the line ℓ_{CA}. Thus for t near 0 the projective line $\ell_{AB}(t)$ containing p_A and $p_B(t)$ does not intersect ∂X. For each $t \neq 0$, the image $\phi_t(\pi)$ is not solvable. Since $\ell_{AB}(t)$ is continuous in t and $\ell_{AB}(1) = \ell_{AB}$ intersects ∂X there must be some t for which $\ell_{AB}(t)$ is tangent to ∂X. However this contradicts Lemma 5.3. Q.E.D.

Remark 5.5. Lemmas 5.3 and 5.4 can be interpreted in terms of the Lorentzian geometry of $G = PSL(2,\mathbb{R})$ as follows. Giving G the bi-invariant Lorentz metric coming from the Killing form, the one-parameter subgroups become geodesics. Since $\exp: \mathcal{G} \to G$ is surjective

For number sequence only.

there is a geodesic triangle with vertices $\phi(A)$, $\phi(B)$, $\phi(C)$. An observer sitting at $1 \in G$ looking out along geodesics sees the triangle $p_A p_B p_C$ in his $\mathbb{RP}^2$ field-of-vision. The preceding lemmas show that there are strong restrictions on how the geodesic triangle may intersect the light cone. If the triangle is nondegenerate 5.3 says no line intersects it tangentially and 5.4 says that if some extended side doesn't intersect the light cone, then no extended side intersects the cone. The various situations which have been ruled out as impossible are shown in Fig. 5.3.

5.6 Proof of Theorem 5.2. We continue to assume $\phi(\pi)$ fixes no point in $\mathbb{RP}^2$. Consider the triangle Δ at its extended sides ℓ_{CA}, etc. 5.3 implies that for each extended side ℓ there is an involution $J \in SO(2,1)$ fixing each point of ℓ (and the dual point ℓ^*). By 5.4 the involutions J_{CA}, J_{AB}, J_{BC} are either all reflections or all symmetries. We claim these involutions satisfy

$$\phi(A) = J_{CA}J_{AB}$$
$$\phi(B) = J_{AB}J_{BC}$$
$$\phi(C) = J_{BC}J_{CA}$$

By symmetry, it suffices to prove only one of these relations.

First, we show there exists some $J \in SO(2,1)$, $J^2 = 1$, with $\phi(A) = J_{CA}J$. If $p_A \in X \cup \partial X$ (so $\phi(A)$ is elliptic or parabolic) then J_{CA} reflects each $\phi(A)$-invariant circle or horocycle. It follows that $J_{CA}\phi(A)J_{CA}^{-1} = \phi(A)^{-1}$, i.e. $(J_{CA}\phi(A))^2 = 1$ as desired. If $\phi(A)$ is hyperbolic, then J_{CA} is symmetry about some point on $\phi(A)$-invariant geodesic; hence it reflects that line and conjugates

Fig. 5.2 A pair of pants doubly covering a right-hexagon orbifold

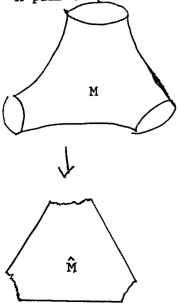
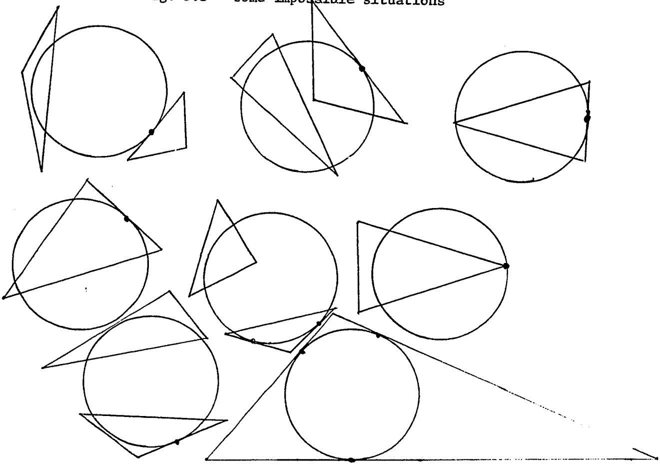


Fig. 5.3 Some impossible situations



$\phi(A)$ to $\phi(A)^{-1}$. Again this implies $J_{CA}\phi(A)$ is an involution.

Now we show $J = J_{AB}$. Certainly J fixes p_A. On the circle of directions about p_A the involution $J = J_{CA}\phi(A)$ reverses orientation. However since an orientation-preserving involution cannot have an isolated fixed point, p_A lies on the fixed line of J. It remains to show that p_B also lies on the fixed line of J. Now

$$J\phi(B)J = J_{CA}\phi(A)\phi(B)J_{CA}\phi(B) = J_{CA}\phi(C)^{-1}J_{CA}\phi(A) = \phi(C)\phi(A) = \phi(B)^{-1}$$

so that J reverses orientation in a neighborhood of p_B, as above; hence the fixed line of J contains p_B. Therefore $J = J_{AB}$ so $\phi(A) = J_{CA}J_{AB}$, completing the proof of Theorem 5.2.

<u>Corollary 5.6</u>. The vertices of Δ cannot be collinear.

<u>Remark 5.7</u>. The main point of Theorem 5.2 is that it gives a geometric object corresponding to a representation $\phi \in \mathrm{Hom}(\pi,G)$, namely a set of triangles with vertices p_A, p_B, p_C in $\mathbb{RP}^2$ which satisfies 5.3, 5.4 and 5.6. Namely, the projective lines $\overleftrightarrow{p_Ap_B}$, $\overleftrightarrow{p_Bp_C}$ and $\overleftrightarrow{p_{AC}}$ are distinct and intersect ∂X in two points. For each line $\overleftrightarrow{p_Ap_B}$ there is a unique involution I_{AB} of $\mathbb{RP}^2$ fixing $\overleftrightarrow{p_Ap_B}$ and leaving X invariant. By defining ϕ by

$$A \mapsto I_{CA}I_{AB}$$
$$B \mapsto I_{AB}I_{BC}$$
$$C \mapsto I_{BC}I_{CA}$$

we have associated a homomorphism ϕ to a triangle in Δ. In this way we establish a one-to-one correspondence between $\phi \in \mathrm{Hom}(\pi,G)$, $\phi(\pi)$ not solvable, and triples of distinct nonconcurrent lines each intersecting ∂X in two points.

5.8 Now we turn to the proof of part (ii) of Theorem 5.1. Continuing to assume $\phi(\pi)$ is not solvable, we see Theorem 5.2 implies that if $e(\phi;\sigma) \neq 0$, then ϕ can be factored into reflections. In that case we can construct a transverse section $M \to E_\phi$ with reasonably good boundary behavior.

Proposition 5.8. Let $\phi \in \mathrm{Hom}(\pi,G)$ satisfy

(a) $\phi(\pi)$ is not solvable

(b) ϕ factors into reflections R_{AB}, R_{BC}, R_{CA}.

Let Q_A (resp. Q_B, Q_C) be a $\phi(A)$ (resp. $\phi(B)$, $\phi(C)$) -invariant region bounded by circles, horocycles or equidistant curves such that the three of them are pairwise disjoint. Then there exists a transverse section $f\colon M \to E_\phi$ and a developing map $\tilde{f}\colon \tilde{M} \to X$, with lifts $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ of A, B, C such that $\tilde{f}(\tilde{A}) = \partial Q_A$, $\tilde{f}(\tilde{B}) = \partial Q_B$, $\tilde{f}(\tilde{C}) = \partial Q_C$.

