Actuarial Mathematics and Life-Table Statistics

Eric V. Slud
Mathematics Department
University of Maryland, College Park

Eric V. Slud
Statistics Program
Mathematics Department
University of Maryland
College Park, MD 20742

Contents

0.1 Preface

This book is a course of lectures on the mathematics of actuarial science. The idea behind the lectures is as far as possible to deduce interesting material on contingent present values and life tables directly from calculus and commonsense notions, illustrated through word problems. Both the Interest Theory and Probability related to life tables are treated as wonderful concrete applications of the calculus. The lectures require no background beyond a third semester of calculus, but the prerequisite calculus courses must have been solidly understood. It is a truism of pre-actuarial advising that students who have not done really well in and digested the calculus ought not to consider actuarial studies.

It is not assumed that the student has seen a formal introduction to probability. Notions of relative frequency and average are introduced first with reference to the ensemble of a cohort life-table, the underlying formal random experiment being random selection from the cohort life-table population (or, in the context of probabilities and expectations for 'lives aged x', from the subset of l_x members of the population who survive to age x). The calculation of expectations of functions of a time-to-death random variables is rooted on the one hand in the concrete notion of life-table average, which is then approximated by suitable idealized failure densities and integrals. Later, in discussing Binomial random variables and the Law of Large Numbers, the combinatorial and probabilistic interpretation of binomial coefficients are derived from the Binomial Theorem, which the student the is assumed to know as a topic in calculus (Taylor series identification of coefficients of a polynomial.) The general notions of expectation and probability are introduced, but for example the Law of Large Numbers for binomial variables is treated (rigorously) as a topic involving calculus inequalities and summation of finite series. This approach allows introduction of the numerically and conceptually useful large-deviation inequalities for binomial random variables to explain just how unlikely it is for binomial (e.g., life-table) counts to deviate much percentage-wise from expectations when the underlying population of trials is large.

The reader is also not assumed to have worked previously with the Theory of Interest. These lectures present Theory of Interest as a mathematical problem-topic, which is rather unlike what is done in typical finance courses.

Getting the typical Interest problems — such as the exercises on mortgage refinancing and present values of various payoff schemes — into correct format for numerical answers is often not easy even for good mathematics students.

The main goal of these lectures is to reach — by a conceptual route — mathematical topics in Life Contingencies, Premium Calculation and Demography not usually seen until rather late in the trajectory of quantitative Actuarial Examinations. Such an approach can allow undergraduates with solid preparation in calculus (not necessarily mathematics or statistics majors) to explore their possible interests in business and actuarial science. It also allows the majority of such students — who will choose some other avenue, from economics to operations research to statistics, for the exercise of their quantitative talents — to know something concrete and mathematically coherent about the topics and ideas actually useful in Insurance.

A secondary goal of the lectures has been to introduce varied topics of applied mathematics as part of a reasoned development of ideas related to survival data. As a result, material is included on statistics of biomedical studies and on reliability which would not ordinarily find its way into an actuarial course. A further result is that mathematical topics, from differential equations to maximum likelihood estimators based on complex life-table data, which seldom fit coherently into undergraduate programs of study, are 'vertically integrated' into a single course.

While the material in these lectures is presented systematically, it is not separated by chapters into unified topics such as Interest Theory, Probability Theory, Premium Calculation, etc. Instead the introductory material from probability and interest theory are interleaved, and later, various mathematical ideas are introduced as needed to advance the discussion. No book at this level can claim to be fully self-contained, but every attempt has been made to develop the mathematics to fit the actuarial applications as they arise logically.

The coverage of the main body of each chapter is primarily 'theoretical'. At the end of each chapter is an Exercise Set and a short section of Worked Examples to illustrate the kinds of word problems which can be solved by the techniques of the chapter. The Worked Examples sections show how the ideas and formulas work smoothly together, and they highlight the most important and frequently used formulas.

Chapter 1

Basics of Probability and the Theory of Interest

The first lectures supply some background on elementary Probability Theory and basic Theory of Interest. The reader who has not previously studied these subjects may get a brief overview here, but will likely want to supplement this Chapter with reading in any of a number of calculus-based introductions to probability and statistics, such as Larson (1982), Larsen and Marx (1985), or Hogg and Tanis (1997) and the basics of the Theory of Interest as covered in the text of Kellison (1970) or Chapter 1 of Gerber (1997).

1.1 Probability, Lifetimes, and Expectation

In the *cohort life-table model*, imagine a number l_0 of individuals born simultaneously and followed until death, resulting in data d_x, l_x for each age $x = 0, 1, 2, \ldots,$ where

$$l_x = \text{number of lives aged } x \quad \text{(i.e. alive at birthday } x \text{)}$$

and

$$d_x \; = \; l_x - l_{x+1} \quad = \text{number dying between ages } x, \, x+1$$

Now, allowing the age-variable x to take all real values, not just whole numbers, treat $S(x) = l_x/l_0$ as a piecewise continuously differentiable nonincreasing function called the "survivor" or "survival" function. Then for all positive real x, $S(x) - S(x+t)$ is the fraction of the initial cohort which fails between time x and $x+t$, and

$$\frac{S(x) - S(x+t)}{S(x)} = \frac{l_x - l_{x+t}}{l_x}$$

denotes the fraction of those alive at exact age x who fail before $x+t$.

Question: what do probabilities have to do with the life table and survival function ?

To answer this, we first introduce probability as simply a relative frequency, using numbers from a cohort life-table like that of the accompanying Illustrative Life Table. In response to a probability question, we supply the fraction of the relevant life-table population, to obtain identities like

$$Pr(\text{life aged 29 dies between exact ages 35 and 41 or between 52 and 60 })$$

$$= S(35) - S(41) + S(52) - S(60) = \Big\{(l_{35} - l_{41}) + (l_{52} - l_{60})\Big\} \Big/ l_{29}$$

where our convention is that a *life aged 29* is one of the cohort surviving to the 29th birthday.

The idea here is that all of the lifetimes covered by the life table are understood to be governed by an identical "mechanism" of failure, and that any probability question about a single lifetime is really a question concerning the fraction of those lives about which the question is asked (e.g., those alive at age x) whose lifetimes will satisfy the stated property (e.g., die either between 35 and 41 or between 52 and 60). This "frequentist" notion of probability of an event as the relative frequency with which the event occurs in a large population of (independent) identical units is associated with the phrase "law of large numbers", which will bediscussed later. For now, remark only that the life table population should be large for the ideas presented so far to make good sense. See Table 1.1 for an illustration of a cohort life-table with realistic numbers.

Note: see any basic probability textbook, such as Larson (1982), Larsen and Marx (1985), or Hogg and Tanis (1997) for formal definitions of the notions of sample space, event, probability, and conditional probability. The main ideas which are necessary to understand the discussion so far are really

Table 1.1: Illustrative Life-Table, simulated to resemble realistic US (Male) life-table. For details of simulation, see Section 3.4 below.

Age x	l_x	d_x	x	l_x	d_x
0	100000	2629	40	92315	295
1	97371	141	41	92020	332
2	97230	107	42	91688	408
3	97123	63	43	91280	414
4	97060	63	44	90866	464
5	96997	69	45	90402	532
6	96928	69	46	89870	587
7	96859	52	47	89283	680
8	96807	54	48	88603	702
9	96753	51	49	87901	782
10	96702	33	50	87119	841
11	96669	40	51	86278	885
12	96629	47	52	85393	974
13	96582	61	53	84419	1082
14	96521	86	54	83337	1088
15	96435	105	55	82249	1213
16	96330	83	56	81036	1344
17	96247	125	57	79692	1423
18	96122	133	58	78269	1476
19	95989	149	59	76793	1572
20	95840	154	60	75221	1696
21	95686	138	61	73525	1784
22	95548	163	62	71741	1933
23	95385	168	63	69808	2022
24	95217	166	64	67786	2186
25	95051	151	65	65600	2261
26	94900	149	66	63339	2371
27	94751	166	67	60968	2426
28	94585	157	68	58542	2356
29	94428	133	69	56186	2702
30	94295	160	70	53484	2548
31	94135	149	71	50936	2677
32	93986	152	72	48259	2811
33	93834	160	73	45448	2763
34	93674	199	74	42685	2710
35	93475	187	75	39975	2848
36	93288	212	76	37127	2832
37	93076	228	77	34295	2835
38	92848	272	78	31460	2803
39	92576	261			

matters of common sense when applied to relative frequency but require formal axioms when used more generally:

- Probabilities are numbers between 0 and 1 assigned to subsets of the entire range of possible outcomes (in the examples, subsets of the interval of possible human lifetimes measured in years).
- The probability $P(A \cup B)$ of the union $A \cup B$ of disjoint (i.e., nonoverlapping) sets A and B is necessarily the sum of the separate probabilities $P(A)$ and $P(B)$.
- When probabilities are requested with reference to a smaller universe of possible outcomes, such as $B =$ *lives aged 29*, rather than all members of a cohort population, the resulting *conditional probabilities* of events A are written $P(A \,|\, B)$ and calculated as $P(A \cap B)/P(B)$, where $A \cap B$ denotes the *intersection* or *overlap* of the two events A, B.
- Two events A, B are defined to be *independent* when $P(A \cap B) = P(A) \cdot P(B)$ or — equivalently, as long as $P(B) > 0$ — the conditional probability $P(A|B)$ expressing the probability of A if B were known to have occurred, is the same as the (unconditional) probability $P(A)$.

The life-table data, and the mechanism by which members of the population die, are summarized first through the survivor function $S(x)$ which at integer values of x agrees with the ratios l_x/l_0. Note that $S(x)$ has values between 0 and 1, and can be interpreted as the probability for a single individual to survive at least x time units. Since fewer people are alive at larger ages, $S(x)$ is a decreasing function of x, and in applications $S(x)$ should be piecewise continuously differentiable (largely for convenience, and because any analytical expression which would be chosen for $S(x)$ in practice *will* be piecewise smooth). In addition, by definition, $S(0) = 1$. Another way of summarizing the probabilities of survival given by this function is to define the **density** function

$$f(x) = -\frac{dS}{dx}(x) = -S'(x)$$

as the (absolute) rate of decrease of the function S. Then, by the fundamental theorem of calculus, for any ages $a < b$,

$$P(\textit{life aged 0 dies between ages a and b}) = (l_a - l_b)/l_0$$

$$= S(a) - S(b) = \int_a^b (-S'(x))\, dx = \int_a^b f(x)\, dx \tag{1.1}$$

which has the very helpful geometric interpretation that the probability of dying within the interval $[a,b]$ is equal to the area under the curve $y = f(x)$ over the x-interval $[a,b]$. Note also that the 'probability' rule which assigns the integral $\int_A f(x)\, dx$ to the set A (which may be an interval, a union of intervals, or a still more complicated set) obviously satisfies the first two of the bulleted axioms displayed above.

The **terminal age** ω of a life table is an integer value large enough that $S(\omega)$ is negligibly small, but no value $S(t)$ for $t < \omega$ is zero. For practical purposes, no individual lives to the ω birthday. While ω is finite in real life-tables and in some analytical survival models, most theoretical forms for $S(x)$ have no finite age ω at which $S(\omega) = 0$, and in those forms $\omega = \infty$ by convention.

Now we are ready to define some terms and motivate the notion of expectation. Think of the age T at which a specified newly born member of the population will die as a **random variable**, which for present purposes means a variable which takes various values x with probabilities governed by the life table data l_x and the survivor function $S(x)$ or density function $f(x)$ in a formula like the one just given in equation (1.1). Suppose there is a contractual amount Y which must be paid (say, to the heirs of that individual) at the time T of death of the individual, and suppose that the contract provides a specific function $Y = g(T)$ according to which this payment depends on (the whole-number part of) the age T at which death occurs. What is the average value of such a payment over all individuals whose lifetimes are reflected in the life-table ? Since $d_x = l_x - l_{x+1}$ individuals (out of the original l_0) die at ages between x and $x+1$, thereby generating a payment $g(x)$, the total payment to all individuals in the life-table can be written as

$$\sum_x (l_x - l_{x+1})\, g(x)$$

Thus the average payment, at least under the assumption that $Y = g(T)$

depends only on the largest whole number $[T]$ less than or equal to T, is

$$\left.\begin{aligned} \textstyle\sum_x (l_x - l_{x+1})\, g(x) \,/\, l_0 \ &= \ \textstyle\sum_x (S(x) - S(x+1)) g(x) \\ = \ \textstyle\sum_x \int_x^{x+1} f(t)\, g(t)\, dt &= \textstyle\int_0^\infty f(t)\, g(t)\, dt \end{aligned}\right\} \tag{1.2}$$

This quantity, the total contingent payment over the whole cohort divided by the number in the cohort, is called the **expectation** of the random payment $Y = g(T)$ in this special case, and can be interpreted as the weighted average of all of the different payments $g(x)$ actually received, where the weights are just the relative frequency in the life table with which those payments are received. More generally, if the restriction that $g(t)$ depends only on the integer part $[t]$ of t were dropped , then the expectation of $Y = g(T)$ would be given by the same formula

$$E(Y) = E(g(T)) = \int_0^\infty f(t)\, g(t)\, dt$$

The last displayed integral, like all expectation formulas, can be understood as a weighted average of values $g(T)$ obtained over a population, with weights equal to the probabilities of obtaining those values. Recall from the Riemann-integral construction in Calculus that the integral $\int f(t)g(t)dt$ can be regarded approximately as the sum over very small time-intervals $[t, t+\Delta]$ of the quantities $f(t)g(t)\Delta$, quantities which are interpreted as the base Δ of a rectangle multiplied by its height $f(t)g(t)$, and the rectangle closely covers the area under the graph of the function $f\,g$ over the interval $[t, t+\Delta]$. The term $f(t)g(t)\Delta$ can alternatively be interpreted as the product of the value $g(t)$ — essentially equal to any of the values $g(T)$ which can be realized when T falls within the interval $[t, t+\Delta]$ — multiplied by $f(t)\,\Delta$. The latter quantity is, by the Fundamental Theorem of the Calculus, approximately equal for small Δ to the area under the function f over the interval $[t, t+\Delta]$, and is by definition equal to the probability with which $T \in [t, t+\Delta]$. In summary, $E(Y) = \int_0^\infty g(t)f(t)dt$ is the average of values $g(T)$ obtained for lifetimes T within small intervals $[t, t+\Delta]$ weighted by the probabilities of approximately $f(t)\Delta$ with which those T and $g(T)$ values are obtained. The expectation is a weighted average because the weights $f(t)\Delta$ sum to the integral $\int_0^\infty f(t)dt = 1$.

