Geodesics in Margulis spacetimes

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Dedicated to the memory of Dan Rudolph

Abstract. Let $M^3$ be a Margulis spacetime whose associated complete hyperbolic surface $\Sigma^2$ has a compact convex core. Generalizing the correspondence between closed geodesics on $M^3$ and closed geodesics on $\Sigma^2$, we establish an orbit equivalence between recurrent spacelike geodesics on $M^3$ and recurrent geodesics on $\Sigma^2$. In contrast, no timelike geodesic recurs in either forward or backward time.

1. Introduction

A Margulis spacetime is a complete flat affine 3-manifold $M^3$ with free non-abelian fundamental group $\Gamma$. It necessarily carries a unique parallel Lorentz metric. Parallelism classes of timelike geodesics form a non-compact complete hyperbolic surface $\Sigma^2$. This complete hyperbolic surface is naturally associated to the flat 3-manifold $M^3$ and we regard $M^3$ as an affine deformation of $\Sigma^2$. This paper relates the dynamics of the geodesic flow of the flat affine manifold $M^3$ to the dynamics of the geodesic flow on the hyperbolic surface $\Sigma^2$.

We restrict ourselves to the case that $\Sigma^2$ has compact convex core (that is, $\Sigma^2$ has finite type and no cusps). Equivalently, the Fuchsian group $\Gamma_0$ corresponding to $\pi_1(\Sigma^2)$ is convex cocompact. In particular, $\Gamma_0$ is finitely generated and contains no parabolic elements. Under this assumption, every free homotopy class of an essential closed curve in $\Sigma^2$ contains a unique closed geodesic. Since $\Sigma^2$ and $M^3$ are homotopy-equivalent, free homotopy classes of essential closed curves in $M^3$ correspond to free homotopy classes of essential closed curves in $\Sigma^2$. Every essential closed curve in $M^3$ is likewise homotopic to a unique closed geodesic in $M^3$.

In her thesis [4, 8], Charette studied the next case of dynamical behavior: geodesics spiralling around closed geodesics both in forward and backward time. She proved bispiralling geodesics in $M^3$ exist, and correspond to bispiralling geodesics in $\Sigma^2$.

This paper extends the above correspondence between geodesics on $\Sigma^2$ and $M^3$ to recurrent geodesics.
A geodesic (either in $\Sigma^2$ or in $M^3$) is \textbf{recurrent} if and only if it (together with its velocity vector) is recurrent in both directions. These correspond to recurrent points for the corresponding geodesic flows as in Katok and Hasselblatt [17, §3.3]. (Our meaning of the term ‘recurrent’ agrees with the term ‘non-wandering’ used by Eberlein [12].) Under our hypotheses on $\Sigma^2$, a geodesic on $\Sigma^2$ is recurrent if and only if the corresponding orbit of the geodesic flow is precompact.

**Theorem 1.** Let $M^3$ be a Margulis spacetime whose associated complete hyperbolic surface $\Sigma$ has compact convex core.

- The recurrent part of the geodesic flow for $\Sigma^2$ is topologically orbit-equivalent to the recurrent spacelike part of the geodesic flow of $M^3$.
- The set of recurrent spacelike geodesics in a Margulis spacetime is the closure of the set of periodic geodesics.
- No timelike geodesic recurs.

A semiconjugacy between these flows was observed by Fried [13].

This paper is the sequel to [15], which characterizes properness of affine deformations by positivity of a marked Lorentzian length spectrum, the \textit{generalized Margulis invariant}. A crucial step in the proof that properness implies positivity is the construction of sections of the associated flat affine bundle, called \textit{neutralized sections}. A further modification of neutralized sections produces an orbit equivalence between recurrent geodesics in $\Sigma$ and recurrent geodesics in $M$.

It follows that the set of recurrent spacelike orbits of the geodesic flow is a Smale hyperbolic set in $TM$.

Null geodesics not parallel to a point in the limit set $\Lambda$ of $\Gamma_0$ do not recur. In this paper, we do not discuss the recurrence of null geodesics parallel to a point of $\Lambda$.