We will construct a transverse section by describing a developing map. To do this we need only specify an immersion $\Pi \to X$ (where Π is a fundamental domain for π on $\tilde{M}$) and identifications of various edges of ∂X which generate the holonomy.

A fundamental domain for $\tilde{M}$ can be formed by taking two hexagons and glueing them together along a side. If H is one of the hexagons, denote the sides A, β, C, α, B, γ in order. Let J denote a reflection in the side γ of H which maps H diffeomorphically onto the other hexagon JH. Then $H \cup \partial JH$ is the fundamental domain corresponding to M and has eight sides $JA \cup A$, β, C, α, $JB \cup B$, $J\alpha$, JC, $J\beta$ (see Fig. 5.4a). The pair-of-pants M is obtained by identifying α

Fig. 5.4a

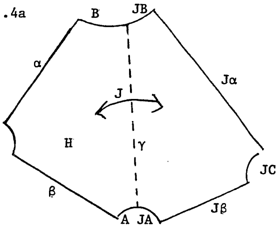
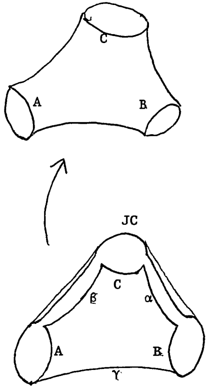


Fig. 5.4b



with $J\alpha$ and β with $J\beta$ (see Fig. 5.4b).

Now let R_α, R_β be the local reflections in a tubular neighborhood of α and β respectively. Then the deck transformation in $\tilde{M}$ which maps the fundamental octagon $H \cup JH$ to the adjacent one which is on the other side of α (resp. β) is given by $R_A J$ (resp. $R_B J$).

A hyperbolic structure on M can be obtained from a hyperbolic structure on the "orbifold" H. That is, first factor ϕ into reflections R_{AB}, R_{BC}, R_{CA}. Then a developing map for H (which determines a developing map for M) is obtained from an immersion $f: H \longrightarrow X$ such that the reflections R_{AB} (resp. R_{BC}, R_{CA}) fix the line γ (resp. α, β). The developing map on JH is just $R_{AB} \circ f \circ J$; if $W(J,J_A,J_B)$ is a word, then the developing map on $W(J,J_A,J_B)$ is just $W(R_{AB},R_{BC},R_{CA}) \circ f \circ W(J,J_A,J_B)^{-1}$. This defines a developing map with holonomy ϕ, and hence a transverse section to E_ϕ.

The boundary conditions we shall require are the following. For a component C of ∂M, we seek a special interior section over C which develops C to some $\phi(C)$-invariance circle, horocycle, or equidistant curve. Hence the developing image of the side of H denoted A (resp. A, C) will be a segment of a $\phi(A)$- (resp. $\phi(B)$, $\phi(C)$) invariant curve.

5.9 Our goal is to build a developing section for M with holonomy ϕ from the triangle Δ. So far we have not specified a triangle Δ but rather the triangular configuration determined by three points or three lines in $\mathbb{RP}^2$. One of our first objectives is to find one of the four triangles with vertices p_A, p_B, p_C and turn it into a fundamental hexagon.

Let $\phi \in \mathrm{Hom}(\pi,G)$ be such that $\phi(\pi)$ is not solvable and factors into reflections. Let p_A, p_B, p_C be the three preferred fixed points of $\phi(A)$, $\phi(B)$, $\phi(C)$ respectively. We shall try to find a triangle Δ with vertices p_A, p_B, p_C which intersects X in a convex set. A fundamental hexagon will be constructed by truncating Δ at its vertices along the corresponding invariant curves.

The model case (when ϕ is Fuchsian and factors into reflections) occurs when all three of the vertices of Δ lie outside X, each side of Δ intersects ∂X, and $\Delta \cap X$ is a convex set. Then $\phi(A)$, $\phi(B)$, $\phi(C)$ are hyperbolic and we truncate Δ at each vertex along the corresponding invariant curve, obtaining a right hexagon in the hyperbolic plane (see Fig. 5.5). The group generated by reflections in the three sides of Δ is isomorphic to $\hat{\pi}$, and the natural representation $\hat{\pi} \longrightarrow SO(2,1)$ is Fuchsian. The quotient $X/\phi(\pi)$ is then a complete hyperbolic pair-of-pants, and the developing image of $\bigcup_{\gamma\in\pi} \gamma H$ is the convex hull of the limit set of $\phi(\pi)$. More generally for each vertex p_A we could choose a $\phi(A)$-invariant equidistant curve c_A, and, assuming that $c_A \cap c_B = c_B \cap c_C = c_C \cap c_A = \emptyset$, truncate Δ along these equidistant curves obtaining as a fundamental hexagon for ϕ a <u>right hexagon with curved edges</u>.

5.10 Now we classify the various possibilities for triangle Δ, hoping to construct a fundamental domain for a hyperbolic structure on M resembling the previous one (two right hexagons). We shall find transverse sections $M \longrightarrow E_\phi$ by constructing polygons satisfying the hypotheses of Proposition 5.8

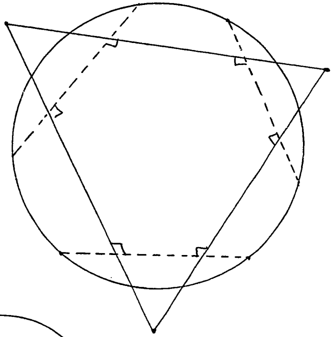


Fig. 5.5

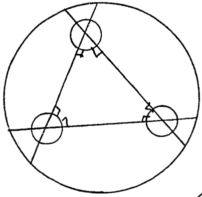


Fig. 5.6

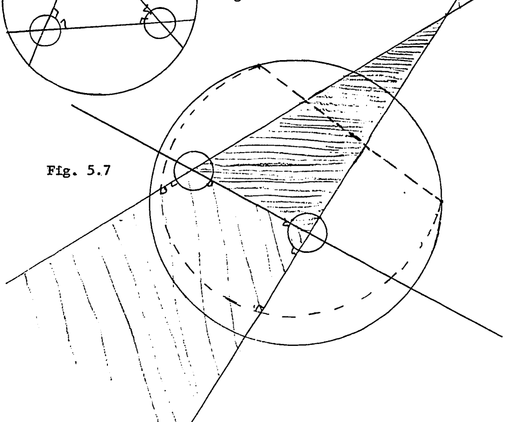


Fig. 5.7

For simplicity, we consider only the case when none of $\phi(A)$, $\phi(B)$, $\phi(C)$ are parabolic. The situation when one of them, say $\phi(A)$, is parabolic is essentially a limiting case of the situation when $\phi(A)$ is elliptic and is handled in the same manner.