The same idea and formula can be applied to the restricted population of lives aged x. The resulting quantity is then called the **conditional**

expected value of $g(T)$ given that $T \geq x$. The formula will be different in two ways: first, the range of integration is from x to ∞, because of the resitriction to individuals in the life-table who have survived to exact age x; second, the density $f(t)$ must be replaced by $f(t)/S(x)$, the so-called **conditional density given** $T \geq x$, which is found as follows. From the definition of conditional probability, for $t \geq x$,

$$P(t \leq T \leq t+\Delta \Big| T \geq x) \;=\; \frac{P(\,[t \leq T \leq t+\Delta] \cap [T \geq x])}{P(T \geq x)}$$

$$= \; \frac{P(t \leq T \leq t+\Delta)}{P(T \geq x)} \;=\; \frac{S(t) - S(t+\Delta)}{S(x)}$$

Thus the density which can be used to calculate conditional probabilities $P(a \leq T \leq b \Big| T \geq x)$ for $x < a < b$ is

$$\lim_{\Delta \to 0} \frac{1}{\Delta} P(t \leq T \leq t+\Delta \Big| T \geq x) \;=\; \lim_{\Delta \to 0} \frac{S(t) - S(t+\Delta)}{S(x)\,\Delta} \;=\; -\,\frac{S'(t)}{S(x)} \;=\; \frac{f(t)}{S(x)}$$

The result of all of this discussion of conditional expected values is the formula, with associated weighted-average interpretation:

$$E(g(T) \Big| T \geq x) \;=\; \frac{1}{S(x)} \int_x^{\infty} g(t)\, f(t)\, dt \qquad (1.3)$$

1.2 Theory of Interest

Since payments based upon unpredictable occurrences or *contingencies* for insured lives can occur at different times, we study next the Theory of Interest, which is concerned with valuing streams of payments made over time. The general model in the case of constant interest is as follows. *Compounding* at time-intervals $h = 1/m$, with *nominal* interest rate $i^{(m)}$, means that a unit amount accumulates to $(1 + i^{(m)}/m)$ after a time $h = 1/m$. The principal or account value $1+i^{(m)}/m$ at time $1/m$ accumulates over the time-interval from $1/m$ until $2/m$, to $(1+i^{(m)}/m)\cdot(1+i^{(m)}/m) = (1+i^{(m)}/m)^2$. Similarly, by induction, a unit amount accumulates to $(1+i^{(m)}/m)^n = (1+i^{(m)}/m)^{Tm}$ after the time $T = nh$ which is a multiple of n whole units of h. In the limit of continuous compounding (i.e., $m \to \infty$), the unit amount compounds to $e^{\delta T}$ after time T, where the instantaneous annualized nominal interest rate $\delta = \lim_m i^{(m)}$ (also called the *force of interest*) will be shown to exist. In either case of compounding, the actual *Annual Percentage Rate* or **APR** or *effective interest rate* is defined as the amount (minus 1, and multiplied by 100 if it is to be expressed as a percentage) to which a unit compounds after a single year, i.e., respectively as

$$i_{\text{APR}} = \Big(1 + \frac{i^{(m)}}{m}\Big)^m - 1 \qquad \text{or} \qquad e^\delta - 1$$

The amount to which a unit invested at time 0 accumulates at the effective interest rate i_{APR} over a time-duration T (still assumed to be a multiple of $1/m$) is therefore

$$\Big(1 + i_{\text{APR}}\Big)^T = \Big(1 + \frac{i^{(m)}}{m}\Big)^{mT} = e^{\delta T}$$

This amount is called the *accumulation factor* operating over the interval of duration T at the fixed interest rate. Moreover, the first and third expressions of the displayed equation also make perfect sense when the duration T is any positive real number, not necessarily a multiple of $1/m$.

All the nominal interest rates $i^{(m)}$ for different periods of compounding are related by the formulas

$$(1 + i^{(m)}/m)^m = 1 + i = 1 + i_{\text{APR}} \quad , \quad i^{(m)} = m \,\Big\{(1+i)^{1/m} - 1\Big\} \tag{1.4}$$

Similarly, interest can be said to be governed by the *discount rates* for various compounding periods, defined by

$$1 - d^{(m)}/m = (1 + i^{(m)}/m)^{-1}$$

Solving the last equation for $d^{(m)}$ gives

$$d^{(m)} = i^{(m)}/(1 + i^{(m)}/m) \tag{1.5}$$

The idea of discount rates is that if \$1 is loaned out at interest, then the amount $d^{(m)}/m$ is the correct amount to be repaid at the *beginning* rather than the end of each fraction $1/m$ of the year, with repayment of the principal of \$1 at the end of the year, in order to amount to the same effective interest rate. The reason is that, according to the definition, the amount $1 - d^{(m)}/m$ accumulates at nominal interest $i^{(m)}$ (compounded m times yearly) to $(1 - d^{(m)}/m) \cdot (1 + i^{(m)}/m) = 1$ after a time-period of $1/m$.

The quantities $i^{(m)}$, $d^{(m)}$ are naturally introduced as the interest payments which must be made respectively at the ends and the beginnings of successive time-periods $1/m$ in order that the principal owed at each time j/m on an amount \$ 1 borrowed at time 0 will always be \$ 1. To define these terms and justify this assertion, consider first the simplest case, $m = 1$. If \$ 1 is to be borrowed at time 0, then the single payment at time 1 which fully compensates the lender, if that lender could alternatively have earned interest rate i, is \$ $(1 + i)$, which we view as a payment of \$ 1 *principal* (the face amount of the loan) and \$ i interest. In exactly the same way, if \$ 1 is borrowed at time 0 for a time-period $1/m$, then the repayment at time $1/m$ takes the form of \$ 1 principal and \$ $i^{(m)}/m$ interest. Thus, if \$ 1 was borrowed at time 0, an interest payment of \$ $i^{(m)}/m$ at time $1/m$ leaves an amount \$ 1 still owed, which can be viewed as an amount borrowed on the time-interval $(1/m, 2/m]$. Then a payment of \$ $i^{(m)}/m$ at time $2/m$ still leaves an amount \$ 1 owed at $2/m$, which is deemed borrowed until time $3/m$, and so forth, until the loan of \$ 1 on the final time-interval $((m-1)/m, 1]$ is paid off at time 1 with a final interest payment of \$ $i^{(m)}/m$ together with the principal repayment of \$ 1. The overall result which we have just proved intuitively is:

> \$ 1 at time 0 is equivalent to the stream of m payments of \$ $i^{(m)}/m$ at times $1/m, 2/m, \ldots, 1$ plus the payment of \$ 1 at time 1.

Similarly, if interest is to be paid at the *beginning* of the period of the loan instead of the end, the interest paid at time 0 for a loan of \$ 1 would be $d = i/(1 + i)$, with the only other payment a repayment of principal at time 1. To see that this is correct, note that since interest d is paid at the same instant as receiving the loan of \$ 1 , the net amount actually received is $1 - d = (1 + i)^{-1}$, which accumulates in value to $(1 - d)(1 + i) =$ \$ 1 at time 1. Similarly, if interest payments are to be made at the beginnings of each of the intervals $(j/m, (j + 1)/m]$ for $j = 0, 1, \ldots, m - 1$, with a final principal repayment of \$ 1 at time 1, then the interest payments should be $d^{(m)}/m$. This follows because the amount effectively borrowed (after the immediate interest payment) over each interval $(j/m, (j+1)/m]$ is \$ $(1 - d^{(m)}/m)$, which accumulates in value over the interval of length $1/m$ to an amount $(1 - d^{(m)}/m)(1 + i^{(m)}/m) = 1$. So throughout the year-long life of the loan, the principal owed at (or just before) each time $(j+1)/m$ is exactly \$ 1. The net result is

> \$ 1 at time 0 is equivalent to the stream of m payments of \$ $d^{(m)}/m$ at times $0, 1/m, 2/m, \ldots, (m-1)/m$ plus the payment of \$ 1 at time 1.

A useful algebraic exercise to confirm the displayed assertions is:

Exercise. Verify that the present values at time 0 of the payment streams with m interest payments in the displayed assertions are respectively

$$\sum_{j=1}^{m} \frac{i^{(m)}}{m} (1+i)^{-j/m} + (1+i)^{-1} \qquad \text{and} \qquad \sum_{j=0}^{m-1} \frac{d^{(m)}}{m} (1+i)^{-j/m} + (1+i)^{-1}$$

and that both are equal to 1. These identities are valid for all $i > 0$.

1.2.1 Variable Interest Rates

Now we formulate the generalization of these ideas to the case of non-constant instantaneously varying, but known or observed, nominal interest rates $\delta(t)$, for which the old-fashioned name would be *time-varying force of interest*. Here, if there is a compounding-interval $[kh, (k+1)h)$ of length $h = 1/m$, one would first use the instantaneous continuously-compounding interest-rate $\delta(kh)$ available at the beginning of the interval to calculate an equivalent annualized nominal interest-rate over the interval, i.e., to find a number $r_m(kh)$ such that

$$\left(1 + \frac{r_m(kh)}{m}\right) = \left(e^{\delta(kh)}\right)^{1/m} = \exp\left(\frac{\delta(kh)}{m}\right)$$

In the limit of large m, there is an essentially constant principal amount over each interval of length $1/m$, so that over the interval $[b, b+t)$, with instantaneous compounding, the unit principal amount accumulates to

$$\lim_{m \to \infty} e^{\delta(b)/m} e^{\delta(b+h)/m} \cdots e^{\delta(b+[mt]h)/m}$$

$$= \exp\left(\lim_m \frac{1}{m} \sum_{k=0}^{[mt]-1} \delta(b+k/m)\right) = \exp\left(\int_0^t \delta(b+s)\,ds\right)$$

The last step in this chain of equalities relates the concept of continuous compounding to that of the Riemann integral. To specify continuous-time varying interest rates in terms of effective or APR rates, instead of the instantaneous nominal rates $\delta(t)$, would require the simple conversion

$$r_{\text{APR}}(t) = e^{\delta(t)} - 1 \quad , \qquad \delta(t) = \ln\left(1 + r_{\text{APR}}(t)\right)$$

Next consider the case of deposits $s_0, s_1, \ldots, s_k, \ldots, s_n$ made at times $0, h, \ldots, kh, \ldots, nh$, where $h = 1/m$ is the given compounding-period, and where nominal annualized instantaneous interest-rates $\delta(kh)$ (with compounding-period h) apply to the accrual of interest on the interval $[kh, (k+1)h)$. If the accumulated bank balance just after time kh is denoted by B_k , then how can the accumulated bank balance be expressed in terms of s_j and $\delta(jh)$? Clearly

$$B_{k+1} = B_k \cdot \left(1 + \frac{i^{(m)}(kh)}{m}\right) + s_{k+1} \quad , \qquad B_0 = s_0$$

The preceding *difference equation* can be solved in terms of successive summation and product operations acting on the sequences s_j and $\delta(jh)$, as follows. First define a function A_k to denote the accumulated bank balance at time kh for a unit invested at time 0 and earning interest with instantaneous nominal interest rates $\delta(jh)$ (or equivalently, nominal rates $r_m(jh)$ for compounding at multiples of $h = 1/m$) applying respectively over the whole compounding-intervals $[jh, (j+1)h)$, $j = 0, \ldots, k-1$. Then by definition, A_k satisfies a *homogeneous equation* analogous to the previous one, which together with its solution is given by

$$A_{k+1} = A_k \cdot \left(1 + \frac{r_m(kh)}{m}\right), \qquad A_0 = 1, \qquad A_k = \prod_{j=0}^{k-1} \left(1 + \frac{r_m(jh)}{m}\right)$$

The next idea is the second basic one in the theory of interest, namely the idea of *equivalent investments* leading to the definition of *present value* of an income stream/investment. Suppose that a stream of deposits s_j accruing interest with annualized nominal rates $r_m(jh)$ with respect to compounding-periods $[jh,(j+1)h)$ for $j=0,\ldots,n$ is such that a single deposit D at time 0 would accumulate by compound interest to give exactly the same final balance F_n at time $T=nh$. Then the present cash amount D in hand is said to be **equivalent** to the value of a contract to receive s_j at time jh, $j=0,1,\ldots,n$. In other words, the **present value** of the contract is precisely D. We have just calculated that an amount 1 at time 0 compounds to an accumulated amount A_n at time $T=nh$. Therefore, an amount a at time 0 accumulates to $a\cdot A_n$ at time T, and in particular $1/A_n$ at time 0 accumulates to 1 at time T. Thus the present value of 1 at time $T=nh$ is $1/A_n$. Now define G_k to be the present value of the stream of payments s_j at time jh for $j=0,1,\ldots,k$. Since B_k was the accumulated value just after time kh of the same stream of payments, and since the present value at 0 of an amount B_k at time kh is just B_k/A_k, we conclude

$$G_{k+1} \;=\; \frac{B_{k+1}}{A_{k+1}} \;=\; \frac{B_k\,(1+r_m(kh)/m)}{A_k\,(1+r_m(kh)/m)}+\frac{s_{k+1}}{A_{k+1}}\;, \quad k\ge 1\;, \qquad G_0=s_0$$

Thus $G_{k+1}-G_k=s_{k+1}/A_{k+1}$, and

$$G_{k+1} \;=\; s_0+\sum_{i=0}^{k}\frac{s_{i+1}}{A_{i+1}} \;=\; \sum_{j=0}^{k+1}\frac{s_j}{A_j}$$

In summary, we have simultaneously found the solution for the accumulated balance B_k just after time kh and for the present value G_k at time 0 :

$$G_k \;=\; \sum_{i=0}^{k}\frac{s_i}{A_i}\;, \qquad B_k \;=\; A_k\cdot G_k\;, \qquad k=0,\ldots,n$$

The intuitive interpretation of the formulas just derived relies on the following simple observations and reasoning:

(a) The present value at fixed interest rate i of a payment of $1 exactly t years in the future, must be equal to the amount which must be put in the bank at time 0 to accumulate at interest to an amount 1 exactly t years later. Since $(1+i)^t$ is the factor by which today's deposit increases in exactly t years, the present value of a payment of $1 delayed t years is $(1+i)^{-t}$. Here t may be an integer or positive real number.