2. Geodesics on affine manifolds

An \textit{affinely flat manifold} is a smooth manifold with a distinguished atlas of local coordinate systems whose charts map to an affine space $E$ such that the coordinate changes are restrictions of affine automorphisms of $E$. Denote the group of affine automorphisms of $E$ by $\text{Aff}(E)$. This structure is equivalent to a flat torsion-free affine connection. The affine coordinate atlas globalizes to a \textit{developing map}

$$\tilde{M} \xrightarrow{\text{dev}} E,$$

where $\tilde{M} \rightarrow M$ denotes a universal covering space of $M$. The coordinate changes globalize to an affine holonomy homomorphism

$$\pi_1(M) \xrightarrow{\rho} \text{Aff}(E),$$

where $\pi_1(M)$ denotes the group of deck transformations of $\tilde{M} \rightarrow M$. The developing map is equivariant with respect to $\rho$.

Denote the vector space of translations $E \rightarrow E$ by $V$. The action of $V$ by translations on $E$ defines a trivialization of the tangent bundle $TM \cong M \times V$. In these local coordinate charts, a geodesic is a path

$$p \mapsto p + tV.$$
where \( p \in E \) and \( v \in V \) is a vector. In terms of the trivialization, the geodesic flow is

\[
E \times V \xrightarrow{\tilde{\psi}} E \times V,
\]

\[
(p, v) \mapsto (p + tv, v),
\]

for \( t \in \mathbb{R} \). Clearly, this \( \mathbb{R} \)-action commutes with \( \text{Aff}(E) \).

Geodesic completeness implies that \( \text{dev} v \) is a diffeomorphism. Thus the universal covering \( \hat{M} \) is affinely isomorphic to the affine space \( E \) and \( M \cong E/\Gamma \), where \( \Gamma := \rho(\pi_1(M)) \) is a discrete group of affine transformations acting properly and freely on \( E \).

3. **Flat Lorentz 3-manifolds**

Let \( \text{Aff}(E) \xrightarrow{L} \text{GL}(V) \) denote the homomorphism given by the linear part, that is, \( L(\gamma) = A \), where

\[
p \xrightarrow{\gamma} A(p) + b.
\]

The differential of \( \gamma \) at any point \( p \) identifies with its linear part \( L(\gamma) \) via the identification \( TM \cong M \times V \).

Any \( L(\Gamma) \)-invariant non-degenerate inner product \( \langle , \rangle \) on \( V \) defines a \( \Gamma \)-invariant flat pseudo-Riemannian structure on \( E \) which descends to \( M = E/\Gamma \). In particular, affine manifolds with \( L(\Gamma) \subset O(n - 1, 1) \) are precisely the **flat Lorentzian manifolds**, and the underlying affine structures their Levi-Civita connections.

For this reason, we henceforth fix the invariant Lorentzian inner product on \( V \), and hence the (parallel) flat Lorentzian structure on \( E \). The group \( \text{Isom}(E) \) of Lorentzian isometries is the semidirect product of the group \( V \) of translations of \( E \) with the orthogonal group \( O(n - 1, 1) \) of linear isometries. The linear part \( \text{Isom}(E) \xrightarrow{L} O(n - 1, 1) \) defines the projection homomorphism for the semidirect product. For \( l \in \mathbb{R} \), define

\[
S_l := \{ v \in V \mid \langle v, v \rangle = l \}.
\]

When \( l > 0 \), \( S_l \) is a Riemannian submanifold of constant curvature \( -l^{-2} \), and when \( l < 0 \), it is a Lorentzian submanifold of constant curvature \( l^{-2} \). In particular, \( S_{-1} \) is a disjoint union of two isometrically embedded copies of hyperbolic \( n - 1 \)-space \( H^{n-1} \) and \( S_1 \) is the **de Sitter space**, a model space of Lorentzian curvature \( +1 \).

The subset \( T_l(M) \) consists of tangent vectors \( v \) such that \( \langle v, v \rangle = l \) is invariant under the geodesic flow. Indeed, using parallel translation, these bundles trivialize over the universal covering \( E \):

\[
T_l(E) \cong E \times S_l.
\]

Abels–Margulis–Soifer \([2, 3]\) proved that if a discrete group of Lorentz isometries acts properly on a Minkowski space \( E \), and \( L(\Gamma) \) is Zariski dense in \( O(n - 1, 1) \), then \( n = 3 \). Consequently, every complete flat Lorentz manifold is a flat Euclidean affine fibration over a complete flat Lorentz 3-manifold. Thus we henceforth restrict to \( n = 3 \).