Suppose first of all that $\phi(A)$, $\phi(B)$, $\phi(C)$ are all elliptic (i.e. when p_A, p_B, p_C lie in X). Then there is a unique triangle Δ with vertices p_A, p_B, p_C such that $\Delta \subset X$ (in fact Δ is just the convex hull of $\{p_A, p_B, p_C\}$). Choose discs centered at the vertices chosen sufficiently small so they are pairwise disjoint. After removing these discs from Δ we obtain a right hexagon Δ', three of whose edges are circular arcs. Choosing a decomposition of M into two hexagons, an involution $J: M \longrightarrow M$ which interchanges the two hexagons H and JH, and a homeomorphism $H \longrightarrow \Delta'$, we construct the desired transverse section $M \longrightarrow E_\phi$ using the procedure 5.9 (see Fig. 5.6).

Now suppose that one of the vertices, say p_A, is ultra-ideal (i.e. $\phi(A)$ is hyperbolic). Then there are exactly two triangles Δ with vertices p_A, p_B, p_C such that $\Delta \cap X$ is connected (Fig. 5.7). Choose one of these triangles Δ; then there exists a region Q_A bounded by a $\phi(A)$-invariant equidistant curve and disc Q_B (resp. Q_C) centered at p_B (resp. p_C) such that $\Delta - (Q_A \cup Q_B \cup Q_C)$ is a right hexagon. By 5.9 we can construct a transverse section $M \longrightarrow E_\phi$ taking ∂M to invariant circles and equidistant curves.

Unfortunately, there may not exist a transverse section which takes A onto a geodesic. Let p_A^* denote the $\phi(A)$-invariant line. If p_A^* separates p_B and p_C, or if p_A^* contains p_B or p_C, then there is no triangle Δ with vertices p_A, p_B, p_C such that

For number sequence only.

Δ - (half-plane bounded by p_A^*) is a quadrilateral contained in X. Consequently it is necessary to use non-geodesic equidistant curves to truncate Δ (see Fig. 5.8)

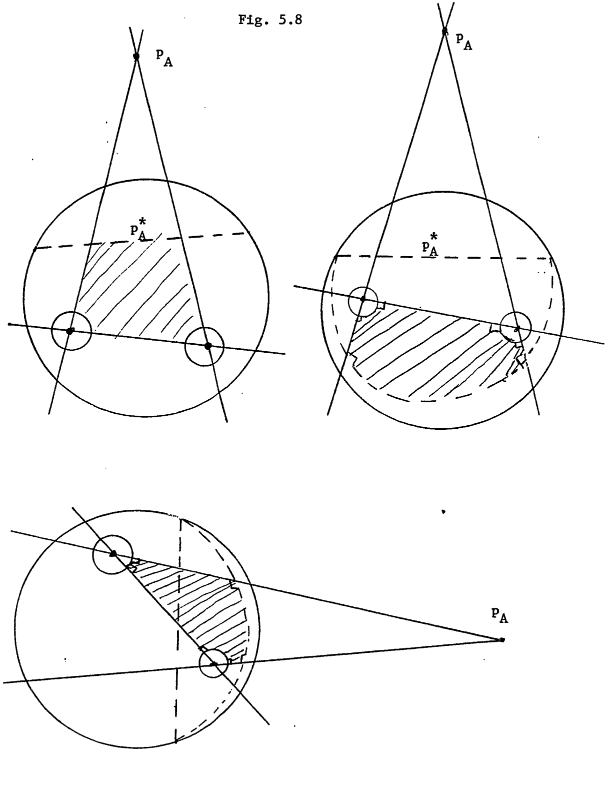
The case when two vertices, say p_A, p_B, are outside $X \cup \partial X$ is handled similarly. In that case the dual lines p_A^* and p_B^* separate X into three regions, and there is a unique triangle Δ with vertices p_A, p_B, p_C such that $\Delta \cap (p_A^* \cup p_B^*) \subset X$ (Fig. 5.9). Then there exist, as before, a disc Q_C centered at p_C and regions Q_A (resp. Q_B) bounded by $\phi(A)$- (resp. $\phi(B)$-) invariant curves such that $\Delta - (Q_A \cup Q_B \cup Q_C)$ is a right hexagon. By 5.9 one may construct a transverse section $M \longrightarrow E_\phi$ which develops A (resp. B, C) $\subset \partial M$ to the invariant curve ∂Q_A (resp. ∂Q_B, ∂Q_C).

If p_C lies in one of the closed regions in X bounded by <u>one</u> of p_A^* and p_B^* then it will be impossible to choose both Q_A and Q_B to be half-planes. As a result, the section constructed will map A and B to geodesics if and only if p_C lies in the interior of the connected region bounded by p_A^* and p_B^* (Fig. 5.10).

<u>Definition 5.11</u>. Let Δ be a triangle in $\mathbb{RP}^2$ and let X denote the interior of a conic in $\mathbb{RP}^2$. Then Δ is <u>acute</u> (with respect to X) if and only if for every vertex p of Δ such that $p \in X$ the vertex angle of Δ at p (measured with respect to the hyperbolic metric of X) is less than $\frac{\pi}{2}$. If one of the vertex angles is greater than $\frac{\pi}{2}$ (resp. equals $\frac{\pi}{2}$), then Δ is said to be <u>obtuse</u> (resp. <u>right</u>).

The importance of acuteness is the following. Suppose $\phi \in \mathrm{Hom}(\pi,G)$ factors into reflections, $\phi(\pi)$ is not solvable, and $\phi(A)$ is not hyperbolic. Then we have shown there exists a transverse

Fig. 5.8



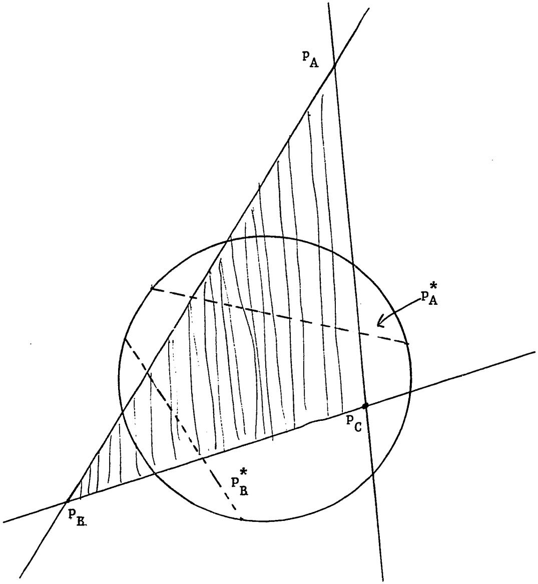


Fig. 5.9

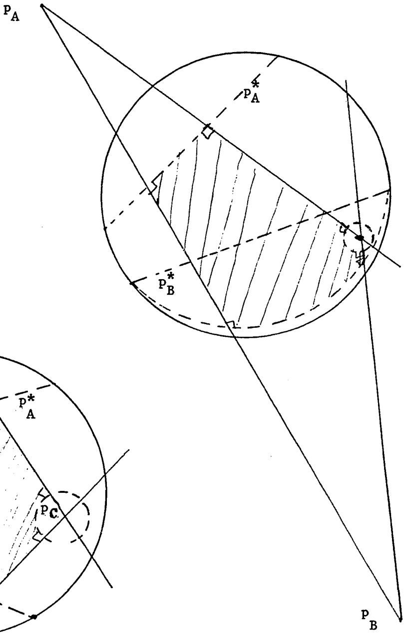
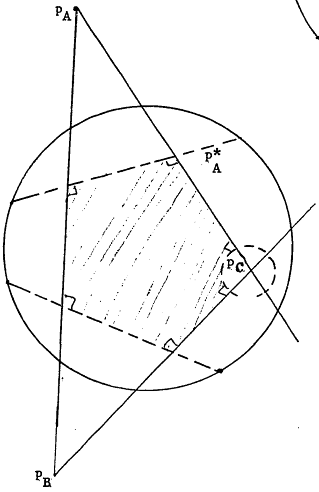