(b) Present values superpose additively: that is, if I am to receive a payment stream C which is the sum of payment streams A and B, then the present value of C is simply the sum of the present value of payment stream A and the present value of payment stream B.

(c) As a consequence of (a) and (b), the present value for constant interest rate i at time 0 of a payment stream consisting of payments s_j at future times t_j, $j = 0, \ldots, n$ must be the summation

$$\sum_{j=0}^{n} s_j \, (1+i)^{-t_j}$$

(d) Finally, to combine present values on distinct time intervals, at possibly different interest rates, remark that if fixed interest-rate i applies to the time-interval $[0, s]$ and the fixed interest rate i' applies to the time-interval $[s, t+s]$, then the present value at time s of a future payment of a at time $t+s$ is $b = a(1+i')^{-t}$, and the present value at time 0 of a payment b at time s is $b\,(1+i)^{-s}$. The idea of present value is that these three payments, a at time $s+t$, $b = a(1+i')^{-t}$ at time s, and $b(1+i)^{-s} = a\,(1+i')^{-t}\,(1+i)^{-s}$ at time 0 , are all equivalent.

(e) Applying the idea of paragraph (d) repeatedly over successive intervals of length $h = 1/m$ each, we find that the present value of a payment of \$1 at time t (assumed to be an integer multiple of h), where $r(kh)$ is the applicable *effective* interest rate on time-interval $[kh, \, (k+1)h]$, is

$$1/A(t) = \prod_{j=1}^{mt} (1 + r(jh))^{-h}$$

where $A(t) = A_k$ is the amount previously derived as the accumulation-factor for the time-interval $[0, t]$.

The formulas just developed can be used to give the *internal rate of return* r over the time-interval $[0, T]$ of a unit investment which pays amount s_k at times t_k, $k = 0, \ldots, n$, $0 \le t_k \le T$. This constant (effective) interest rate r is the one such that

$$\sum_{k=0}^{n} s_k \left(1 + r\right)^{-t_k} = 1$$

With respect to the APR r , the present value of a payment s_k at a time t_k time-units in the future is $s_k \cdot (1+r)^{-t_k}$. Therefore the stream of payments s_k at times t_k, $(k = 0, 1, \ldots, n)$ becomes equivalent, for the uniquely defined interest rate r, to an immediate (time-0) payment of 1.

Example 1 *As an illustration of the notion of effective interest rate, or internal rate of return, suppose that you are offered an investment option under which a \$ 10,000 investment made now is expected to pay \$ 300 yearly for 5 years (beginning 1 year from the date of the investment), and then \$ 800 yearly for the following five years, with the principal of \$ 10,000 returned to you (if all goes well) exactly 10 years from the date of the investment (at the same time as the last of the \$ 800 payments. If the investment goes as planned, what is the effective interest rate you will be earning on your investment ?*

As in all calculations of effective interest rate, the present value of the payment-stream, at the unknown interest rate $r = i_{APR}$, must be balanced with the value (here \$ 10,000) which is invested. (That is because the indicated payment stream is being regarded as equivalent to bank interest at rate r.) The balance equation in the Example is obviously

$$10,000 \;=\; 300 \sum_{j=1}^{5} (1+r)^{-j} \;+\; 800 \sum_{j=6}^{10} (1+r)^{-j} \;+\; 10,000\,(1+r)^{-10}$$

The right-hand side can be simplified somewhat, in terms of the notation $x = (1+r)^{-5}$, to

$$\frac{300}{1+r}\left(\frac{1-x}{1-(1+r)^{-1}}\right) + \frac{800x}{(1+r)}\left(\frac{1-x}{1-(1+r)^{-1}}\right) + 10000\,x^2$$

$$= \frac{1-x}{r}\,(300+800x) \;+\; 10000x^2 \qquad (1.6)$$

Setting this simplified expression equal to the left-hand side of 10,000 does not lead to a closed-form solution, since both $x = (1+r)^{-5}$ and r involve the unknown r. Nevertheless, we can solve the equation roughly by 'tabulating' the values of the simplified right-hand side as a function of r ranging in increments of 0.005 from 0.035 through 0.075. (We can guess that the correct answer lies between the minimum and maximum payments expressed as a fraction of the principal.) This tabulation yields:

r	.035	.040	.045	.050	.055	.060	.065	.070	.075
(1.6)	11485	11018	10574	10152	9749	9366	9000	8562	8320

From these values, we can see that the right-hand side is equal to $\$ 10,000$ for a value of r falling between 0.05 and 0.055. Interpolating linearly to approximate the answer yields $r = 0.050 + 0.005 * (10000 - 10152)/(9749 - 10152) = 0.05189$, while an accurate equation-solver (the one in the **Splus** function *uniroot*) finds $r = 0.05186$.

1.2.2 Continuous-time Payment Streams

There is a completely analogous development for continuous-time deposit streams with continuous compounding. Suppose $D(t)$ to be the **rate** per unit time at which savings deposits are made, so that if we take m to go to ∞ in the previous discussion, we have $D(t) = \lim_{m\to\infty} m s_{[mt]}$, where $[\cdot]$ again denotes greatest-integer. Taking $\delta(t)$ to be the time-varying nominal interest rate with continuous compounding, and $B(t)$ to be the accumulated balance as of time t (analogous to the quantity $B_{[mt]} = B_k$ from before, when $t = k/m$), we replace the previous difference-equation by

$$B(t+h) \;=\; B(t)\,(1 + h\,\delta(t)) \;+\; h\,D(t) \;+\; o(h)$$

where $o(h)$ denotes a remainder such that $o(h)/h \to 0$ as $h \to 0$. Subtracting $B(t)$ from both sides of the last equation, dividing by h, and letting h decrease to 0, yields a *differential equation* at times $t > 0$:

$$B'(t) \;=\; B(t)\,\delta(t) + D(t)\;, \qquad A(0) = s_0 \tag{1.7}$$

The method of solution of (1.7), which is the standard one from differential equations theory of multiplying through by an *integrating factor*, again has a natural interpretation in terms of present values. The integrating factor $1/A(t) = \exp(-\int_0^t \delta(s)\,ds)$ is the present value at time 0 of a payment of 1 at time t, and the quantity $B(t)/A(t) \;=\; G(t)$ is then the present value of the deposit stream of s_0 at time 0 followed by continuous deposits at rate $D(t)$. The ratio-rule of differentiation yields

$$G'(t) \;=\; \frac{B'(t)}{A(t)} - \frac{B(t)\,A'(t)}{A^2(t)} \;=\; \frac{B'(t) \;-\; B(t)\,\delta(t)}{A(t)} \;=\; \frac{D(t)}{A(t)}$$

where the substitutuion $A'(t)/A(t) \equiv \delta(t)$ has been made in the third expression. Since $G(0) = B(0) = s_0$, the solution to the differential equation (1.7) becomes

$$G(t) = s_0 + \int_0^t \frac{D(s)}{A(s)}\, ds\ , \qquad B(t) = A(t)\, G(t)$$

Finally, the formula can be specialized to the case of a constant unit-rate payment stream ($D(x) = 1$, $\delta(x) = \delta = \ln(1+i)$, $0 \le x \le T$) with no initial deposit (i.e., $s_0 = 0$). By the preceding formulas, $A(t) = \exp(t\ln(1+i)) = (1+i)^t$, and the present value of such a payment stream is

$$\int_0^T 1 \cdot \exp(-t\ln(1+i))\, dt = \frac{1}{\delta}\left(1 - (1+i)^{-T}\right)$$

Recall that the *force of interest* $\delta = \ln(1+i)$ is the limiting value obtained from the nominal interest rate $i^{(m)}$ using the difference-quotient representation:

$$\lim_{m\to\infty} i^{(m)} \;=\; \lim_{m\to\infty} \frac{\exp((1/m)\ln(1+i)) \;-\; 1}{1/m} \;=\; \ln(1+i)$$

The present value of a payment at time T in the future is, as expected,

$$\left(1 + \frac{i^{(m)}}{m}\right)^{-mT} \;=\; (1+i)^{-T} \;=\; \exp(-\delta\, T)$$

1.3 *Exercise Set 1*

The first homework set covers the basic definitions in two areas: (i) probability as it relates to events defined from *cohort life-tables*, including the theoretical machinery of population and conditional survival, distribution, and density functions and the definition of expectation; (ii) the theory of interest and present values, with special reference to the idea of income streams of equal value at a fixed rate of interest.

(1). For how long a time should \$100 be left to accumulate at 5% interest so that it will amount to twice the accumulated value (over the same time period) of another \$100 deposited at 3% ?

(2). Use a calculator to answer the following numerically:

(a) Suppose you sell for \$6,000 the right to receive for 10 years the amount of \$1,000 per year payable quarterly (beginning at the end of the first quarter). What effective rate of interest makes this a fair sale price ? (You will have to solve numerically or graphically, or interpolate a tabulation, to find it.)

(b) \$100 deposited 20 years ago has grown at interest to \$235. The interest was compounded twice a year. What were the nominal and effective interest rates ?

(c) How much should be set aside (the same amount each year) at the beginning of each year for 10 years to amount to \$1000 at the end of the 10th year at the interest rate of part (b) ?

In the following problems, $S(x)$ denotes the probability for a newborn in a designated population to survive to exact age x . If a *cohort life table* is under discussion, then the probability distribution relates to a randomly chosen member of the newborn cohort.

(3). Assume that a population's survival probability function is given by $S(x) = 0.1(100 - x)^{1/2}$, for $0 \le x \le 100$.

(a) Find the probability that a life aged 0 will die between exact ages 19 and 36.

(b) Find the probability that a life aged 36 will die before exact age 51.

(4). (a) Find the expected age at death of a member of the population in problem (3).

(b) Find the expected age at death of a life aged 20 in the population of problem (3).

(5). Use the Illustrative Life-table (Table 1.1) to calculate the following probabilities. (In each case, assume that the indicated span of years runs from birthday to birthday.) Find the probability

(a) that a life aged 26 will live at least 30 more years;

(b) that a life aged 22 will die between ages 45 and 55;

(c) that a life aged 25 will die either before age 50 or after the 70'th birthday.

(6). In a certain population, you are given the following facts:

(i) The probability that two independent lives, respectively aged 25 and 45, *both* survive 20 years is 0.7.

(ii) The probability that a life aged 25 will survive 10 years is 0.9.

Then find the probability that a life aged 35 will survive to age 65.

(7). Suppose that you borrowed \$1000 at 6% APR, to be repaid in 5 years in a lump sum, and that after holding the money idle for 1 year you invested the money to earn 8% APR for the remaining four years. What is the effective interest rate you have earned (ignoring interest costs) over 5 years on the \$1000 which you borrowed ? Taking interest costs into account, what is the present value of your profit over the 5 years of the loan ? Also re-do the problem if instead of repaying all principal and interest at the end of 5 years, you must make a payment of accrued interest at the end of 3 years, with the additional interest and principal due in a single lump-sum at the end of 5 years.

(8). Find the total present value at 5% APR of payments of \$1 at the end of 1, 3, 5, 7, and 9 years and payments of \$2 at the end of 2, 4, 6, 8, and 10 years.

1.4 Worked Examples

Example 1. How many years does it take for money to triple in value at interest rate i ?

The equation to solve is $3 = (1+i)^t$, so the answer is $\ln(3)/\ln(1+i)$, with numerical answer given by

$$t = \begin{cases} 22.52 & \text{for} \quad i = 0.05 \\ 16.24 & \text{for} \quad i = 0.07 \\ 11.53 & \text{for} \quad i = 0.10 \end{cases}$$

Example 2. Suppose that a sum of \$1000 is borrowed for 5 years at 5%, with interest deducted immediately in a lump sum from the amount borrowed, *and principal due in a lump sum at the end of the* 5 *years. Suppose further that the amount received is invested and earns 7%. What is the value of the net profit at the end of the* 5 *years ? What is its present value (at 5%) as of time 0 ?*

First, the amount received is $1000\,(1-d)^5 \;=\; 1000/(1.05)^5 \;=\; 783.53$, where $d = .05/1.05$, since the amount received should compound to precisely the principal of \$1000 at 5% interest in 5 years. Next, the compounded value of 783.53 for 5 years at 7% is $783.53\,(1.07)^5 = 1098.94$, so the net profit at the end of 5 years, after paying off the principal of 1000, is \$98.94. The present value of the profit ought to be calculated with respect to the 'going rate of interest', which in this problem is presumably the rate of 5% at which the money is borrowed, so is $98.94/(1.05)^5 = 77.52$.

Example 3. For the following small cohort life-table (first 3 columns) with 5 age-categories, find the probabilities for all values of $[T]$, *both unconditionally and conditionally for lives aged* 2, *and find the expectation of both* $[T]$ *and* $(1.05)^{-[T]-1}$.

