Playing pool on curved surfaces
and the wrong way to add fractions

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Mathematics: a *MOST* exact science

- Natural phenomena understood through quantitative measurements
  - Which are abstracted into mathematics.
  - These abstract ideas can be manipulated rigorously to make predictions.
  - Mathematical statements form a language in which measurements can be processed.
  - Mathematics represents an *ideal* situation which approximates the everyday world.

- For example:
  - Rates of change governed by laws of calculus.
  - Force = Mass \cdot Acceleration.
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Billiards on a square

- A billiard ball starts moving once it is subjected to the initial force, and changes direction when it bounces off the side of a billiard table.
  - Here is an example of a billiard ball on a square billiard table, which follows a *periodic* path.
  - Here is a longer periodic path. When the slope is rational (a fraction of two whole numbers), the path is periodic.
  - When the slope is irrational, the path never closes up, and eventually fills the whole square.

- Example of the inter-relationship between seemingly *different* subjects of mathematics: arithmetic (number theory), and differential equations (mechanics).
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Looking for universal patterns

- The same kind of differential equations that govern the motion of a moving ball can govern population growth, financial markets, chemical reactions...
  - Because they exhibit similar patterns.

- Mathematics is scalable:
  - What's true in the small is true in the large.

- Mathematics is reproducible:
  - Governed only by abstract logic,
  - And does not need special equipment, just working conditions conducive for clear thinking.
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Promote recurring patterns into primitive concepts.
- Break complicated relationships into simpler ones.
- Consolidating definitions creates new concepts.

Sometimes finding the right question is just as important as finding the right answer!

Asking and answering questions about the simpler concepts creates new mathematics.
- And it keeps on going...
- And growing.

More mathematics created in the last 50 years than before.

Challenge: How can you learn enough of what has already been done to create new mathematics?
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Art: beauty in the simplicity of ideas

- Sensing a familiar pattern in an unexpected setting;
- Familiarity is not only reassuring but empowering.
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The Golden Ratio

- The Parthenon is in the proportion of the *Golden Ratio*:

\[ \phi = \frac{1 + \sqrt{5}}{2} \approx 1.6180339887498948482045868343656381177203091798... \]

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A fraction which continues...

- $\phi \approx 1.618 \ldots$ satisfies the algebraic equation

$$\phi = 1 + \frac{1}{\phi}$$

- Replacing $\phi$ by $1 + \frac{1}{\phi}$ in this expression:

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What does this infinite fraction mean?

- This infinite expression is **meaningless** until we give it meaning!
  - Mathematicians change the questions to fit the answers!
- For example, define it to be the *limit* of the sequence
  
  \[
  1, \quad 1 + \frac{1}{1} = 2, \quad 1 + \frac{1}{2} = \frac{3}{2}, \quad 1 + \frac{1}{3/2} = \frac{5}{3}, \quad 1 + \frac{1}{5/3} = \frac{8}{5}, \quad 1 + \frac{1}{8/5} = \frac{13}{8}, \ldots
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- Numerators and denominators are *Fibonacci numbers*:

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  \[ 1 + 1 = 2, 3, 5, 8, 13, 21, 34, \ldots, \] obtained by successively adding the two previous numbers in the sequence.

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Approximate $\phi$ with billiards!
The wrong way to add fractions

Notice a pattern in the sequence of fractions approximating $\phi$:

$$\frac{1}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \cdots$$

Each fraction is obtained from the preceding pair by adding numerators and denominators:

$$\frac{a}{b} \boxplus \frac{c}{d} = \frac{a + c}{b + d}.$$
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Each fraction is obtained from the preceding pair by adding numerators and denominators:

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a + c}{b + d}.$$
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\[
1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad 55 \quad 89 \\
1' \quad 1' \quad 2' \quad 3' \quad 5' \quad 8' \quad 13' \quad 21' \quad 34' \quad 55' \\
\]

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\]

\[
\frac{1}{1}, \frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{5}{5}, \frac{8}{8}, \frac{13}{13}, \frac{21}{21}, \frac{34}{34}, \frac{55}{55}, \ldots
\]

- Each fraction is obtained from the preceding pair by *adding numerators and denominators*:

\[
\frac{a}{b} \oplus \frac{c}{d} = \frac{a + c}{b + d}.
\]

\[
1 \oplus 2 = 3, 5, 8, 13, 21, 34, 55, 89, \ldots
\]

\[
1 \oplus 1 = 2, 3, 5, 8, 13, 21, 34, 55, \ldots
\]
The wrong way to add fractions

- Notice a pattern in the sequence of fractions approximating \( \phi \):

\[
\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \cdots
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\[
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\]
The wrong way to add fractions

- Notice a pattern in the sequence of fractions approximating $\phi$:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 89 \\
1' & 1' & 2' & 3' & 5' & 8' & 13' & 21' & 34' & 55' \\
\end{array}
\]