Let \( M^3 \) be a complete affinely flat 3-manifold. By Fried and Goldman \([14]\), either \( \Gamma \) is solvable or \( L \circ h \) embeds \( \Gamma \) as a discrete subgroup in (a conjugate of) the orthogonal group

\[
\text{SO}(2, 1) \subset \text{GL}(3, \mathbb{R}).
\]

The cases when \( \Gamma \) is solvable are easily classified (see \([14]\)) and we assume we are in the latter case. In that case, \( M^3 \) is a complete flat Lorentz 3-manifold.
In the early 1980s, Margulis, answering a question of Milnor [22], constructed the first examples [19, 20], which are now called Margulis spacetimes. Explicit geometric constructions of these manifolds have been given by Drumm [9, 10] and his coauthors [4–7, 11]. For an excellent survey of this subject, see Abels [1].

Since the hyperbolic plane \( H^2 \) is the symmetric space of SO(2, 1), \( \Gamma \) acts properly and discretely on \( H^2 \). Since \( M^3 \) is aspherical, its fundamental group \( \pi_1(M^3) \cong \Gamma \) is torsion-free, so \( \Gamma \) acts freely as well. Therefore the quotient \( H^2/L(\Gamma) \) is a complete hyperbolic surface \( \Sigma^2 \). Furthermore, by Mess [21], \( \Sigma \) is non-compact. (See Goldman and Margulis [16] and Labourie [18] for alternative proofs.) Furthermore, every non-compact complete hyperbolic surface occurs for a Margulis spacetime (Drumm [9]).

The points of \( \Sigma^2 \) correspond to parallelism classes of (unoriented) timelike geodesics on \( M^3 \) as follows. It suffices to identify \( H^2 \) with the parallelism classes of (unoriented) timelike geodesics in \( E \), equivariantly respecting \( \text{Isom}(E) \xrightarrow{L} \text{SO}(2, 1) \). The velocity of a unit-speed timelike geodesic in \( E \) is a \( \psi \)-orbit in

\[ T_{−1}E \cong (E \times S_{−1}). \]

The two components of \( S_{−1} \) correspond to future-pointing timelike geodesics and past-pointing timelike geodesics respectively. Points in \( S_{−1} \) correspond to points in \( H^2 \) (the projectivization of \( S_{−1} \)) together with an orientation of \( H^2 \). The geodesic flow \( \tilde{\psi} \) gives \( T_{−1}E \), the structure of a principal \( \mathbb{R} \)-bundle over the quotient. The quotient identifies with an affine bundle over \( S_{−1} \cong H^2 \times \{±1\} \), whose associated vector bundle is the tangent bundle, as follows: the fiber over the line spanned by a fixed timelike vector \( v \) is the affine space quotient of the space of lines parallel to \( v \); the associated vector space is \( V/(v) \cong (v)^\perp \). The tangent space to \( S_{−1} \) at \( v \) is \( v^\perp \) proving the claim.

Passing to the quotient by \( \Gamma \),

\[ T_{−1}M \cong (E \times H^2)/\Gamma. \]

Since \( \Gamma \xrightarrow{L} \text{SO}(2, 1) \) is a discrete embedding [14], \( \text{SO}(2, 1) \) acting properly on \( H^2 \) implies that \( \Gamma \) acts properly on \( H^2 \). The Cartesian projection \( E \times H^2 \to H^2 \) induces a projection

\[ T_{−1}M \longrightarrow H^2/L(\Gamma) = \Sigma, \]

invariant under the restriction of the geodesic flow \( \psi \) to \( T_{−1}M \), which defines an \( E \)-bundle over \( \Sigma \). Its fiber over the orbit \( \Gamma v \) of a fixed future-pointing unit-timelike vector \( v \) is the union of geodesics in \( M = E/\Gamma \) parallel to \( \Gamma v \). In particular, properness of the \( L(\Gamma) \)-action on \( H^2 \) implies non-recurrence of timelike geodesics, the last statement in Theorem 1.

More generally, any \( L(\Gamma) \)-invariant subset \( \Omega \subset V \) defines a subset \( T_\Omega(M) \subset TM \) invariant under the geodesic flow. If \( \Omega \) is an open set upon which \( L(\Gamma) \) acts properly, then the geodesic flow defines a proper \( \mathbb{R} \)-action on \( T_\Omega(M) \). In particular, every geodesic whose velocity lies in \( \Omega \) is properly immersed and is neither positively nor negatively recurrent. An important example is the following. The lines in \( S_0 \) form the ideal boundary (the circle-at-infinity), \( \partial H^2 \), of \( H^2 \). The limit set of \( L(\Gamma) \) consists of endpoints of recurrent geodesic rays in \( \Sigma \). Furthermore, \( \Lambda_{L(\Gamma)} \) is the unique minimal \( L(\Gamma) \)-invariant closed
subset of $\partial \mathbb{H}^2$. In particular, the set of fixed points of elements of $L(\Gamma)$ is dense in $\Lambda_{L(\Gamma)}$. Moreover, $L(\Gamma)$ acts properly on the complement

$$\Omega := S_0 \setminus \Lambda_{L(\Gamma)}.$$ 

Applying the above discussion, no geodesic tangent to $T_\Omega(M)$ recurs, that is, a lightlike recurrent geodesic ray must be parallel to $\Lambda_{L(\Gamma)}$.