The basic information in the table is the first column l_x of numbers surviving. Then $d_x = l_x - l_{x+1}$ for $x = 0, 1, \ldots, 4$. The random variable T is the life-length for a randomly selected individual from the age=0 cohort, and therefore $P([T] = x) = P(x \le T < x+1) = d_x/l_0$. The conditional probabilities given survivorship to age-category 2 are simply the ratios with numerator d_x for $x \ge 2$, and with denominator $l_2 = 65$.

x	l_x	d_x	$P([T]=x)$	$P([T]=x \mid T \ge 2)$	1.05^{-x-1}
0	100	20	0.20	0	0.95238
1	80	15	0.15	0	0.90703
2	65	10	0.10	0.15385	0.86384
3	55	15	0.15	0.23077	0.82770
4	40	40	0.40	0.61538	0.78353
5	0	0	0	0	0.74622

In terms of the columns of this table, we evaluate from the definitions and formula (1.2)

$$E([T]) \;=\; 0\cdot(0.20) + 1\cdot(0.15) + 2\cdot(0.10) + 3\cdot(0.15) + 4\cdot(0.40) \;=\; 2.4$$

$$E([T] \,|\, T \geq 2) \;=\; 2 \cdot (0.15385) + 3 \cdot (0.23077) + 4 \cdot (0.61538) \;=\; 3.4615$$

$$E(1.05^{-[T]-1}) \;=\; 0.95238 \cdot 0.20 \;+\; 0.90703 \cdot 0.15 \;+\; 0.86384 \cdot 0.10 + $$
$$+\, 0.8277 \cdot 0.15 \;+\; 0.78353 \cdot 0.40 \;=\; 0.8497$$

The expectation of $[T]$ is interpreted as the average per person in the cohort life-table of the number of completed whole years before death. The quantity $(1.05)^{-[T]-1}$ can be interpreted as the present value at birth of a payment of \$1 to be made at the end of the year of death, and the final expectation calculated above is the average of that present-value over all the individuals in the cohort life-table, if the going rate of interest is 5%.

Example 4. Suppose that the death-rates $q_x = d_x/l_x$ *for integer ages* x *in a cohort life-table follow the functional form*

$$q_x \;=\; \begin{cases} 4 \cdot 10^{-4} & \text{for } 5 \leq x < 30 \\ 8 \cdot 10^{-4} & \text{for } 30 \leq x \leq 55 \end{cases}$$

between the ages x *of 5 and 55 inclusive. Find analytical expressions for* $S(x)$, l_x, d_x *at these ages if* $l_0 = 10^5$, $S(5) = .96$.

The key formula expressing survival probabilities in terms of death-rates q_x is:

$$\frac{S(x+1)}{S(x)} \;=\; \frac{l_{x+1}}{l_x} \;=\; 1 - q_x$$

or

$$l_x \;=\; l_0 \,\cdot\, S(x) \;=\; (1-q_0)(1-q_1)\cdots(1-q_{x-1})$$

So it follows that for $x = 5, \ldots, 30$,

$$\frac{S(x)}{S(5)} \;=\; (1 - .0004)^{x-5} \;, \qquad l_x \;=\; 96000 \cdot (0.9996)^{x-5}$$

so that $S(30) \;=\; .940446$, and for $x = 31, \ldots, 55$,

$$S(x) \;=\; S(30) \cdot (.9992)^{x-30} \;=\; .940446\,(.9992)^{x-30}$$

The death-counts d_x are expressed most simply through the preceding expressions together with the formula $d_x \;=\; q_x\, l_x$.

1.5 Useful Formulas from Chapter 1

$$S(x) \;=\; \frac{l_x}{l_0} \quad , \qquad d_x = l_x - l_{x+1}$$

p. 1

$$P(x \le T \le x+k) \;=\; \frac{S(x) - S(x+k)}{S(x)} \;=\; \frac{l_x - l_{x+k}}{l_x}$$

p. 2

$$f(x) \;=\; -S'(x) \quad , \qquad S(x) - S(x+k) \;=\; \int_x^{x+t} f(t)\,dt$$

pp. 4, 5

$$E\left(g(T) \,\middle|\, T \ge x \right) \;=\; \frac{1}{S(x)} \int_x^{\infty} g(t)\, f(t)\, dt$$

p. 7

$$1 + i_{\text{APR}} \;=\; \left(1 + \frac{i^{(m)}}{m}\right)^m \;=\; \left(1 - \frac{d^{(m)}}{m}\right)^{-m} \;=\; e^{\delta}$$

p. 8

Chapter 2

Theory of Interest and Force of Mortality

The parallel development of Interest and Probability Theory topics continues in this Chapter. For application in Insurance, we are preparing to value uncertain payment streams in which times of payment may also be uncertain. The interest theory allows us to express the present values of *certain* payment streams compactly, while the probability material prepares us to find and interpret average or expected values of present values expressed as functions of random lifetime variables.

This installment of the course covers: (a) further formulas and topics in the pure (i.e., non-probabilistic) theory of interest, and (b) more discussion of lifetime random variables, in particular of *force of mortality* or hazard-rates, and theoretical families of life distributions.

2.1 More on Theory of Interest

The objective of this subsection is to define notations and to find compact formulas for present values of some standard payment streams. To this end, newly defined payment streams are systematically expressed in terms of previously considered ones. There are two primary methods of manipulating one payment-stream to give another for the convenient calculation of present values:

- First, if one payment-stream can be obtained from a second one precisely by delaying all payments by the same amount t of time, then the present value of the first one is v^t multiplied by the present value of the second.
- Second, if one payment-stream can be obtained as the superposition of two other payment streams, i.e., can be obtained by paying the total amounts at times indicated by either of the latter two streams, then the present value of the first stream is the sum of the present values of the other two.

The following subsection contains several useful applications of these methods. For another simple illustration, see Worked Example 2 at the end of the Chapter.

2.1.1 Annuities & Actuarial Notation

The general present value formulas above will now be specialized to the case of constant (instantaneous) interest rate $\delta(t) \equiv \ln(1+i) = \delta$ at all times $t \geq 0$, and some very particular streams of payments s_j at times t_j, related to periodic premium and annuity payments. The effective interest rate or APR is always denoted by i, and as before the m-times-per-year equivalent nominal interest rate is denoted by $i^{(m)}$. Also, from now on the standard and convenient notation

$$v \;\equiv\; 1/(1+i) \;=\; 1/\left(1+\frac{i^{(m)}}{m}\right)^m$$

will be used for the present value of a payment of \$1 in one year.

(i) If $s_0 = 0$ and $s_1 = \cdots = s_{nm} = 1/m$ in the discrete setting, where m denotes the number of payments per year, and $t_j = j/m$, then the payment-stream is called an **immediate annuity**, and its present value G_n is given the notation $a^{(m)}_{\overline{n}|}$ and is equal, by the geometric-series summation formula, to

$$m^{-1}\sum_{j=1}^{nm}\left(1+\frac{i^{(m)}}{m}\right)^{-j} = \frac{1-(1+i^{(m)}/m)^{-nm}}{m(1+i^{(m)}/m-1)} = \frac{1}{i^{(m)}}\Big(1-\Big(1+\frac{i^{(m)}}{m}\Big)^{-nm}\Big)$$

This calculation has shown

$$a^{(m)}_{\overline{n}|} \;=\; \frac{1 - v^n}{i^{(m)}} \tag{2.1}$$

All of these immediate annuity values, for fixed v, n but varying m, are roughly comparable because all involve a total payment of 1 per year. Formula (2.1) shows that all of the values $a^{(m)}_{\overline{n}|}$ differ only through the factors $i^{(m)}$, which differ by only a few percent for varying m and fixed i, as shown in Table 2.1. Recall from formula (1.4) that $i^{(m)} = m\{(1+i)^{1/m} - 1\}$.

If instead $s_0 = 1/m$ but $s_{nm} = 0$, then the notation changes to $\ddot{a}^{(m)}_{\overline{n}|}$, the payment-stream is called an **annuity-due**, and the value is given by any of the equivalent formulas

$$\ddot{a}^{(m)}_{\overline{n}|} \;=\; (1 + \frac{i^{(m)}}{m})\, a^{(m)}_{\overline{n}|} \;=\; \frac{1 - v^n}{m} \,+\, a^{(m)}_{\overline{n}|} \;=\; \frac{1}{m} + a^{(m)}_{\overline{n-1/m}|} \tag{2.2}$$

The first of these formulas recognizes the annuity-due payment-stream as identical to the annuity-immediate payment-stream shifted earlier by the time $1/m$ and therefore worth more by the accumulation-factor $(1+i)^{1/m} = 1 + i^{(m)}/m$. The third expression in (2.2) represents the annuity-due stream as being equal to the annuity-immediate stream with the payment of $1/m$ at $t = 0$ added and the payment of $1/m$ at $t = n$ removed. The final expression says that if the time-0 payment is removed from the annuity-due, the remaining stream coincides with the annuity-immediate stream consisting of $nm - 1$ (instead of nm) payments of $1/m$.

In the limit as $m \to \infty$ for fixed n, the notation $\overline{a}_{\overline{n}|}$ denotes the present value of an annuity paid instantaneously at constant unit rate, with the limiting nominal interest-rate which was shown at the end of the previous chapter to be $\lim_m \, i^{(m)} \;=\; i^{(\infty)} = \delta$. The limiting behavior of the nominal interest rate can be seen rapidly from the formula

$$i^{(m)} \;=\; m\Big((1+i)^{1/m} \,-\, 1\Big) \;=\; \delta \,\cdot\, \frac{\exp(\delta/m) - 1}{\delta/m}$$

since $(e^z - 1)/z$ converges to 1 as $z \to 0$. Then by (2.1) and (2.2),

$$\overline{a}_{\overline{n}|} \;=\; \lim_{m\to\infty} \ddot{a}^{(m)}_{\overline{n}|} \;=\; \lim_{m\to\infty} a^{(m)}_{\overline{n}|} \;=\; \frac{1 - v^n}{\delta} \tag{2.3}$$

Table 2.1: Values of nominal interest rates $i^{(m)}$ (upper number) and $d^{(m)}$ (lower number), for various choices of effective annual interest rate i and number m of compounding periods per year.

$i =$	.02	.03	.05	.07	.10	.15
$m = 2$	.0199	.0298	.0494	.0688	.0976	.145
	.0197	.0293	.0482	.0665	.0931	.135
3	.0199	.0297	.0492	.0684	.0968	.143
	.0197	.0294	.0484	.0669	.0938	.137
4	.0199	.0297	.0491	.0682	.0965	.142
	.0198	.0294	.0485	.0671	.0942	.137
6	.0198	.0296	.0490	.0680	.0961	.141
	.0198	.0295	.0486	.0673	.0946	.138
12	.0198	.0296	.0489	.0678	.0957	.141
	.0198	.0295	.0487	.0675	.0949	.139

A handy formula for annuity-due present values follows easily by recalling that

$$1 - \frac{d^{(m)}}{m} = \left(1 + \frac{i^{(m)}}{m}\right)^{-1} \quad \text{implies} \quad d^{(m)} = \frac{i^{(m)}}{1 + i^{(m)}/m}$$

Then, by (2.2) and (2.1),

$$\ddot{a}^{(m)}_{\overline{n}|} = (1 - v^n) \cdot \frac{1 + i^{(m)}/m}{i^{(m)}} = \frac{1 - v^n}{d^{(m)}} \tag{2.4}$$

In case m is 1, the superscript $^{(m)}$ is omitted from all of the annuity notations. In the limit where $n \to \infty$, the notations become $a^{(m)}_{\overline{\infty}|}$ and $\ddot{a}^{(m)}_{\overline{\infty}|}$, and the annuities are called **perpetuities** (respectively immediate and due) with present-value formulas obtained from (2.1) and (2.4) as:

$$a^{(m)}_{\overline{\infty}|} = \frac{1}{i^{(m)}} \quad , \quad \ddot{a}^{(m)}_{\overline{\infty}|} = \frac{1}{d^{(m)}} \tag{2.5}$$

Let us now build some more general annuity-related present values out of the standard functions $a^{(m)}_{\overline{n}|}$ and $\ddot{a}^{(m)}_{\overline{n}|}$.

(ii). Consider first the case of the **increasing perpetual annuity-due**, denoted $(I^{(m)}\ddot{a})^{(m)}_{\overline{\infty}|}$, which is defined as the present value of a stream of payments $(k+1)/m^2$ at times k/m, for $k = 0, 1, \ldots$ forever. Clearly the present value is

$$(I^{(m)}\ddot{a})^{(m)}_{\overline{\infty}|} \;=\; \sum_{k=0}^{\infty} m^{-2}\,(k+1)\left(1+\frac{i^{(m)}}{m}\right)^{-k}$$

Here are two methods to sum this series, the first purely mathematical, the second with actuarial intuition. First, without worrying about the strict justification for differentiating an infinite series term-by-term,

$$\sum_{k=0}^{\infty} (k+1)\,x^k \;=\; \frac{d}{dx}\sum_{k=0}^{\infty} x^{k+1} \;=\; \frac{d}{dx}\,\frac{x}{1-x} \;=\; (1-x)^{-2}$$

for $0 < x < 1$, where the geometric-series formula has been used to sum the second expression. Therefore, with $x = (1+i^{(m)}/m)^{-1}$ and $1-x = (i^{(m)}/m)/(1+i^{(m)}/m)$,

$$(I^{(m)}\ddot{a})^{(m)}_{\overline{\infty}|} = m^{-2}\Big(\frac{i^{(m)}/m}{1+i^{(m)}/m}\Big)^{-2} \;=\; \left(\frac{1}{d^{(m)}}\right)^2 \;=\; \left(\ddot{a}^{(m)}_{\overline{\infty}|}\right)^2$$

and (2.5) has been used in the last step. Another way to reach the same result is to recognize the increasing perpetual annuity-due as $1/m$ multiplied by the superposition of perpetuities-due $\ddot{a}^{(m)}_{\overline{\infty}|}$ paid at times $0, 1/m, 2/m, \ldots$, and therefore its present value must be $\ddot{a}^{(m)}_{\overline{\infty}|} \cdot \ddot{a}^{(m)}_{\overline{\infty}|}$. As an aid in recognizing this equivalence, consider each annuity-due $\ddot{a}^{(m)}_{\overline{\infty}|}$ paid at a time j/m as being equivalent to a stream of payments $1/m$ at time j/m, $1/m$ at $(j+1)/m$, etc. Putting together all of these payment streams gives a total of $(k+1)/m$ paid at time k/m, of which $1/m$ comes from the annuity-due starting at time 0, $1/m$ from the annuity-due starting at time $1/m$, up to the payment of $1/m$ from the annuity-due starting at time k/m.