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$$
\begin{align*}
1 & \quad 2 & \quad 3 & \quad 5 & \quad 8 & \quad 13 & \quad 21 & \quad 34 & \quad 55 & \quad 89 \\
\frac{1}{1} & \quad \frac{1}{1} & \quad \frac{2}{2} & \quad \frac{3}{3} & \quad \frac{5}{5} & \quad \frac{8}{8} & \quad \frac{13}{13} & \quad \frac{21}{21} & \quad \frac{34}{34} & \quad \frac{55}{55} \\
\end{align*}
$$

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$$

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\end{align*}
$$
The wrong way to add fractions

- Notice a pattern in the sequence of fractions approximating $\phi$:

  $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \cdots$

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  $$\frac{a}{b} \oplus \frac{c}{d} = \frac{a + c}{b + d}.$$

  $\frac{1}{1} \oplus \frac{13}{8} = \frac{21}{13}, \frac{2}{1} \oplus \frac{8}{5} = \frac{21}{13}, \frac{3}{2} \oplus \frac{13}{8} = \frac{34}{21}, \frac{5}{3} \oplus \frac{21}{13} = \frac{55}{34}, \frac{8}{5} \oplus \frac{34}{21} = \frac{89}{55}, \cdots$
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\[
\begin{align*}
&\quad 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 \\
&\quad \frac{1}{1}, \frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{5}{5}, \frac{8}{8}, \frac{13}{13}, \frac{21}{21}, \frac{34}{34}, \frac{55}{55}, \cdots
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\]

\[
\begin{align*}
1, 2, 3, 5, 8, 13 \oplus 21 &= 34, 55, 89 \\
\frac{1}{1}, \frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{5}{5}, \frac{8}{8} \oplus \frac{21}{13} &= \frac{21}{21}, \frac{34}{34}, \frac{55}{55}, \cdots
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Farey series

- List the fractions (in order) with denominator \( \leq n \):
- Each fraction is obtained from the two closest ones above by adding *numerators* and *denominators*: \( \frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d} \).

- \( n = 6 \):

  0, 1, 1, 1, 2, 1, 3, 2, 3, 4, 5, 1, 7, 6, 5, 4, 7, 3, 8, 5, 7, 9, 11, 2

  1, 5, 6, 4, 3, 5, 2, 5, 3, 4, 5, 6, 1, 6, 5, 4, 3, 5, 2, 5, 3, 4, 5, 6, 1

- John Farey, Sr. (1766–1826), a British geologist, was led to these discoveries through his interest in the mathematics of sound. *(Philosophical Magazine 1816).*
List the fractions (in order) with denominator $\leq n$:

Each fraction is obtained from the two closest ones above by adding numerators and denominators: $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$.

$n = 6$:

\[
\begin{array}{cccccccccccccccc}
0 & 1 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & 4 & 5 & 1 & 7 & 6 & 5 & 4 & 7 & 3 & 8 & 5 & 7 & 9 & 11 & 2 \\
1 & 5 & 6 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 5 & 6 & 1 & 6 & 5 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 5 & 6 & 1
\end{array}
\]

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  \[
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  \]

- \( n = 6 \):
  
  \[
  \begin{align*}
  0 & \quad \frac{1}{5} & \quad \frac{1}{6} & \quad \frac{1}{4} & \quad \frac{1}{3} & \quad \frac{1}{2} & \quad \frac{1}{3} & \quad \frac{2}{5} & \quad \frac{3}{4} & \quad \frac{4}{5} & \quad \frac{5}{6} & \quad \frac{1}{7} & \quad \frac{7}{6} & \quad \frac{6}{5} & \quad \frac{4}{3} & \quad \frac{7}{5} & \quad \frac{3}{2} & \quad \frac{8}{5} & \quad \frac{7}{4} & \quad \frac{9}{5} & \quad \frac{11}{6} & \quad \frac{2}{1}
  \end{align*}
  \]

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Each fraction is obtained from the two closest ones above by adding numerators and denominators:
\[
\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}.
\]

- \( n = 1 \):
  \[
  0, \frac{1}{1}, \frac{2}{1}
  \]

- \( n = 6 \):
  \[
  0, \frac{1}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{2}{4}, \frac{2}{3}, \frac{3}{5}, \frac{3}{4}, \frac{3}{3}, \frac{4}{5}, \frac{4}{4}, \frac{4}{3}, \frac{5}{5}, \frac{5}{4}, \frac{5}{3}, \frac{6}{5}, \frac{6}{4}, \frac{6}{3}, \frac{7}{5}, \frac{7}{4}, \frac{7}{3}, \frac{8}{5}, \frac{8}{4}, \frac{8}{3}, \frac{9}{5}, \frac{9}{4}, \frac{11}{6}, \frac{11}{5}, \frac{1}{1}
  \]