Fig. 5.10



section $M \to E_\phi$ which, in local foliation charts, takes boundary components A, B, C into appropriately chosen corresponding invariant curves. The section is constructed by finding a triangle Δ whose vertices p_A, p_B, p_C are the preferred fixed points of $\phi(A)$, $\phi(B)$, $\phi(C)$. If $\phi(B)$, say, is hyperbolic, then we have seen that it may or may not be possible to find a section $f: M \to E_\phi$ such that $f(B)$ is geodesic. However, if Δ is <u>acute</u>, then such a section may always be found. Recall that if $\phi(C)$ is elliptic then a section f with $f(B)$ geodesic exists precisely when p_C and p_A lie in the same $\phi(B)$-invariant open half-plane in X. Depending on the choice of Δ, the vertex angle of Δ at either p_A or p_C will be obtuse (Fig. 5.11). When $\phi(C)$ is hyperbolic, then there is a unique triangle Δ such that $(p_A^* \cup p_C^*) \cap X \subset \Delta$ and the vertex angle at p_B is obtuse (or right) whenever p_B does not lie in the open region bounded by the invariant lines p_A^* and p_C^* (Fig. 5.12). This is precisely the condition needed for the existence of a transverse section f with $f(A)$ and $f(C)$ geodesic.

Now we come to the case when all three of p_A, p_B, p_C are ultra-ideal. Since ϕ factors into reflections and not symmetries, the projective line containing p_A and p_B, etc. intersects ∂X in two points. Let p_A^*, p_B^*, p_C^* be the lines dual to p_A, p_B, p_C, etc. respectively. Then <u>either</u> there is one line, say p_A^*, such that each of the two components of $X - p_A^*$ contains one of p_B^* and p_C^* (Fig. 5.13a) <u>or</u> each of the three lines bounds a region not containing another line (Fig. 5.13b). In the first case (Fig. 5.13a) no triangular region intersects X in a hexagon and there is no natural way to find and truncate a triangle Δ to produce a transverse section of the desired sort.

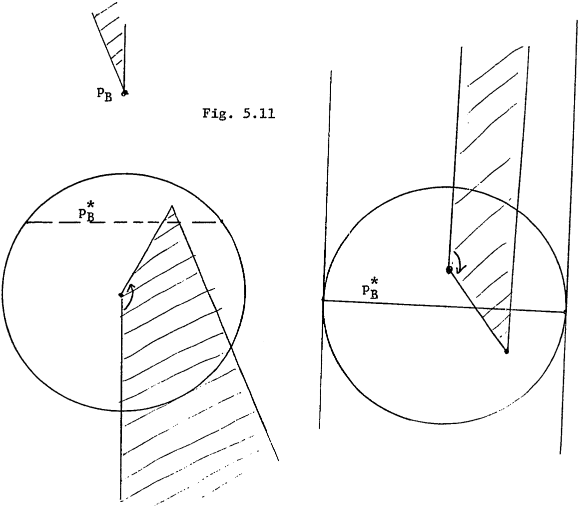
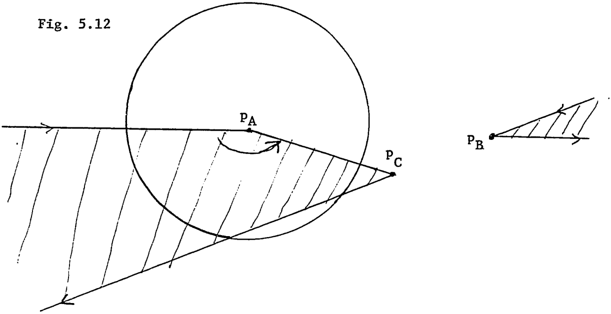


Fig. 5.11

Fig. 5.12



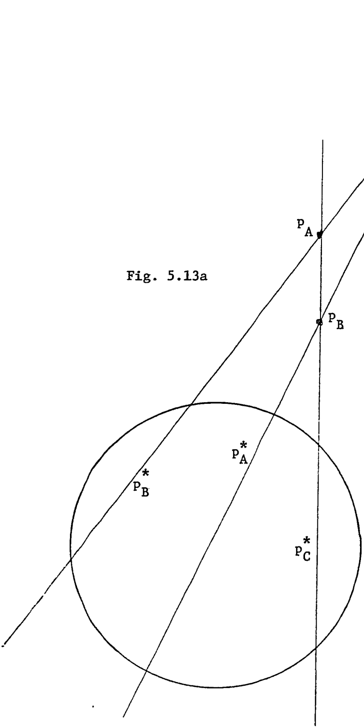


Fig. 5.13a

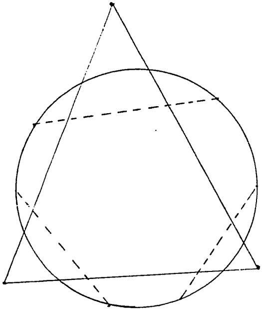


Fig. 5.13b

In the second case (Fig. 5.13b) there is a natural transverse section mapping ∂M to geodesics; indeed this is the model case treated in 5.9.

5.12 Now we compute the relative Euler class $e(\phi;\sigma)$ of $\phi \in \mathrm{Hom}(\pi,G)$ with respect to a special ideal section σ over ∂M. We shall compute $e(\phi;\sigma)$ by constructing deformations of ϕ to "canonical" examples, watching carefully how σ changes. Deformations of representations ϕ can be visualized as deformations of the corresponding triangles, by Proposition 5.2. We remind the reader that the special ideal section $\sigma_t = \sigma(\phi_t)$ corresponding to a deformation ϕ_t is continuous provided that $\phi_t(C) \notin \mathrm{Sym}$ for all t and all components $C \subset \partial M$. In that case we say ϕ_t is "admissible".

Since π is free on two generators, $\mathrm{Hom}(\pi,G) \approx G \times G$ is connected. Since the map

$$\begin{aligned} \mathrm{Hom}(\pi,G) &\longrightarrow \mathbb{Z} \\ \phi &\longmapsto e(\phi;\sigma(\phi)) \end{aligned}$$

has disconnected image, it cannot be continuous. On the other hand we shall really be considering the open subset $W = \{\phi \in \mathrm{Hom}(\pi,G)\colon \phi(C) \notin \mathrm{Sym}$ for all components $C \subset \partial M\}$ where the map $\phi \mapsto e(\phi;\sigma(\phi))$ is continuous. Indeed, we show that it is onto $\{-1,0,+1\}$.

To begin with consider the trivial representation ϕ_0, $\phi_0(\pi) = 1$. Clearly a special ideal condition over ∂M extends over M, so $e(\phi_0;\sigma) = 0$.