(iii). The **increasing perpetual annuity-immediate** $(I^{(m)}a)^{(m)}_{\overline{\infty}|}$ — the same payment stream as in the increasing annuity-due, but deferred by a time $1/m$ — is related to the perpetual annuity-due in the obvious way

$$(I^{(m)}a)^{(m)}_{\overline{\infty}|} \;=\; v^{1/m}\,(I^{(m)}\ddot{a})^{(m)}_{\overline{\infty}|} \;=\; (I^{(m)}\ddot{a})^{(m)}_{\overline{\infty}|}\Big/(1+i^{(m)}/m) \;=\; \frac{1}{i^{(m)}\,d^{(m)}}$$

(iv). Now consider the **increasing annuity-due of finite duration** n years. This is the present value $(I^{(m)}\ddot{a})^{(m)}_{\overline{n}|}$ of the payment-stream of $(k+1)/m^2$ at time k/m, for $k = 0, \ldots, nm-1$. Evidently, this payment-stream is equivalent to $(I^{(m)}\ddot{a})^{(m)}_{\overline{\infty}|}$ minus the sum of n multiplied by an annuity-due $\ddot{a}^{(m)}_{\overline{\infty}|}$ starting at time n together with an increasing annuity-due $(I^{(m)}\ddot{a})^{(m)}_{\overline{\infty}|}$ starting at time n. (To see this clearly, equate the payments $0 = (k+1)/m^2 - n \cdot \frac{1}{m} - (k - nm + 1)/m^2$ received at times k/m for $k \geq nm$.) Thus

$$
\begin{aligned}
(I^{(m)}\ddot{a})^{(m)}_{\overline{n}|} &= (I^{(m)}\ddot{a})^{(m)}_{\overline{\infty}|} \left(1 - (1 + i^{(m)}/m)^{-nm}\right) - n\ddot{a}^{(m)}_{\overline{\infty}|}(1 + i^{(m)}/m)^{-nm} \\
&= \ddot{a}^{(m)}_{\overline{\infty}|} \left(\ddot{a}^{(m)}_{\overline{\infty}|} - (1 + i^{(m)}/m)^{-nm}\left[\ddot{a}^{(m)}_{\overline{\infty}|} + n\right]\right) \\
&= \ddot{a}^{(m)}_{\overline{\infty}|} \left(\ddot{a}^{(m)}_{\overline{n}|} - n\, v^n\right)
\end{aligned}
$$

where in the last line recall that $v = (1+i)^{-1} = (1 + i^{(m)}/m)^{-m}$ and that $\ddot{a}^{(m)}_{\overline{n}|} = \ddot{a}^{(m)}_{\overline{\infty}|}(1 - v^n)$. The latter identity is easy to justify either by the formulas (2.4) and (2.5) or by regarding the annuity-due payment stream as a superposition of the payment-stream up to time $n - 1/m$ and the payment-stream starting at time n. As an exercise, fill in details of a second, intuitive verification, analogous to the second verification in pargraph (ii) above.

(v). The **decreasing annuity** $(D^{(m)}\ddot{a})^{(m)}_{\overline{n}|}$ is defined as (the present value of) a stream of payments starting with n/m at time 0 and decreasing by $1/m^2$ every time-period of $1/m$, with no further payments at or after time n. The easiest way to obtain the present value is through the identity

$$
(I^{(m)}\ddot{a})^{(m)}_{\overline{n}|} + (D^{(m)}\ddot{a})^{(m)}_{\overline{n}|} = (n + \frac{1}{m})\, \ddot{a}^{(m)}_{\overline{n}|}
$$

Again, as usual, the method of proving this is to observe that in the payment-stream whose present value is given on the left-hand side, the payment amount at each of the times j/m, for $j = 0, 1, \ldots, nm-1$, is

$$
\frac{j+1}{m^2} + (\frac{n}{m} - \frac{j}{m^2}) = \frac{1}{m}(n + \frac{1}{m})
$$

2.1.2 Loan Amortization & Mortgage Refinancing

The only remaining theory-of-interest topic to cover in this unit is the breakdown between principal and interest payments in repaying a loan such as a mortgage. Recall that the present value of a payment stream of amount c per year, with c/m paid at times $1/m, 2/m, \ldots, n-1/m, n/m$, is $c\, \mathrm{a}^{(m)}_{\overline{n}|}$. Thus, if an amount *Loan-Amt* has been borrowed for a term of n years, to be repaid by equal installments at the end of every period $1/m$, at fixed nominal interest rate $i^{(m)}$, then the installment amount is

$$\text{Mortgage Payment} \;=\; \frac{\textit{Loan-Amt}}{m\, \mathrm{a}^{(m)}_{\overline{n}|}} \;=\; \textit{Loan-Amt}\; \frac{i^{(m)}}{m\,(1-v^n)}$$

where $v = 1/(1+i) = (1+i^{(m)}/m)^{-m}$. Of the payment made at time $(k+1)/m$, how much can be attributed to interest and how much to principal ? Consider the present value at 0 of the debt per unit of *Loan-Amt* less accumulated amounts paid up to and including time k/m :

$$1 - m\, \mathrm{a}^{(m)}_{\overline{k/m}|}\; \frac{1}{m\, \mathrm{a}^{(m)}_{\overline{n}|}} \;=\; 1 - \;\frac{1-v^{k/m}}{1-v^n} \;=\; \frac{v^{k/m}-v^n}{1-v^n}$$

The remaining debt, per unit of *Loan-Amt*, valued just after time k/m, is denoted from now on by $B_{n,\,k/m}$. It is greater than the displayed present value at 0 by a factor $(1+i)^{k/m}$, so is equal to

$$B_{n,\,k/m} \;=\; (1+i)^{k/m}\, \frac{v^{k/m}-v^n}{1-v^n} = \frac{1-v^{n-k/m}}{1-v^n} \tag{2.6}$$

The amount of interest for a Loan Amount of 1 after time $1/m$ is $(1+i)^{1/m} - 1 \;=\; i^{(m)}/m$. Therefore the interest included in the payment at $(k+1)/m$ is $i^{(m)}/m$ multiplied by the value $B_{n,\,k/m}$ of outstanding debt just after k/m. Thus the next total payment of $i^{(m)}/(m(1-v^n))$ consists of the two parts

$$\text{Amount of interest} \;=\; m^{-1}\, i^{(m)}\, (1-v^{n-k/m})/(1-v^n)$$

$$\text{Amount of principal} \;= m^{-1} i^{(m)} v^{n-k/m}/(1-v^n)$$

By definition, the principal included in each payment is the amount of the payment minus the interest included in it. These formulas show in particular that the amount of principal repaid in each successive payment increases geometrically in the payment number, which at first seems surprising. Note as a check on the displayed formulas that the outstanding balance $B_{n,(k+1)/m}$ immediately after time $(k+1)/m$ is re-computed as $B_{n,\,k/m}$ minus the interest paid at $(k+1)/m$, or

$$\frac{1 - v^{n-k/m}}{1 - v^n} - \frac{i^{(m)}}{m} \, \frac{v^{n-k/m}}{1 - v^n} = \frac{1 - v^{n-k/m}(1 + i^{(m)}/m)}{1 - v^n}$$

$$= \frac{1 - v^{n-(k+1)/m}}{1 - v^n} = \left(1 - \frac{a^{(m)}_{\overline{(k+1)/m}|}}{a^{(m)}_{\overline{n}|}} \right) v^{-(k+1)/m} \tag{2.7}$$

as was derived above by considering the accumulated value of amounts paid. The general definition of the principal repaid in each payment is the excess of the payment over the interest since the past payment on the total balance due immediately following that previous payment.

2.1.3 Illustration on Mortgage Refinancing

Suppose that a 30–year, nominal-rate 8%, $100,000 mortgage payable monthly is to be refinanced at the end of 8 years for an additional 15 years (instead of the 22 which would otherwise have been remaining to pay it off) at 6%, with a refinancing closing-cost amount of $1500 and *2 points*. (The points are each 1% of the refinanced balance including closing costs, and costs plus points are then extra amounts added to the initial balance of the refinanced mortgage.) Suppose that the **new** pattern of payments is to be valued at each of the nominal interest rates 6%, 7%, or 8%, due to uncertainty about what the interest rate will be in the future, and that these valuations will be taken into account in deciding whether to take out the new loan.

The monthly payment amount of the initial loan in this example was $\$100,000(.08/12)/(1-(1+.08/12)^{-360}) = \$733.76$, and the present value as of time 0 (the beginning of the old loan) of the payments made through the end of the 8^{th} year is $(\$733.76)\cdot(12 a^{(12)}_{\overline{8}|}) = \$51,904.69$. Thus the present value, *as of the end of 8 years*, of the payments still to be made under the *old* mortgage, is $\$(100,000 - 51,904.69)(1+.08/12)^{96} = \$91,018.31$. Thus, if the loan were to be refinanced, the new refinanced loan amount would be

$\$91,018.31 + 1,500.00 = \$92,518.31$. If *2 points* must be paid in order to lock in the rate of 6% for the refinanced 15-year loan, then this amount is $(.02)92518.31 = \$1850.37$. The new principal balance of the refinanced loan is $92518.31 + 1850.37 = \$94,368.68$, and this is the present value at a nominal rate of 6% of the future loan payments, no matter what the term of the refinanced loan is. The new monthly payment (for a 15-year duration) of the refinanced loan is $\$94,368.68(.06/12)/(1-(1+.06/12)^{-180}) = \$796.34$.

For purposes of comparison, what is the present value at the current going rate of 6% (nominal) of the continuing stream of payments under the old loan ? That is a 22-year stream of monthly payments of \$733.76, as calculated above, so the present value at 6% is $\$733.76 \cdot (12 a_{\overline{22}|}^{(12)}) = \$107,420.21$. Thus, if the new rate of 6% were really to be the correct one for the next 22 years, and each loan would be paid to the end of its term, then it would be a financial disaster *not* to refinance. Next, suppose instead that right after re-financing, the economic rate of interest would be a nominal 7% for the next 22 years. In that case *both* streams of payments would have to be re-valued — the one before refinancing, continuing another 22 years into the future, and the one after refinancing, continuing 15 years into the future. The respective present values (as of the end of the 8^{th} year) at nominal rate of 7% of these two streams are:

Old loan: $733.76\,(12 a_{\overline{22}|}^{(12)}) = \$98,700.06$

New loan: $796.34\,(12 a_{\overline{15}|}^{(12)}) = \$88,597.57$

Even with these different assumptions, and despite closing-costs and points, it is well worth re-financing.

Exercise: Suppose that you can forecast that you will in fact sell your house in precisely 5 more years after the time when you are re-financing. At the time of sale, you would pay off the cash principal balance, whatever it is. Calculate and compare the present values (at each of 6%, 7%, and 8% nominal interest rates) of your payment streams to the bank, (a) if you continue the old loan without refinancing, and (b) if you re-finance to get a 15-year 6% loan including closing costs and points, as described above.

2.1.4 Computational illustration in Splus

All of the calculations described above are very easy to program, in any language from Pascal to Mathematica, and also on a programmable calculator; but they are also very handily organized within a spreadsheet, which seems to be the way that MBA's, bank-officials, and actuaries will learn to do them from now on.

In this section, an **Splus** function (*cf.* Venables & Ripley 1998) is provided to do some comparative refinancing calculations. Concerning the syntax of **Splus**, the only explanation necessary at this point is that the symbol $<\!-$ denotes assignment of an expression to a variable: $A <\!- B$ means that the variable A is assigned the value of expression B. Other syntactic elements used here are common to many other computer languages: * denotes multiplication, and ^ denotes exponentiation.

The function *RefExmp* given below calculates mortgage payments, balances for purposes of refinancing both before and after application of administrative costs and points, and the present value under any interest rate (not necessarily the ones at which either the original or refinanced loans are taken out) of the stream of repayments to the bank up to and including the lump-sum payoff which would be made, for example, at the time of selling the house on which the mortgage loan was negotiated. The output of the function is a list which, in each numerical example below, is displayed in 'unlisted' form, horizontally as a vector. Lines beginning with the symbol # are comment-lines.

The outputs of the function are as follows. *Oldpayment* is the monthly payment on the original loan of face-amount *Loan* at nominal interest $i^{(12)} =$ *OldInt* for a term of *OldTerm* years. *NewBal* is the balance $B_{n,\,k/m}$ of formula (2.6) for $n =$ *OldTerm*, $m = 12$, and $k/m =$ *RefTim*, and the refinanced loan amount is a multiple $1+$ *Points* of *NewBal*, which is equal to *RefBal* + *Costs*. The new loan, at nominal interest rate *NewInt*, has monthly payments *Newpaymt* for a term of *NewTerm* years. The loan is to be paid off *PayoffTim* years after *RefTim* when the new loan commences, and the final output of the function is the present value at the start of the refinanced loan with nominal interest rate *ValInt* of the stream of payments made under the refinanced loan up to and including the lump sum payoff.

```
 Splus FUNCTION CALCULATING REFINANCE PAYMENTS & VALUES

RefExmp
function(Loan, OldTerm, RefTim, NewTerm, Costs, Points,
       PayoffTim, OldInt, NewInt, ValInt)
{
# Function calculates present value of future payment stream
#      underrefinanced loan.
# Loan = original loan amount;
# OldTerm = term of initial loan in years;
# RefTim = time in years after which to refinance;
# NewTerm = term of refinanced loan;
# Costs = fixed closing costs for refinancing;
# Points = fraction of new balance as additional costs;
# PayoffTim (no bigger than NewTerm) = time (from refinancing-
#      time at which new loan balance is to be paid off in
#      cash (eg at house sale);
# The three interest rates OldInt, NewInt, ValInt are
#      nominal 12-times-per-year, and monthly payments
#      are calculated.
   vold <- (1 + OldInt/12)^(-12)
   Oldpaymt <- ((Loan * OldInt)/12)/(1 - vold^OldTerm)
   NewBal <- (Loan * (1 - vold^(OldTerm - RefTim)))/
       (1 - vold^OldTerm)
   RefBal <- (NewBal + Costs) * (1 + Points)
   vnew <- (1 + NewInt/12)^(-12)
   Newpaymt <- ((RefBal * NewInt)/12)/(1 - vnew^NewTerm)
   vval <- (1 + ValInt/12)^(-12)
   Value <- (Newpaymt * 12 * (1 - vval^PayoffTim))/ValInt +
        (RefBal *  vval^PayoffTim * (1 - vnew^(NewTerm -
         PayoffTim)))/(1 -  vnew^NewTerm)
   list(Oldpaymt = Oldpaymt, NewBal = NewBal,
       RefBal = RefBal, Newpaymt = Newpaymt, Value = Value)
}
```

We begin our illustration by reproducing the quantities calculated in the previous subsection:

```
> unlist(RefExmp(100000, 30, 8, 15, 1500, 0.02, 15,
         0.08, 0.06, 0.06))
 Oldpaymt NewBal RefBal Newpaymt Value
   733.76  91018  94368   796.33 94368
```

Note that, since the payments under the new (refinanced) loan are here valued at the same interest rate as the loan itself, the present value Value of all payments made under the loan must be equal to the the refinanced loan amount *RefBal*.