4. **From geodesics in $\Sigma^2$ to geodesics in $M^3$**

While timelike directions correspond to points of $\Sigma^2$, spacelike directions correspond to geodesics in $\mathbb{H}^2$. The recurrent geodesics in $\Sigma$ intimately relate to the recurrent spacelike geodesics on $M^3$.

Denote the set of oriented spacelike geodesics in $E$ by $\mathcal{S}$. It identifies with the orbit space of the geodesic flow $\tilde{\psi}$ on $T_{+1}E \cong E \times S_{+1}$. The natural map $\mathcal{S} \rightarrow S_{+1}$ associating to a spacelike vector its direction is equivariant with respect to $\text{Isom}(E) \rightarrow SO(2, 1)$.

The identity component of $SO(2, 1)$ simply acts transitively on the unit tangent bundle $U\mathbb{H}^2$, and therefore we identify $SO(2, 1)^0$ with $U\mathbb{H}^2$ by choosing a basepoint $u_0$ in $U\mathbb{H}^2$. Unit-spacelike vectors in $S_{+1}$ correspond to oriented geodesics in $\mathbb{H}^2$. Explicitly, if $v \in S_{+1}$, then there is a one-parameter subgroup $a(t) \in SO(2, 1)$, having $v$ as a fixed vector, and such that

$$\det(v, v^-, v^+) > 0,$$

where $v^+$ is an expanding eigenvector of $a(t)$ (for $t > 0$) and $v^-$ is the contracting eigenvector. Choose a basepoint $v_0 \in S_{+1}$ corresponding to the orbit of $u_0$ under the geodesic flow on $U\Sigma$. Geodesics in $\mathbb{H}^2$ relate to spacelike directions by an equivariant mapping

$$U\mathbb{H}^2 \rightarrow S_{+1},$$

$$g(u_0) \mapsto g(v_0).$$

The unit tangent bundle $U\Sigma$ of $\Sigma$ identifies with the quotient

$$L(\Gamma) \setminus U\mathbb{H}^2 \cong L(\Gamma) \setminus SO(2, 1)^0,$$

where the geodesic flow $\psi$ corresponds to the right-action of $a(-t)$ (see, for example, [15, §1.2]).

Observe that a geodesic in $\Sigma^2$ is recurrent if and only if the endpoints of any of its lifts to $\tilde{\Sigma} \cong \mathbb{H}^2$ lie in the limit set $\Lambda_{L(\Gamma)}$ of $L(\Gamma)$. If the convex core of $\Sigma^2$ is compact, then the union $U_{\text{rec}}\Sigma$ of recurrent $\phi$-orbits is compact.

**Lemma 2.** There exists an orbit-preserving map

$$U_{\text{rec}} \Sigma \xrightarrow{\tilde{N}} T_{+1}(M)$$

mapping $\phi$-orbits injectively to recurrent $\psi$-orbits.

**Proof.** The associated flat affine bundle $E_{\Gamma}$ over $U\Sigma$ associated to the affine deformation $\Gamma$ is defined as follows. The affine representation of $\Gamma$ defines a diagonal action of $\Gamma$
on $\widetilde{U}\Sigma \times E$. Its total space is the quotient of the product $\widetilde{U}\Sigma \times E$ by the diagonal action
of $\pi_1(U\Sigma)$:
$$\pi_1(U\Sigma) \twoheadrightarrow \pi_1(\Sigma) \twoheadrightarrow \text{Isom}(E).$$
Similarly, the flat vector bundle $V_\Gamma$ over $U\Sigma$ is the quotient of $\widetilde{U}\Sigma \times V$ by the diagonal action
$$\pi_1(U\Sigma) \twoheadrightarrow \pi_1(\Sigma) \twoheadrightarrow \text{Isom}(E) \xrightarrow{L} \text{SO}(2, 1).$$
According to [15], the neutral section of $V_\Gamma$ is a $\text{SO}(2, 1)$-invariant section which is parallel with respect to the geodesic flow on $U\Sigma$, and arises from the graph of the $\text{SO}(2, 1)$-equivariant map
$$U\Sigma \cong UH^2 \twoheadrightarrow V$$
with image $S_{+1}$, the space of unit-spacelike vectors in $V$.