Now consider ϕ such that $\phi(\pi)$ fixes a point of ∂X. Then none of $\phi(A)$, $\phi(B)$, and $\phi(C)$ are elliptic. Consider the deformation ϕ_t defined by

$$\phi_t(A) = \exp t \log \phi(A)$$
$$\phi_t(B) = \exp t \log \phi(B)$$
$$\phi_t(C) = \phi_t(B)^{-1}\phi_t(A)^{-1}$$

Then $\phi = \phi_1$ and ϕ_0 is the trivial representation. Since none of $\phi_t(A)$, $\phi_t(B)$, $\phi_t(C)$ are ever elliptic, ϕ_t is admissible, so $e(\phi;\sigma) = e(\phi_0;\sigma) = 0$. This proves conditions (b) and (c) of Theorem 5.1(ii).

5.13 The next case we discuss occurs when $\phi(A)$, $\phi(B)$, $\phi(C)$ are all hyperbolic. We can assume $\phi(\pi)$ is not solvable; then by 5.2 either ϕ factors into reflections or factors into symmetries. This gives three cases for the associated triangular configurations, depending on whether ϕ factors into symmetries, ϕ factors into reflections and no triangle intersects X in a hexagon (see Fig. 5.13a), or ϕ is Fuchsian and factors into reflections. The three cases are pictured below in Fig. 5.14.

We claim that if either of the first two possibilities occur then $e(\phi;\sigma) = 0$. If ϕ factors into symmetries, there exists a triangle Δ which is disjoint from X. There exists a one-parameter family of triangles Δ_t $(\frac{1}{2} \leq t \leq 1)$, $\Delta_0 = \Delta$, such that $\bigcap_{t>1/2} (\Delta_t)$ is a point outside $X \cup \partial X$ and no extended side of any Δ_t meets ∂X (Fig. 5.14). From this we obtain a deformation ϕ_t which for each t factors into symmetries and $\phi_{1/2}(\pi)$ lies in a hyperbolic one-parameter subgroup. Since $\phi_t(A)$, $\phi_t(B)$, $\phi_t(C)$ are never elliptic, ϕ_t is admissible. Thus $e(\phi;\sigma) = e(\phi_{1/2};\sigma) = 0$.

Similarly in the second case (Fig. 5.13a) there exists a triangle Δ such that exactly two sides meet ∂X. Then there is a continuous

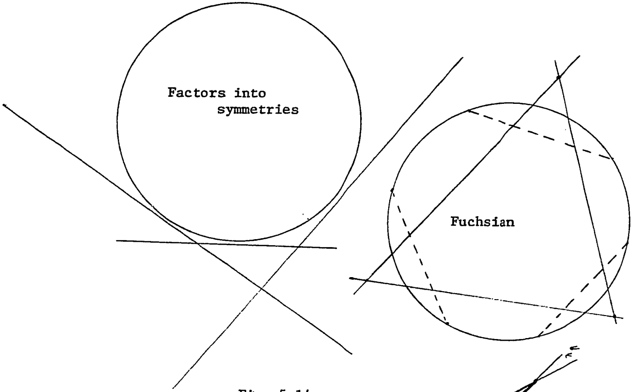
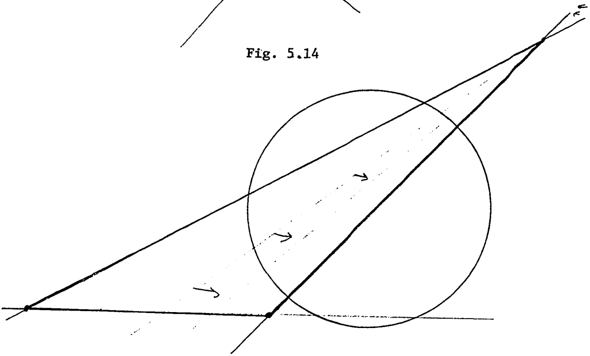


Fig. 5.14



shrinking of Δ to one of these sides (Fig. 5.14) which defines, as before, an admissible deformation. The limiting case is the trivial representation, so $e(\phi;\sigma) = 0$.

It remains to show that if ϕ factors into reflections and is Fuchsian, then $e(\phi;\sigma) = \pm 1$. One can show this by similar deformation arguments, keeping careful track of the special ideal sections. However, we prefer a different proof. Let Δ be the unique triangle intersecting $X \cup \partial X$ in a hexagon and let p_A^*, p_B^* and p_C^* be the three geodesics along which we truncate Δ into a right hexagon H (5.9). Choose one of them, say p_A^*, and reflect Δ in it, obtaining a right octagon $H \cup R_A(H)$ (Fig. 5.15). Now $H \cup R_A H$ is the fundamental domain for a hyperbolic structure on the sphere-minus-four-discs M'. There are unique hyperbolic elements which take p_C^* to $R_A p_C^*$ with invariant axis $\overleftrightarrow{p_C(R_A p_C)}$ (resp. p_B^* to $R_A p_B^*$ with invariant axis $\overleftrightarrow{p_B(R_A p_B)}$). Since these isometries identify the corresponding boundary components of the hyperbolic surface M', we have constructed a hyperbolic structure on the double $2M$, a closed surface of genus two. Let ϕ_2 denote the holonomy of this hyperbolic structure; then $e(\phi;\sigma) = \frac{1}{2}e(\phi_2) = \pm 1$. The sign of $e(\phi;\sigma)$ depends on the orientation of this hyperbolic structure which in turn depends on the cyclic ordering of the vertices of Δ as viewed from a point inside X.

We have just completed the proof of part (iv) of Theorem 5.1.

5.14 If $\phi(A)$, $\phi(B)$, $\phi(C)$ are allowed to be parabolic, as well as hyperbolic, the situation changes little. Although ϕ factors into symmetries only if all three are hyperbolic, ϕ might determine a triangle with one or two ideal vertices which cannot be truncated in the desired manner (Fig. 5.16).

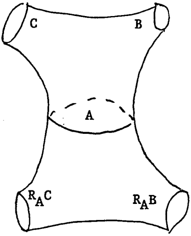
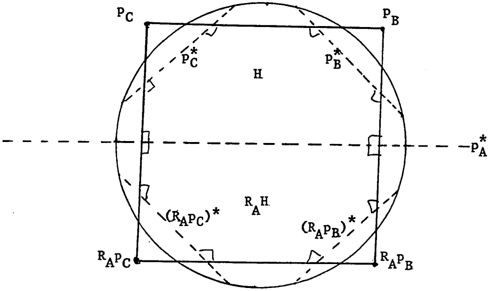
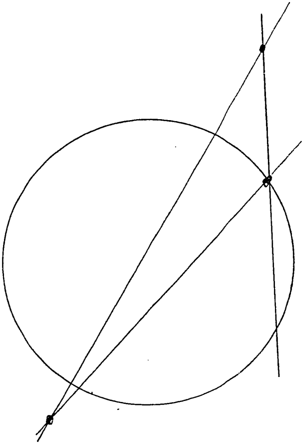


Fig. 5.15





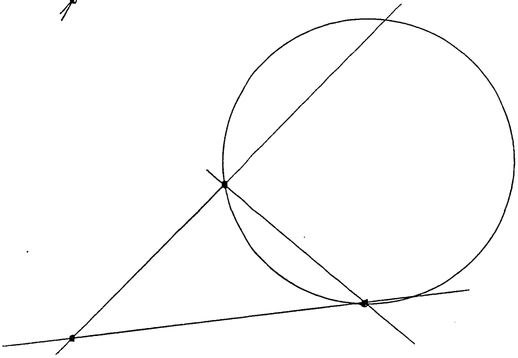


Fig. 5.16

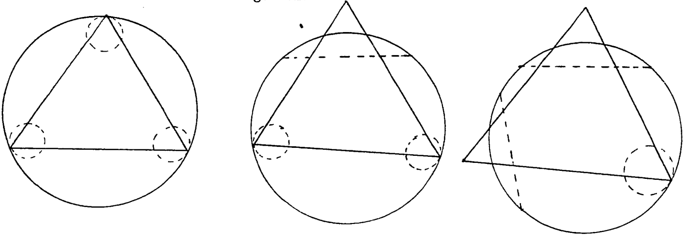
These triangles with some ideal vertices cannot be truncated into right hexagons. The relative Euler class is zero.