The comparisons of the previous Section between the original and refinanced loans, at (nominal) interest rates of 6, 7, and 8 %, are all recapitulated easily using this function. To use it, for example, in valuing the old loan at 7%, the arguments must reflect a 'refinance' with no costs or points for a period of 22 years at nominal rate 6%, as follows:

```
> unlist(RefExmp(100000,30,8,22,0,0,22,0.08,0.08,0.07))
 Oldpaymt  NewBal RefBal Newpaymt  Value
   733.76  91018  91018   733.76   98701
```

(The small discrepancies between the values found here and in the previous subsection are due to the rounding used there to express payment amounts to the nearest cent.)

We consider next a numerical example showing break-even point for refinancing by balancing costs versus time needed to amortize them.

Suppose that you have a 30-year mortage for \$100,000 at nominal 9% ($= i^{(12)}$*), with level monthly payments, and that after 7 years of payments you refinance to obtain a new 30-year mortgage at 7% nominal interest (* $= i^{(m)}$ *for* $m = 12$*), with closing costs of \$1500 and 4 points (i.e., 4% of the total refinanced amount including closing costs added to the initial balance), also with level monthly payments. Figuring present values using the new interest rate of 7%, what is the time* K *(to the nearest month) such that if both loans — the old and the new — were to be paid off in exactly* K *years after the time (the 7-year mark for the first loan) when you would have refinanced,* *then the remaining payment-streams for both loans from the time when you refinance are equivalent (i.e., have the same present value from that time) ?*

We first calculate the present value of payments under the new loan. As remarked above in the previous example, since the same interest rate is being used to value the payments as is used in figuring the refinanced loan, the valuation of the new loan *does not depend upon the time* K *to payoff.* (It is figured here as though the payoff time K were 10 years.)

```
> unlist(RefExmp(1.e5, 30,7,30, 1500,.04, 10, 0.09,0.07,0.07))
 Oldpaymt NewBal RefBal Newpaymt Value
   804.62  93640  98946   658.29 98946
```

Next we compute the value of payments under the old loan, at 7% nominal rate, also at payoff time $K = 10$. For comparison, the value under the old loan for payoff time 0 (i.e., for cash payoff at the time when refinancing would have occurred) coincides with the New Balance amount of \$93640.

```
> unlist(RefExmp(1.e5, 30,7,23, 0,0, 10, 0.09,0.09,0.07))
 Oldpaymt NewBal RefBal Newpaymt  Value
   804.62  93640  93640   804.62 106042
```

The values found in the same way when the payoff time K is successively replaced by 4, 3, 3.167, 3.25 are 99979, 98946, 98593, 98951. Thus, the payoff-time K at which there is essentially no difference in present value at nominal 7% between the old loan or the refinanced loan with costs and points (which was found to have Value 98946), is 3 years and 3 months after refinancing.

2.1.5 Coupon & Zero-coupon Bonds

In finance, an investor assessing the present value of a *bond* is in the same situation as the bank receiving periodic level payments in repayment of a loan. If the payments are made every $1/m$ year, with nominal *coupon interest rate* $i^{(m)}$, for a bond with face value \$1000, then the payments are precisely the interest on \$1000 for $1/m$ year, or $1000 \cdot i^{(m)}/m$. For most corporate or government bonds, $m = 4$, although some bonds have $m = 2$. If the bond is *uncallable*, which is assumed throughout this discussion, then it entitles the holder to receive the stream of such payments every $1/m$ year until a fixed final *redemption date*, at which the final interest payment coincides with the repayment of the principal of \$1000. Suppose that the time remaining on a bond until redemption is R (assumed to be a whole-number multiple of $1/m$ years), and that the nominal annualized m-period-per-year interest rate, taking into account the credit-worthiness of the bond issuer together with current economic conditions, is $r^{(m)}$ which will typically not be equal to $i^{(m)}$. Then the current price P of the bond is

$$P \; = \; 1000\, i^{(m)}\, a^{(m)}_{\overline{R|}\,,\, r^{(m)}} \; + \; 1000\, (1 + \frac{r^{(m)}}{m})^{-Rm}$$

In this equation, the value P represents cash on hand. The first term on the right-hand side is the present value at nominal interest rate $r^{(m)}$ of the payments of $i^{(m)}\, 1000/m$ every $1/m$ year, which amount to $1000 i^{(m)}$ every year for R years. The final repayment of principal in R years contributes the second term on the right-hand side to the present value. As an application of this formula, it is easy to check that a 10-year \$1000 bond with nominal annualized quarterly interest rate $i^{(4)} = 0.06$ would be priced at \$863.22 if the going nominal rate of interest were $r^{(4)} = 0.08$.

A slightly different valuation problem is presented by the *zero-coupon bond*, a financial instrument which pays all principal and interest, at a declared interest rate $i = i_{\text{APR}}$, at the end of a term of n years, but pays nothing before that time. When first issued, a zero-coupon bond yielding i_{APR} which will pay \$1000 at the end of n years is priced at its present value

$$\Pi_n \; = \; 1000 \cdot (1 + i)^{-n} \tag{2.8}$$

(Transaction costs are generally figured into the price before calculating the yield i.) At a time, $n - R$ years later, when the zero-coupon bond has R years left to run and the appropriate interest rate for valuation has changed to $r = r_{\text{APR}}$, the correct price of the bond is the present value of a payment of 1000 R years in the future, or $1000(1 + r)^{-R}$.

For tax purposes, at least in the US, an investor is required (either by the federal or state government, depending on the issuer of the bond) to declare the amount of interest income received or *deemed to have been received* from the bond in a specific calendar year. For an ordinary or coupon bond, the year's income is just the total $i^{(m)} \cdot 1000$ actually received during the year. For a zero-coupon bond assumed to have been acquired when first issued, at price Π_n, if the interest rate has remained constant at $i = i_{\text{APR}}$ since the time of acquisition, then the interest income deemed to be received in the year, also called the *Original Issue Discount* (OID), would simply be the year's interest $\Pi_n \, i^{(m)}$ on the initial effective face amount Π_n of the bond. That is because all accumulated value of the bond would be attributed to OID interest income received year by year, with the principal remaining the same at Π_n. Assume next that the actual year-by-year APR interest rate is $r(j)$ throughout the year $[j, j+1)$, for $j = 0, 1, \ldots n-1$, with $r(0)$ equal to i. Then, again because accumulated value over the initial price is deemed to have been received in the form of yearly interest, the OID should be $\Pi_n \, r(j)$ in the year $[j, j+1)$. The problematic aspect of this calculation is that, when interest rates have fluctuated a lot over the times $j = 0, 1, \ldots, n-R-1$, the zero-coupon bond investor will be deemed to have received income $\Pi_n \, r(j)$ in successive years $j = 0, 1, \ldots, n-R-1$, corresponding to a total accumulated value of

$$\Pi_n \, (1 + r(0)) \, (1 + r(1)) \cdots (1 + r(n-R-1))$$

while the price $1000 \, (1 + r(n-R))^{-R}$ for which the bond could be sold may be very different. The discrepancy between the 'deemed received' accumulated value and the final actual value when the bond is redeemed or sold must presumably be treated as a capital gain or loss. However, the present author makes no claim to have presented this topic according to the views of the Internal Revenue Service, since he has never been able to figure out authoritatively what those views are.

2.2 Force of Mortality & Analytical Models

Up to now, the function $S(x)$ called the "survivor" or "survival" function has been defined to be equal to the life-table ratio l_x/l_0 at all integer ages x, and to be piecewise continuously differentiable for all positive real values of x. Intuitively, for all positive real x and t, $S(x) - S(x+t)$ is the fraction of the initial life-table cohort which dies between ages x and $x+t$, and $(S(x) - S(x+t))/S(x)$ represents the fraction of those alive at age x who fail before $x+t$. An equivalent representation is $S(x) = \int_x^\infty f(t)\, dt$, where $f(t) \equiv -S'(t)$ is called the *failure density*. If T denotes the random variable which is the age at death for a newly born individual governed by the same causes of failure as the life-table cohort, then $P(T \geq x) = S(x)$, and according to the Fundamental Theorem of Calculus,

$$\lim_{\epsilon \to 0+} \frac{P(x \leq T \leq x + \epsilon)}{\epsilon} = \lim_{\epsilon \to 0+} \int_x^{x+\epsilon} f(u)\, du = f(x)$$

as long as the failure density is a continuous function.

Two further useful actuarial notations, often used to specify the theoretical lifetime distribution, are:

$$_tp_x = P(T \geq x + t \,|\, T \geq x\,) = S(x+t)/S(x)$$

and

$$_tq_x = 1 - {}_tp_x = P(T \leq x + t \,|\, T \geq x\,) = (S(x) - S(x+t))/S(x)$$

The quantity $_tq_x$ is referred to as the *age-specific death rate* for periods of length t. In the most usual case where $t = 1$ and x is an integer age, the notation $_1q_x$ is replaced by q_x, and $_1p_x$ is replaced by p_x. The rate q_x would be estimated from the cohort life table as the ratio d_x/l_x of those who die between ages x and $x+1$ as a fraction of those who reached age x. The way in which this quantity varies with x is one of the most important topics of study in actuarial science. For example, one important way in which numerical analysis enters actuarial science is that one wishes to interpolate the values q_x smoothly as a function of x. The topic called "Graduation Theory" among actuaries is the mathematical methodology of Interpolation and Spline-smoothing applied to the raw function $q_x = d_x/l_x$.

To give some idea what a realistic set of death-rates looks like, Figure 2.1 pictures the age-specific 1-year death-rates q_x for the simulated life-table given as Table 1.1 on page 3. Additional granularity in the death-rates can be seen in Figure 2.2, where the logarithms of death-rates are plotted. After a very high death-rate during the first year of life (26.3 deaths per thousand live births), there is a rough year-by-year decline in death-rates from 1.45 per thousand in the second year to 0.34 per thousand in the eleventh year. (But there were small increases in rate from ages 4 to 7 and from 8 to 9, which are likely due to statistical irregularity rather than real increases in risk.) Between ages 11 and 40, there is an erratic but roughly linear increase of death-rates per thousand from 0.4 to 3.0. However, at ages beyond 40 there is a rapid increase in death-rates as a function of age. As can be seen from Figure 2.2, the values q_x seem to increase roughly as a power c^x where $c \in [1.08, 1.10]$. (Compare this behavior with the Gompertz-Makeham Example (v) below.) This exponential behavior of the age-specific death-rate for large ages suggests that the death-rates pictured could reasonably be extrapolated to older ages using the formula

$$q_x \;\approx\; q_{78} \cdot (1.0885)^{x-78} \quad , \qquad x \geq 79 \tag{2.9}$$

where the number 1.0885 was found as $log(q_{78}/q_{39})/(78-39)$.

Now consider the behavior of ${}_\epsilon q_x$ as ϵ gets small. It is clear that ${}_\epsilon q_x$ must also get small, roughly proportionately to ϵ, since the probability of dying between ages x and $x+\epsilon$ is approximately $\epsilon\, f(x)$ when ϵ gets small.

<u>Definition</u>: The limiting death-rate ${}_\epsilon q_x/\epsilon$ per unit time as $\epsilon \searrow 0$ is called by actuaries the **force of mortality** $\mu(x)$. In reliability theory or biostatistics, the same function is called the *failure intensity, failure rate*, or *hazard intensity.*

The reasoning above shows that for small ϵ,

$$\frac{{}_\epsilon q_x}{\epsilon} \;=\; \frac{1}{\epsilon S(x)} \int_x^{x+\epsilon} f(u)\, du \;\longrightarrow\; \frac{f(x)}{S(x)} \;, \qquad \epsilon \searrow 0$$