Here is the main construction of [15]. To every section $\sigma$ of $E_\Gamma$ continuously differentiable along $\phi$, associate the function
$$F_\sigma := \langle \nabla_\phi \sigma, \nu \rangle$$
on $U\Sigma$. (Here the covariant derivative of a section of $E_\Gamma$ along a vector field $\phi$ in the base is a section of the associated vector bundle $V_\Gamma$.) Different choices of section $\sigma$ yield cohomologous functions $F_\sigma$. (Recall that two functions $f_1, f_2$ are cohomologous, written $f_1 \sim f_2$, if
$$f_1 - f_2 = \phi g$$
for a function $g$ which is differentiable with respect to the vector field $\phi$ [17, §2.2]).

Restrict the affine bundle $E_\Gamma$ to $U_{\text{rec}} \Sigma$. Goldman et al [15, Lemma 8.4] guarantees the existence of a neutralized section, that is, a section $N$ of $(E_\Gamma)|_{U_{\text{rec}} \Sigma}$ satisfying
$$\nabla_\phi N = f \nu,$$
for some function $f$.

Although the following lemma is well known, we could not find a proof in the literature. For completeness, we supply a proof in the appendix.

**Lemma 3.** Let $X$ be a compact space equipped with a flow $\phi$. Let $f \in C(X)$, such that, for all $\phi$-invariant measures $\mu$ on $X$,
$$\int f \ d\mu > 0.$$Then $f$ is cohomologous to a positive function.

Since $\Gamma$ acts properly, [15, Proposition 8.1] implies that $\int F_\sigma \ d\mu \neq 0$ for all $\phi$-invariant probability measures $\mu$ on $U_{\text{rec}} \Sigma$. Since the set of invariant measures is connected, $\int F_\sigma \ d\mu$ is either positive for all $\phi$-invariant probability measures $\mu$ on $U_{\text{rec}} \Sigma$ or negative for all $\phi$-invariant probability measures $\mu$ on $U_{\text{rec}} \Sigma$. Conjugating by $-I$ if necessary, we may assume that $\int F_\sigma \ d\mu > 0$. Lemma 3 implies $F_\sigma + \phi g > 0$ for some function $g$. Write
$$\hat{N} = N + g \nu,$$$\hat{N}$ remains neutralized, and $\nabla_\phi \hat{N}$ vanishes nowhere.
Let $\tilde{U}_{\text{rec}} \Sigma$ be the preimage of $U_{\text{rec}} \Sigma$ in $UH^2$. Then $\tilde{N}$ determines a $\Gamma$-equivariant map

$$\tilde{U}_{\text{rec}} \Sigma \xrightarrow{\tilde{N}} E.$$ 

Each $\phi$-orbit injectively maps to a spacelike geodesic. The map

$$U_{\text{rec}} \Sigma \xrightarrow{\tilde{N}} (E \times S^1)/\Gamma,$$

$$x \mapsto [(\tilde{N}(x), v(x))]$$

is the desired orbit equivalence $U_{\text{rec}} \Sigma \longrightarrow T_{+1}(M)$.

**Lemma 4.** Any spacelike recurrent geodesic parallel to a geodesic $\gamma$ in the image of $\tilde{N}$ coincides with $\gamma$.

**Proof.** Let $t \mapsto g \xi(v)$ be an orbit in $U_{\text{rec}} \Sigma$. A geodesic $\xi$ parallel to $\tilde{N}(g)$ determines a parallel section $u$ of $V$ along $g$. Since $g$ recurs, the resulting parallel section is a bounded invariant parallel section along the closure of $g$. By the Anosov property, such a section is along $v$, and, therefore, up to reparametrization, $\gamma = \tilde{N}(g)$.