However, in either of these cases pictured in Fig. 5.16, one proves easily that $e(\phi;\sigma) = 0$. Similarly ϕ may factor into reflections and be Fuchsian, in which case $e(\phi;\sigma) = \pm 1$ (Fig. 5.17).

5.15 Now we consider the general case when ϕ factors into reflections and $\phi(\pi)$ is not solvable. Choose a triangle Δ with vertices p_A, p_B, p_C such that $\Delta \subset X$ if possible, and for each ultra-ideal vertex p_A, we have $p_A^* \cap \Delta \subset X$. For each vertex $p_C \in X$ we consider deforming Δ, gradually moving p_C until it lies on ∂X. In the limiting case the sides containing the limiting ideal vertices lie inside X. That such a deformation exists, which induces a deformation ϕ_t $(\frac{1}{2} \leq t \leq 1)$ of $\phi_1 = \phi$ may be seen from Fig. 5.18. Since the ideal vertices of $\Delta_{1/2}$ lie on sides interior to X, $e(\phi_{1/2};\sigma) = \pm 1$, by the previous section. Unfortunately, the deformation ϕ_t might not be admissible. If $p_A \in X$ is a vertex, then the rotation angle of $\phi(A)$ is exactly twice the vertex angle of Δ at p_A. Hence $\phi(A) \in \mathrm{Sym}$ precisely when Δ has a right angle at p_A. In the deformation described, the rotation angle of $\phi_t(A)$ steadily decreases to 0, so $\phi_t(A)$ is an admissible deformation if and only if Δ has an acute angle at p_A. Therefore if Δ is acute, then $e(\phi;\sigma) = \pm 1$.

Suppose Δ is obtuse. By our assumptions on Δ, there is at most one obtuse angle. By an admissible deformation we may assume the other vertices are ideal or ultra-ideal. By a further admissible deformation we may assume the other vertices p_B, p_C ideal (Fig. 5.19). Then there is an admissible deformation, to the trivial representation, such that p_B and p_C are fixed and p_A approaches the line $\overleftrightarrow{p_B p_C}$. Hence if Δ is obtuse, $e(\phi;\sigma) = 0$.

Fig. 5.17



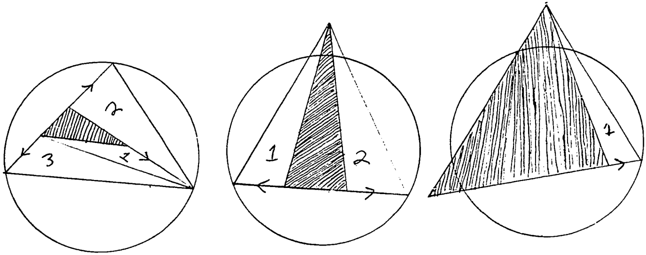
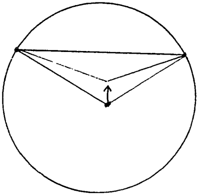


Fig. 5.18 Deforming away interior vertices

Fig. 5.19



For number sequence only.

Combining these calculations with the construction of transverse sections (5.10-5.11) we conclude the following:

Proposition 5.15. Suppose ϕ is not solvable and factors into reflections. Let Δ be the associated triangle. Then Δ is obtuse if and only if $e(\phi;\sigma) = 0$, and Δ is acute or right if $e(\phi;\sigma) = \pm 1$. If Δ is acute or right, then there exists a transverse section to E_ϕ which restricts to a special interior section over ∂M.

Thus we have proved part (iii) of Theorem 5.1.

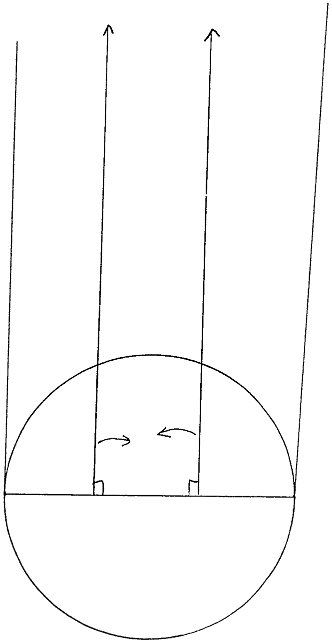
5.16 In order to prove part (i) and part (a) of (ii) of Theorem 5.1, we must now consider the case $\phi(\pi)$ is solvable. First suppose that $\phi(\pi)$ lies on an elliptic one-parameter subgroup. Let the rotation angles of $\phi(A)$, $\phi(B)$, $\phi(C)$ be θ_A, θ_B, θ_C respectively, where $-\pi < \theta \leq \pi$. Then $(\theta_A+\theta_B+\theta_C) \equiv 0 \pmod{2\pi}$ and $|\theta_A+\theta_B+\theta_C| \leq 3\pi$. It follows from Proposition 4.9 that $e(\phi;\sigma) = \frac{1}{2\pi}(\theta_A+\theta_B+\theta_C)$. Hence $e(\phi;\sigma) = -1$, 0, or +1.

If, say, $\phi(A) = 1$, then $\theta_A = 0$ and $e(\phi;\sigma) = 0$ unless $\phi(B)$ (and hence $\phi(C)$) is a symmetry (in that case $e(\phi;\sigma) = 1$). This proves (a) of Theorem 5.1(ii).

The only remaining case to consider is when $\phi(B)$ and $\phi(C)$ are both symmetries. In this case the associated triangle has two right angles (at p_B and p_C) and $p_A^* = \overleftrightarrow{p_B p_C}$. By moving p_B and p_C towards each other, we admissibly deform ϕ until its image lies in an elliptic one-parameter subgroup (Fig. 5.20). In any case, $|e(\phi;\sigma)| \leq 1$, proving 5.1(i).

Fig. 5.20

Here is a triangle with two right angles. Although $\phi(\pi)$ is solvable, the relative Euler number is ± 1. There is an admissible deformation to the highly degenerate case when $\phi(\pi)$ has order two.



§6. Constructing Hyperbolic Structures

Let M be a closed oriented surface, $\chi(M) = 2 - 2g < 0$, and $\pi = \pi_1(M)$. Our goal is to prove Theorem A: for $\phi \in \mathrm{Hom}(\pi,G)$, $|e(\phi)| \leq |\chi(M)|$; equality holds, $e(\phi) = \pm\chi(M)$, if and only if E_ϕ admits a transverse section.

We decompose M into subsurfaces M_i, where each M_i is homeomorphic to the pair-of-pants of §5. There are many such pairs-of-pants decompositions (such as Fig. 6.1) but for our purposes any one of them will do. Let V be the union of all the ∂M_i; then $M - V = \bigcup_{i=1}^{2g-2} \mathrm{int}\, M.$

The idea will be to construct a sufficiently small special interior section to E_ϕ over V and, using 5.1, extend it over the $\mathrm{int}(M_i)$.