Thus

$$\mu(x) \;=\; \frac{f(x)}{S(x)} \;=\; \frac{-S'(x)}{S(x)} \;=\; -\frac{d}{dx} \ln(S(x))$$

where the chain rule for differentiation was used in the last step. Replacing x by y and integrating both sides of the last equation between 0 and x, we find

$$\int_0^x \mu(y)\, dy \;=\; \Big(-\ln(S(y))\Big)_0^x \;=\; -\ln(S(x))$$

since $S(0) = 1$. Similarly,

$$\int_x^{x+t} \mu(y)\, dy \;=\; \ln S(x) - \ln S(x+t)$$

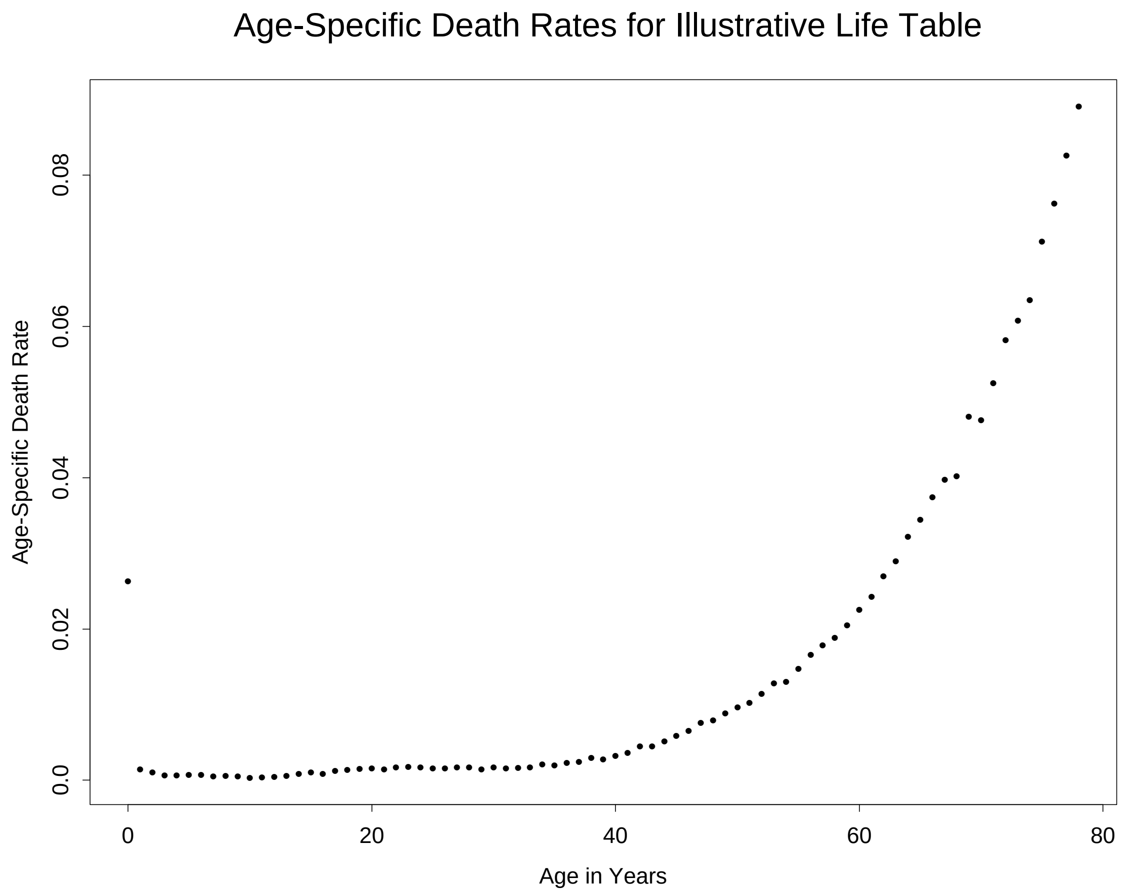

Figure 2.1: Plot of age-specific death-rates q_x versus x, for the simulated illustrative life table given in Table 1.1, page 3.

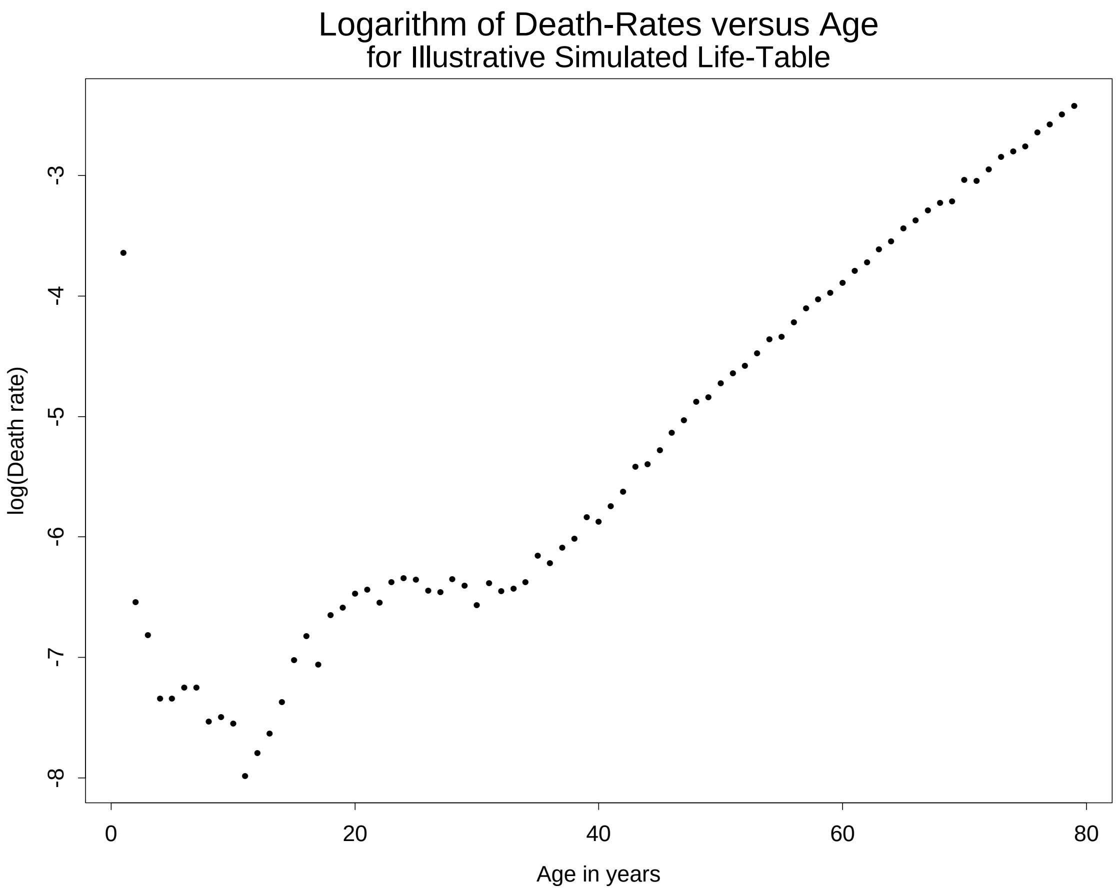

Figure 2.2: Plot of logarithm $\log(q_x)$ of age-specific death-rates as a function of age x, for the simulated illustrative life table given in Table 1.1, page 3. The rates whose logarithms are plotted here are the same ones shown in Figure 2.1.

Now exponentiate to obtain the useful formulas

$$S(x) = \exp\left\{-\int_0^x \mu(y)\,dy\right\} \ , \qquad {}_tp_x = \frac{S(x+t)}{S(x)} = \exp\left\{-\int_x^{x+t} \mu(y)\,dy\right\}$$

Examples:

(i) If $S(x) = (\omega - x)/\omega$ for $0 \le x \le \omega$ (the *uniform failure distribution* on $[0, \omega]$), then $\mu(x) = (\omega - x)^{-1}$. Note that this hazard function increases to ∞ as x increases to ω.

(ii) If $S(x) = e^{-\mu x}$ for $x \ge 0$ (the *exponential failure distribution* on $[0, \infty)$), then $\mu(x) = \mu$ is constant.

(iii) If $S(x) = \exp(-\lambda x^\gamma)$ for $x \ge 0$, then mortality follows the *Weibull life distribution model* with *shape parameter* $\gamma > 0$ and *scale parameter* λ. The force of mortality takes the form

$$\mu(x) \ = \ \lambda\,\gamma\,x^{\gamma-1}$$

This model is very popular in engineering reliability. It has the flexibility that by choice of the shape parameter γ one can have

(a) failure rate increasing as a function of x ($\gamma > 1$),

(b) constant failure rate ($\gamma = 1$, the exponential model again),
or

(c) decreasing failure rate ($0 < \gamma < 1$).

But what one cannot have, in the examples considered so far, is a force-of-mortality function which decreases on part of the time-axis and increases elsewhere.

(iv) Two other models for positive random variables which are popular in various statistical applications are the **Gamma**, with

$$S(x) \ = \ \int_x^\infty \beta^\alpha\,y^{\alpha-1}\,e^{-\beta y}\,dy \Big/ \int_0^\infty z^{\alpha-1}\,e^{-z}\,dz \ , \qquad \alpha,\,\beta > 0$$

and the **Lognormal**, with

$$S(x) \ = \ 1 - \Phi\Big(\frac{\ln x - m}{\sigma}\Big) \quad , \qquad m \ \text{real}, \ \ \sigma > 0$$

where

$$\Phi(z) \equiv \frac{1}{2} + \int_0^z e^{-u^2/2} \frac{du}{\sqrt{2\pi}}$$

is called the *standard normal distribution function.* In the Gamma case, the expected lifetime is α/β, while in the Lognormal, the expectation is $\exp(\mu + \sigma^2/2)$. Neither of these last two examples has a convenient or interpretable force-of-mortality function.

Increasing force of mortality intuitively corresponds to aging, where the causes of death operate with greater intensity or effect at greater ages. Constant force of mortality, which is easily seen from the formula $S(x) = \exp(-\int_0^x \mu(y)\, dy)$ to be equivalent to exponential failure distribution, would occur if mortality arose only from pure accidents unrelated to age. Decreasing force of mortality, which really does occur in certain situations, reflects what engineers call "burn-in", where after a period of initial failures due to loose connections and factory defects the nondefective devices emerge and exhibit high reliability for a while. The decreasing force of mortality reflects the fact that the devices known to have functioned properly for a short while are known to be correctly assembled and are therefore highly likely to have a standard length of operating lifetime. In human life tables, infant mortality corresponds to burn-in: risks of death for babies decrease markedly after the one-year period within which the most severe congenital defects and diseases of infancy manifest themselves. Of course, human life tables also exhibit an aging effect at high ages, since the high-mortality diseases like heart disease and cancer strike with greatest effect at higher ages. Between infancy and late middle age, at least in western countries, hazard rates are relatively flat. This pattern of initial decrease, flat middle, and final increase of the force-of-mortality, seen clearly in Figure 2.1, is called a *bathtub shape* and requires new survival models.

As shown above, the failure models in common statistical and reliability usage *either* have increasing force of mortality functions *or* decreasing force of mortality, but not both. Actuaries have developed an analytical model which is somewhat more realistic than the preceding examples for human mortalty at ages beyond childhood. While the standard form of this model does not accommodate a bathtub shape for death-rates, a simple modification of it does.

Example (v). (*Gompertz-Makeham* forms of the force of mortality). Suppose that $\mu(x)$ is defined directly to have the form $A + B\,c^x$. (The Bc^x term was proposed by Gompertz, the additive constant A by Makeham. Thus the *Gompertz* force-of-mortality model is the special case with $A = 0$, or $\mu(x) = Bc^x$.) By choice of the parameter c as being respectively greater than or less than 1, one can arrange that the force-of-mortality curve either be increasing or decreasing. Roughly realistic values of c for human mortality will be only slightly greater than 1: if the Gompertz (non-constant) term in force-of-mortality were for example to quintuple in 20 years, then $c \approx 5^{1/20} = 1.084$, which may be a reasonable value except for very advanced ages. (Compare the comments made in connection with Figures 2.1 and 2.2: for middle and higher ages in the simulated illustrative life table of Table 1.1, which corresponds roughly to US male mortality of around 1960, the figure of c was found to be roughly 1.09.) Note that in any case the Gompertz-Makeham force of mortality is strictly convex (i.e., has strictly positive second derivative) when $B > 0$ and $c \neq 1$. The Gompertz-Makeham family could be enriched still further, with further benefits of realism, by adding a linear term Dx. If $D < -B\,\ln(c)$, with $0 < A < B$, $c > 1$, then it is easy to check that

$$\mu(x) \;=\; A + B\,c^x + Dx$$

has a bathtub shape, initially decreasing and later increasing.

Figures 2.3 and 2.4 display the shapes of force-of-mortality functions (iii)-(v) for various parameter combinations chosen in such a way that the expected lifetime is 75 years. This restriction has the effect of reducing the number of free parameters in each family of examples by 1. One can see from these pictures that the Gamma and Weibull families contain many very similar shapes for force-of-mortality curves, but that the lognormal and Makeham families are quite different.

Figure 2.5 shows survival curves from several analytical models plotted on the same axes as the 1959 US male life-table data from which Table 1.1 was simulated. The previous discussion about bathtub-shaped force of mortality functions should have made it clear that none of the analytical models presented could give a good fit at all ages, but the Figure indicates the rather good fit which can be achieved to realistic life-table data at ages 40 and above. The models fitted all assumed that $S(40) = 0.925$ and that for lives aged 40, $T - 40$ followed the indicated analytical form. Parameters for all models were determined from the requirements of median age 72 at death (equal by definition to the value t_m for which $S(t_m) = 0.5$) and probability 0.04 of surviving to age 90. Thus, all four plotted survival curves have been designed to pass through the three points $(40, 0.925)$, $(72,\ 0.5)$, $(90,\ 0.04)$. Of the four fitted curves, clearly the Gompertz agrees most closely with the plotted points for 1959 US male mortality. The Gompertz curve has parameters $B = 0.00346$, $c = 1.0918$, the latter of which is close to the value 1.0885 used in formula (2.9) to extrapolate the 1959 life-table death-rates to older ages.

2.2.1 Comparison of Forces of Mortality

What does it mean to say that one lifetime, with associated survival function $S_1(t)$, has hazard (i.e. force of mortality) $\mu_1(t)$ which is a constant multiple κ at all ages of the force of mortality $\mu_2(t)$ for a second lifetime with survival function $S_2(t)$? It means that the cumulative hazard functions are *proportional*, i.e.,

$$-\ln S_1(t) \;=\; \int_0^t \mu_1(x)dx \;=\; \int_0^t \kappa\, \mu_2(x)dx \;=\; \kappa(-\ln S_2(t))$$

and therefore that

$$S_1(t) \;=\; (S_2(t))^\kappa \quad , \qquad \text{all} \quad t \geq 0$$

This remark is of especial interest in biostatistics and epidemiology when the factor κ is allowed to depend (e.g., by a regression model $\ln(\kappa) = \beta \cdot Z$) on other measured variables (*covariates*) Z. This model is called the *(Cox) Proportional-Hazards model* and is treated at length in books on survival data analysis (Cox and Oakes 1984, Kalbfleisch and Prentice 1980) or biostatistics (Lee 1980).