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- \( n = 2 \):
  \[
  0, \frac{1}{1}, \frac{1}{2}, \frac{1}{1}, 2
  \]
- \( n = 6 \):
  \[
  0, 1, 1, 1, 1, 1, 2, 1, 3, 2, 3, 2, 4, 5, 1, 7, 6, 5, 1, 5, 3, 4, 5, 4, 7, 3, 8, 5, 7, 9, 11, 2
  \]
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Farey series

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- $n = 3$:
  
  $\begin{align*}
  0, & 1, 1, 2, 1, 4, 3, 5, 2 \\
  1, & 3, 2, 3, 2, 3, 2, 3, 5
  \end{align*}$

- $n = 6$:
  
  $\begin{align*}
  0, & 1, 1, 1, 1, 2, 1, 3, 2, 3, 4, 5, 1, 7, 6, 5, 4, 7, 3, 8, 5, 7, 9, 11, 2 \\
  1, & 5, 6, 4, 3, 5, 2, 5, 3, 4, 5, 6, 1, 6, 5, 4, 3, 5, 2, 5, 3, 4, 5, 6, 1
  \end{align*}$

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- $n = 4$:
  
<table>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>1</th>
<th>5</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

- $n = 6$:
  
  | 0 | 1 | 1 | 1 | 2 | 1 | 3 | 2 | 3 | 4 | 1 | 7 | 6 | 5 | 4 | 7 | 3 | 8 | 5 | 7 | 9 | 11 | 2 |
  | 1 | 5 | 6 | 4 | 3 | 5 | 2 | 5 | 3 | 4 | 5 | 6 | 1 | 6 | 5 | 4 | 3 | 5 | 2 | 5 | 3 | 4 | 5 | 6 | 1 |

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- \( n = 5 \):
  \[
  0, 1, 1, 1, 2, 1, 3, 2, 3, 4, 1, 6, 5, 4, 7, 3, 8, 5, 7, 9, 2
  \]
  \[
  1, 5, 4, 3, 5, 2, 5, 3, 4, 5, 1, 5, 4, 3, 5, 2, 5, 3, 4, 5
  \]
- \( n = 6 \):
  \[
  0, 1, 1, 1, 2, 1, 3, 2, 3, 4, 5, 1, 7, 6, 5, 4, 7, 3, 8, 5, 7, 9, 11, 2
  \]
  \[
  1, 5, 6, 4, 3, 5, 2, 5, 3, 4, 5, 6, 1, 5, 4, 3, 5, 2, 5, 3, 4, 5, 6, 1
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\[
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\[
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0 & 1 & 1 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & 4 & 5 & 1 & 7 & 6 & 5
\end{array}
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How a mathematical concept is created

- A pattern is isolated.
  - Focus on its essential qualities.
- Promote it to a new concept
  - Give it a definition.
- Relate it to already defined concepts through theorems,
  - which must be rigorously proved!
  - The right definitions may make the theorems much easier to prove.
- Similar to art: a human representation of an abstract pattern.
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Challenges to doing mathematics

Its unique nature leads to basic challenges in its teaching, communication, and dissemination, unlike any other intellectual discipline.
A remarkably *successful* discipline

- Mathematics goes back thousands of years, and ...
  - continues to grow.
- Old mathematics is *not* discarded ...
  - but condensed.
- Leading to challenges in disseminating, organizing, teaching ...
- As more common relationships are discovered, ideas *generalize* ...
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Going out of control?

- Too many subdivisions...
  - Despite basic unity, a natural tendency to splinter.
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- Last 30 years: remarkable confluence of mathematical ideas.
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The Tower of Babel
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![The Tower of Babel](image)

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The Tower of Babel
Investing in mathematics is investing in *people*!

- The speakers of a specialized language...
  - Are the audience ...
  - And the practitioners...
  - And the developers ...
  - And the first users.
- Build a community of technically literate and creative *people*. 
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Mathematics: A fundamentally *human* activity.

Terrapins work out the equations of straight lines on curved surfaces.
Potomac High School students visit the Experimental Geometry Lab.
Why support mathematics?

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  - Learn and work with abstract ideas,
  - Communicate them effectively
  - ... all in a short period of time...
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A community activity
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Mathematics:

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- **A Language:** a collection of ideas, represented symbolically and organized into units of communication.
- **An art:** an esthetic activity, characterized by elegance and simplicity, despite its innate complexity.
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These three roles complement each other in a unique way.

And the growth of mathematics leads to serious challenges in
- Training,
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Mathematics: A fundamentally human activity.

Let’s enrich our society with communities of literate, knowledgeable and creative mathematicians
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Playing pool on curved surfaces...