We shall want to speak of $\pi_1(M_i) \subset \pi_1(M)$ so we make the following convention concerning basepoints. Choose a basepoint $x_i \in \mathrm{int}(M_i)$ for each M_i; then connect the x_i by an embedded contractible 1-complex K (i.e. a tree) in M. By allowing translations only along K we obtain a unique collection of canonical isomorphisms $h_{ij}\colon \pi_1(M;x_i) \to \pi_1(M;x_j)$ satisfying $h_{ii} = \mathrm{id}$, $h_{ij}h_{jk} = h_{ik}$. In this way we obtain well-defined canonical inclusions $\pi_1(M_i) = \pi_1(M_i;x_i) \hookrightarrow \pi$ (Fig. 6.2).

6.2 If $\phi \in \mathrm{Hom}(\pi,G)$ we have by additivity

$$e(\phi) = \sum_{i=1}^{-\chi} e(\phi|_{\pi_1(M_i)};\sigma|_{\partial M_i}) ,$$

where σ is a special ideal section over V. From 5.1(i) follows

$$(1) \qquad |e(\phi|_{\pi_1(M_i)};\sigma|_{\partial M_i})| \leq 1$$

Fig. 6.1

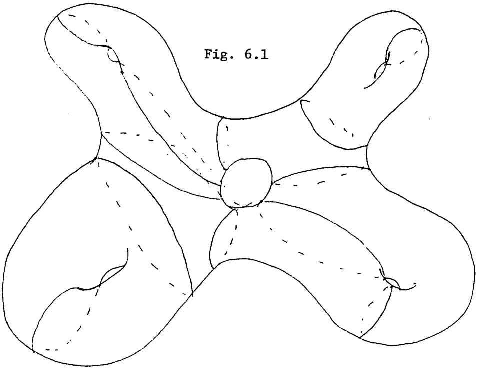
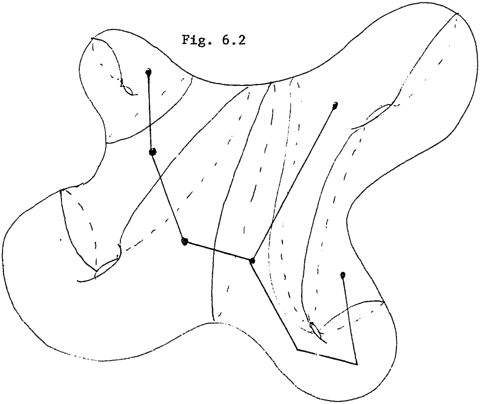


Fig. 6.2



For number sequence only.

proving the Milnor-Wood inequality

(2) $$|e(\phi)| \leq |\chi(M)| \ .$$

Now suppose equality holds in (2). By reversing the orientation on M if necessary we may assume that $e(\phi) = \chi(M)$. Then the only way that $e(\phi)$ can assume its minimum value in (2) is if $e(\phi|_{\pi_1(M_i)}; \sigma|_{\partial M_i})$ achieves its minimum value in (1). Hence from $e(\phi) = \chi(M)$ we conclude that

(3) $$e(\phi|_{\pi_1(M_i)}; \sigma|_{\partial M_i}) = -1 \ .$$

By 5.1(ii) this equality implies that <u>either</u> $\phi(\pi_1(M_i))$ is solvable and is generated by nontrivial elliptic elements <u>or</u> there exists an orientation-preserving transverse section f_i over M_i such that $f|_{\partial M_i}$ is a sufficiently small special interior section.

We eventually prove that the first possibility (when $\phi(\pi_1(M_i))$ is solvable) cannot occur if $e(\phi) = \chi(M)$. However, for the moment, let us just be content to say that the first possibility does not <u>generically</u> occur. This follows immediately from the genericity results 3.19, since, for example, a generic ϕ is injective whereby $\phi(\pi_1 M_i)$ is free and hence not solvable. Thus, after perturbing ϕ, Theorem 5.1 guarantees the existence of a transverse section $f_i \colon M_i \longrightarrow E_\phi|_{M_i}$. Furthermore, the perturbation may be chosen as small as desired.

6.3 The next step, still assuming ϕ generic, involves using 5.1 to glue the transverse sections f_i together to give a global section $f\colon M \longrightarrow E_\phi$. To this end we must choose, for each $A \subset \partial M_i$ a

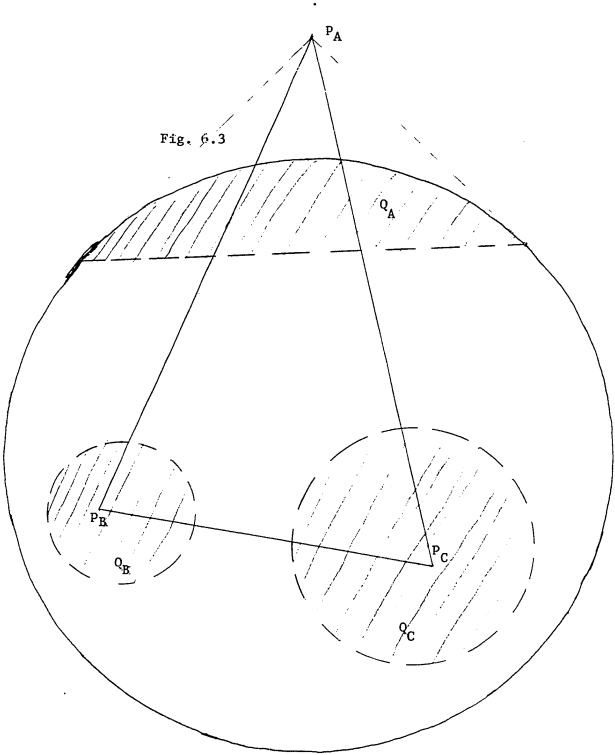
$\phi(A)$-invariant disc, horodisc, or half-plane Q_A such that the f_i may be chosen so that a collar neighborhood of A in M_i develops under f_i to the complement $X-Q_A$. By 5.1(iii) as long as Q_A, Q_B and Q_C are pairwise disjoint for $A \cup B \cup C = \partial M_i$, there exists such a transverse section. Since a given $A \subset V$ lies on at most two M_i, we may consistently choose the smaller Q_A, except when $\phi(A)$ is hyperbolic (Fig. 6.3). In that case 5.1(iii) tells us that there exists an f_i such that $f_i(A)$ is geodesic. Therefore there exists a special interior section over V which extends to a transverse section f_i over each M_i and maps a collar neighborhood of any $A \subset \partial M_i$ to the complement of a $\phi(A)$-invariant disc, horodisc, or half-plane.

Choose such a special ideal section $f: V \to E_\phi|_V$; then 5.1 implies that f_0 extends to a section $f: M \to E_\phi$ such that $f|_{M_i}$ is transverse and orientation-preserving.

6.4 Since the section f of E_ϕ has the property that it is transverse over each M_i, its worst singularities are folds along subsets of V. However since over each M_i the section is orientation-preserving, the section must actually be transverse over V as well (Proposition 1.12). Thus f is transverse.