**Proposition 5.** $\tilde{N}$ is injective and its image is the set of recurrent spacelike geodesics.

**Proof.** An orbit of the geodesic flow $\phi$ recurs if and only if the corresponding $\Gamma$-orbit in the space $\mathcal{S}$ of spacelike geodesics in $E$ recurs. Similarly a $\phi$-orbit in $T_{+1}(M)$ recurs if and only if the corresponding $L(\Gamma)$-orbit in $S^1$ recurs. The map $\mathcal{S} \xrightarrow{\Upsilon} S^1$ recording the direction of a spacelike geodesic is $L$-equivariant. If the $\Gamma$-orbit of $g \in \mathcal{S}$ corresponds to a recurrent spacelike geodesic in $M$, then the $L(\Gamma)$-orbit of $\Upsilon(g)$ corresponds to a recurrent $\phi$-orbit in $U\Sigma$. $\tilde{N}$ is injective along orbits of the geodesic flow. Thus it suffices to prove that the restriction of $\Upsilon$ to the subset of $\Gamma$-recurrent geodesics in $\mathcal{S}$ is injective. Since the fibers of $\Upsilon$ are parallelism classes of spacelike geodesics, Lemma 4 implies injectivity of $\tilde{N}$.

Finally, let $g$ be a $\psi$-recurrent point in $T_{+1}(M)$, corresponding to a spacelike recurrent geodesic $\gamma$ in $M$. It corresponds to a recurrent $\Gamma$-orbit $\Gamma g$ in $\mathcal{S}$. Then $\Upsilon(\Gamma g)$ is a recurrent $L(\Gamma)$-orbit in $S^1$, and corresponds to a recurrent $\phi$-orbit in $U\Sigma$. The image of this $\phi$-orbit under $\tilde{N}$ is a spacelike recurrent geodesic in $T_{+1}(M)$ parallel to $\gamma$. Now apply Lemma 4 again to conclude that $g$ lies in the image of $\tilde{N}$.

The proof of Theorem 1 is complete.

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**A. Appendix. Cohomology and positive functions**

Let $X$ be a smooth manifold equipped with a smooth flow $\phi$. A function $g \in C(X)$ is continuously differentiable along $\phi$ if, for each $x \in X$, the function

$$t \mapsto g(\phi_t(x))$$
is a continuously differentiable map $\mathbb{R} \to X$. Denote the subspace of $C(X)$ consisting of functions continuously differentiable along $\phi$ by $C_\phi(X)$. For $g \in C_\phi(X)$, denote its directional derivative by
\[
\phi(g) := \frac{d}{dt} \bigg|_{t=0} g \circ \phi_t.
\]
The proof of Lemma 3 will be based on two lemmas.

**Lemma A.1.** Let $f \in C_\phi(X)$. For any $T > 0$, define
\[
f_T(x) := \frac{1}{T} \int_0^T f(\phi_s(x)) \, ds.
\]
Then $f \sim f_T$.

**Proof.** We must show that there exists a function $g \in C_\phi(X)$ such that
\[
f_T - f = \phi g.
\]
By the fundamental theorem of calculus,
\[
f \circ \phi_t = f + \int_0^t (\phi f \circ \phi_s) \, ds.
\]
Writing
\[
g = \frac{1}{T} \int_0^T \int_0^t (f \circ \phi_s) \, ds \, dt,
\]
then
\[
f_T - f = \frac{1}{T} \int_0^T \int_0^t (f \circ \phi_t - f) \, dt
= \frac{1}{T} \int_0^T \int_0^{t'} \phi(f \circ \phi_s) \, ds \, dt
= \phi g.
\]
as desired. \(\square\)

**Lemma A.2.** Assume that for all $\phi$-invariant measures $\mu$,
\[
\int f \, d\mu > 0.
\]
Then $f_T > 0$ for some $T > 0$.

**Proof.** Otherwise, sequences $\{T_m\}_{m \in \mathbb{N}}$ of positive real numbers and sequences $\{x_m\}_{m \in \mathbb{N}}$ of points in $M$ exist such that
\[
f_{T_m}(x_m) \leq 0.
\]
Using the flow $\phi_t$, push forward the (normalized) Lebesgue measure
\[
\frac{1}{T_m} \mu_{[0,T_m]}
\]
on the interval $[0, T_m]$ to $X$, to obtain a sequence of probability measures $\mu_n$ on $X$ such that
\[
\int f \, d\mu_n \leq 0.
\]
As in [15, §7], a subsequence weakly converges to a $\phi$-invariant measure $\mu$ for which
\[ \int f \, d\mu \leq 0, \]
contradicting our hypotheses. \hfill \Box

Proof of Lemma 3. By Lemma A.1, $f \sim f_T$ for any $T > 0$, and Lemma A.2 implies that $f_T > 0$ for some $T$. \hfill \Box

References