6.5 So far we have proved that if $\phi \in \mathrm{Hom}(\pi,G)$, $e(\phi) = \chi(M)$, there exists ϕ' arbitrarily near ϕ such that $E_{\phi'}$ admits an orientation-preserving transverse section. If E_ϕ itself does not admit a transverse section, then for some i, the group $\phi(\pi_1(M_i))$ is solvable; in fact there is some component C of ∂M_i such that $\phi(C)$ is elliptic by 5.1(iii). Since ϕ' approximates ϕ, it

Fig. 6.3



follows that $\phi'(C)$ is elliptic.

Now there exists some M_j such that $C = \bar{M}_i \cap \bar{M}_j$. As we have seen, there is a section f over $M_i \cup M_j$ which is transverse. By construction a developing map for f maps C to a $\phi(C)$-invariant circle. Moreover the respective collar neighborhoods of C in M_i and M_j are mapped to the <u>exterior</u> of the $\phi(C)$-invariant circle. Clearly this means that not both of $f|_{M_i}$ and $f|_{M_j}$ are orientation-preserving; thus f suffers a fold along C, a contradiction.

This concludes the proof of Theorem A.

6.6 If M is compact but $\partial M \neq \emptyset$ a relative version of Theorem A can be proved by the same methods. Let $\pi = \pi_1(M)$, $\phi \in \mathrm{Hom}(\pi, G)$ and σ a special ideal section of E_ϕ over ∂M.

Suppose that $e(\phi;\sigma) = \chi(M)$. Decompose M along a closed 1-submanifold into pairs-of-pants M_i. Suppose further for each component C of ∂M that $\phi(C) \notin \mathrm{Sym}$. This insures that there is an admissible deformation ϕ' of ϕ such that $\phi'(\pi_1(M_i))$ is not solvable. (To see this, we may embed M in a closed surface M^* and ϕ in a representation $\phi^*\colon \pi_1(M^*) \longrightarrow G$. The condition "$\phi(C) \notin \mathrm{Sym}$ for components $C \subset \partial M$" being open, together with the density (3.19) of isomorphisms in $\mathrm{Hom}(\pi, G)$, implies that there is a sufficiently near ϕ' which is injective on $\pi_1(M)$ and is deformable to ϕ along a path ϕ_t such that $\phi_t(C) \notin \mathrm{Sym}$ for all $C \subset \partial M$.)

It follows from additivity (4.2) and 5.1 that there exists a section $f\colon M \longrightarrow E_{\phi'}$ which restricts to a special interior section over $\partial M \cup V$ and such that over each M_i, the section f is transverse and preserves orientation. By 1.12, the fact that for adjacent

M_i and M_j, the orientations of $f|_{M_i}$ and $f|_{M_j}$ agree implies that f is a transverse section over M. Thus $E_{\phi'}$ admits an orientation-preserving transverse section which restricts to a special interior section over ∂M.

Let $C \subset V$ be $\partial M_i \cap \partial M_j$ for adjacent M_i and M_j. Unless $\phi'(C)$ is hyperbolic, a developing map for f maps collar neighborhoods of C in M_i and M_j to the <u>same</u> side of a $\phi'(C)$-invariant curve. Hence <u>if</u> $f|_{M_i}$ <u>and</u> $f|_{M_j}$ <u>both preserve orientation</u>, $\phi'(C)$ <u>must be hyperbolic</u>.

Now we return to ϕ. If the perturbation ϕ' were necessary, then by 5.1(iii) for some such C, $\phi(C)$ would be elliptic. Since elliptic elements are open in G, $\phi'(C)$ is elliptic, in contradiction to what has just been proved. (Of course for this argument to work, we must find such a simple closed curve C not homotopic to ∂M_j such a curve exists unless M is a pair-of-pants.)

In summary we have proved the following extension of Theorem A to manifolds-with-boundary:

<u>Theorem 6.6</u>. Let M be a closed surface, not homeomorphic to a pair-of-pants, $\chi(M) < 0$, and let $\pi = \pi_1(M)$. Suppose $\phi \in \mathrm{Hom}(\pi,G)$ and $e(\phi;\sigma) = \chi(M)$ where σ is a special ideal section. Suppose that for no simple closed curve C not homotopic to ∂M, $\phi(C) \notin \mathrm{Sym}$. Then E_ϕ admits an orientation-preserving transverse section which restricts to a special interior section over ∂M. If C is a component of ∂M with $\phi(C)$ hyperbolic, then a developing map for f takes C to an invariant geodesic.

In general $e(\phi;\sigma) = \chi(M)$ does not imply that ϕ is Fuchsian. It does, however, if for every component C of ∂M, $\phi(C)$ is hyperbolic. This may be seen by embedding M in a closed surface M_1 and ϕ in a homomorphism $\phi_1 \colon \pi_1(M_1) \to G$ with $e(\phi_1) = \chi(M_1)$. There are many ways of accomplishing this, for example by "doubling" M.

Explicitly, let $2M$ denote the double of M and $J\colon 2M \to 2M$ an involution, fixing ∂M, with fundamental domain M. Let C be a component of ∂M, and let R_C be the reflection in the $\phi(C)$-invariant geodesic. Define the holonomy ϕ_2 of a hyperbolic foliated bundle over $2M$ by

$$\phi_2|_{\pi_1(M)} = \phi$$
$$\phi_2|_{\pi_1(JM)} = R_C \circ \phi \circ J_* \;;$$

then the hyperbolic foliated bundle E_{ϕ_2} defined over $(\text{int } M) \cup C \cup (\text{int } JM)$ extends uniquely to a hyperbolic foliated bundle over $2M$. Now $e(\phi|_{\partial\pi_1(JM)};\sigma|_{\partial JM}) = e(\phi;\sigma) = \chi(M)$ so $e(\phi_2) = 2e(\phi;\sigma) = 2\chi(M) = \chi(2M)$. By Theorem A, it follows that ϕ_2, and hence $\phi = \phi_2|_{\pi_1(M)}$, is Fuchsian.

If ϕ satisfies the hypotheses of Theorem 6.6 and $M' \subset M$ is a compact connected subsurface such that no component of $\partial M'$ is homotopic to ∂M, then we have shown that for each component C of $\partial M'$, $\phi(C)$ is hyperbolic. Although ϕ itself may not be Fuchsian it follows that $\phi|_{\pi_1(M')}$ is Fuchsian.

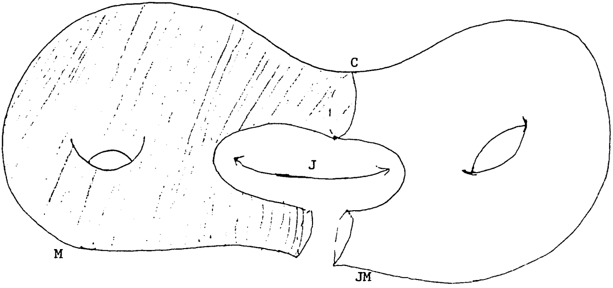
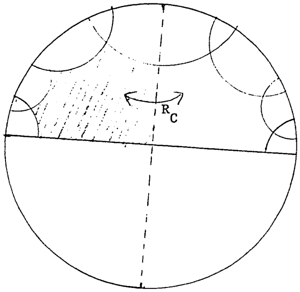


Fig. 6.4. Doubling a hyperbolic structure



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