Example. Consider a setting in which there are four subpopulations of the general population, categorized by the four combinations of values of two binary covariates $Z_1,\ Z_2 = 0,\ 1$. Suppose that these four combinations have

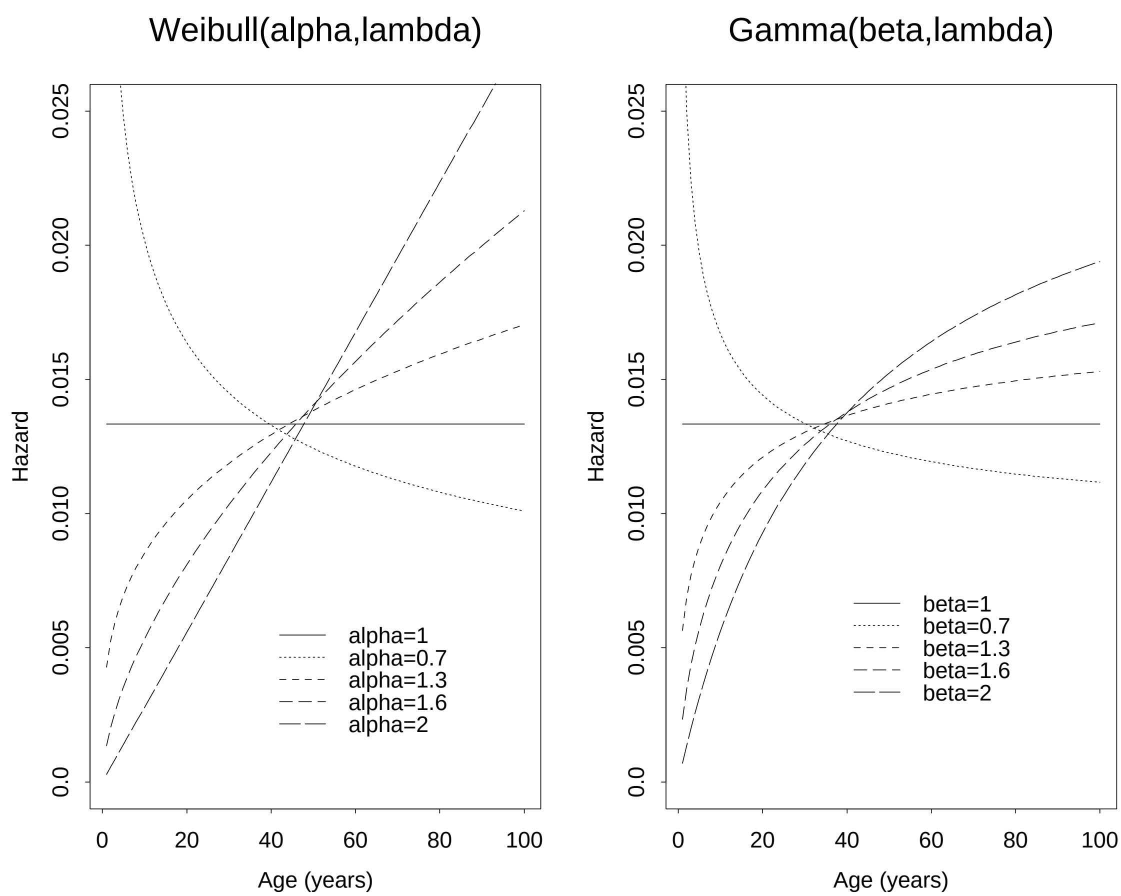

Figure 2.3: Force of Mortality Functions for Weibull and Gamma Probability Densities. In each case, the parameters are fixed in such a way that the expected survival time is 75 years.

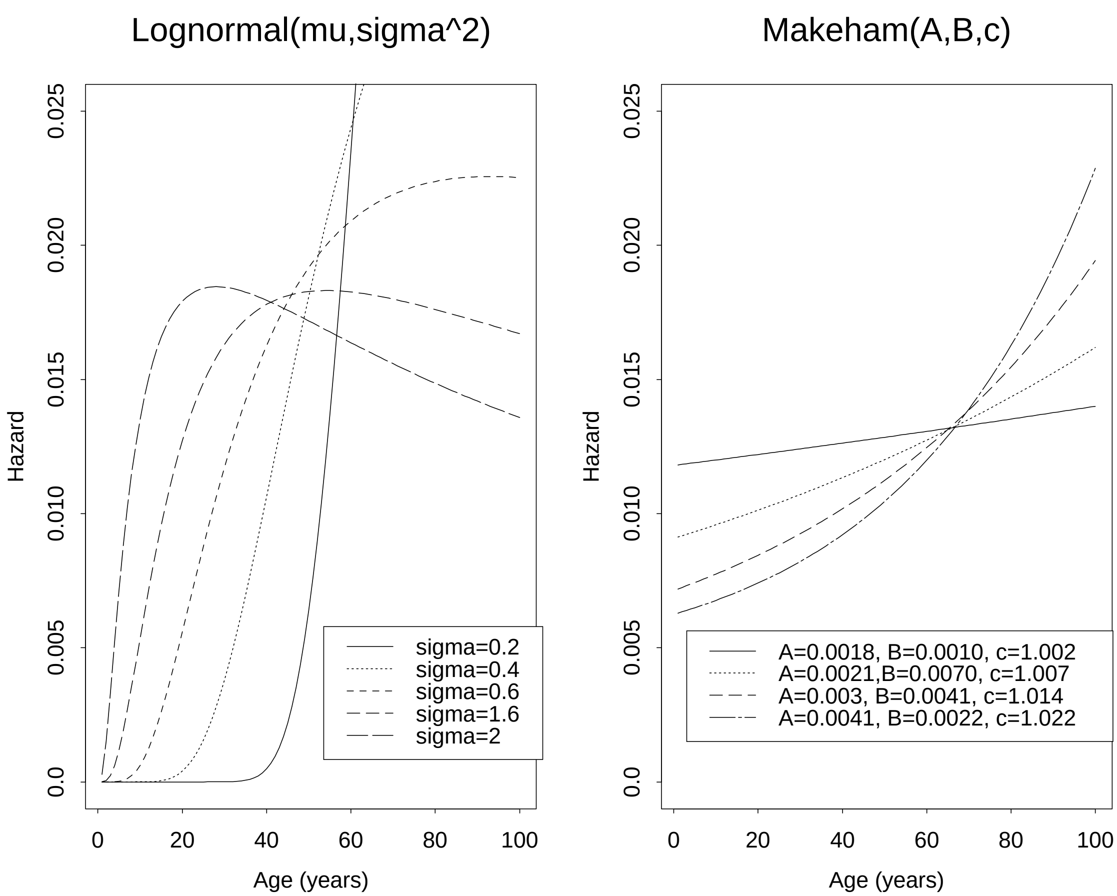

Figure 2.4: Force of Mortality Functions for Lognormal and Makeham Densities. In each case, the parameters are fixed in such a way that the expected survival time is 75 years.

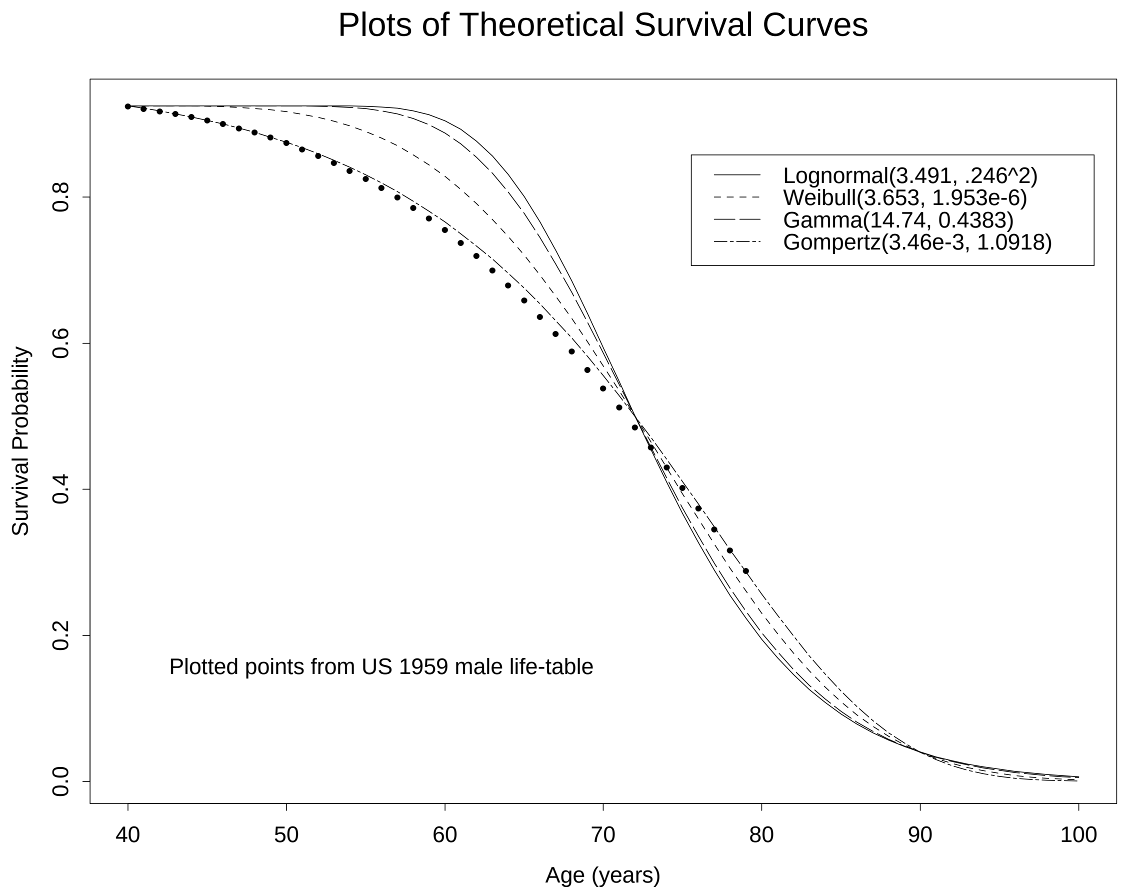

Figure 2.5: Theoretical survival curves, for ages 40 and above, plotted as lines for comparison with 1959 US male life-table survival probabilities plotted as points. The four analytical survival curves — Lognormal, Weibull, Gamma, and Gompertz — are taken as models for age-at-death minus 40, so if $S_{\rm theor}(t)$ denotes the theoretical survival curve with indicated parameters, the plotted curve is $(t,\ 0.925 \cdot S_{\rm theor}(t-40))$. The parameters of each analytical model were determined so that the plotted probabilities would be $0.925,\ 0.5,\ 0.04$ respectively at $t = 40,\ 72,\ 90$.

respective conditional probabilities for lives aged x (or relative frequencies in the general population aged x)

$$P_x(Z_1 = Z_2 = 0) = 0.15 \quad , \quad P_x(Z_1 = 0,\, Z_2 = 1) = 0.2$$
$$P_x(Z_1 = 1,\, Z_2 = 0) = 0.3 \quad , \quad P_x(Z_1 = Z_2 = 1) = 0.35$$

and that for a life aged x and all $t > 0$,

$$P(T \geq x + t \,|\, T \geq x,\ Z_1 = z_1,\ Z_2 = z_2) \;=\; \exp(-2.5\, e^{0.7 z_1 - .8 z_2}\, t^2/20000)$$

It can be seen from the conditional survival function just displayed that the forces of mortality at ages greater than x are

$$\mu(x+t) \;=\; (2.5\, e^{0.7 z_1 - .8 z_2})\, t/10000$$

so that the force of mortality at all ages is multiplied by $e^{0.7} = 2.0138$ for individuals with $Z_1 = 1$ versus those with $Z_1 = 0$, and is multiplied by $e^{-0.8} = 0.4493$ for those with $Z_2 = 1$ versus those with $Z_2 = 0$. The effect on age-specific death-rates is approximately the same. Direct calculation shows for example that the ratio of age-specific death rate at age x+20 for individuals in the group with $(Z_1 = 1, Z_2 = 0)$ versus those in the group with $(Z_1 = 0, Z_2 = 0)$ is not precisely $e^{0.7} = 2.014$, but rather

$$\frac{1 - \exp(-2.5 e^{0.7}((21^2 - 20^2)/20000)}{1 - \exp(-2.5((21^2 - 20^2)/20000)} \;=\; 2.0085$$

Various calculations, related to the fractions of the surviving population at various ages in each of the four population subgroups, can be performed easily . For example, to find

$$P(Z_1 = 0,\, Z_2 = 1 \,|\, T \geq x + 30)$$

we proceed in several steps (which correspond to an application of Bayes' rule, *viz.* Hogg and Tanis 1997, sec. 2.5, or Larson 1982, Sec. 2.6):

$$P(T \geq x{+}30,\, Z_1 = 0\, Z_2 = 1 |\, T \geq x) \;=\; 0.2\, \exp(-2.5 e^{-0.8} \frac{30^2}{20000}) \;=\; 0.1901$$

and similarly

$$P(T \geq x + 30 \,|\, T \geq x) \;=\; 0.15\, \exp(-2.5(30^2/20000)) \;+\; 0.1901 \;+$$

$$+ \, 0.3 \, \exp(-2.5 * e^{0.7} \frac{30^2}{20000}) \; + \; 0.35 \, \exp(-2.5 e^{0.7-0.8} \frac{30^2}{20000}) \; = \; 0.8795$$

Thus, by definition of conditional probabilities (restricted to the cohort of lives aged x), taking ratios of the last two displayed quantities yields

$$P(Z_1 = 0, \, Z_2 = 1 \,|\, T \ge x + 30) \; = \; \frac{0.1901}{0.8795} \; = \; 0.2162$$

$\square$.

In biostatistics and epidemiology, the measured variables $\underline{Z} = (Z_1, \ldots, Z_p)$ recorded for each individual in a survival study might be: indicator of a specific disease or diagnostic condition (e.g., diabetes, high blood pressure, specific electrocardiogram anomaly), quantitative measurement of a risk-factor (dietary cholesterol, percent caloric intake from fat, relative weight-to-height index, or exposure to a toxic chemical), or indicator of type of treatment or intervention. In these fields, the objective of such detailed models of covariate effects on survival can be: to correct for incidental individual differences in assessing the effectiveness of a treatment; to create a prognostic index for use in diagnosis and choice of treatment; or to ascertain the possible risks and benefits for health and survival from various sorts of life-style interventions. The multiplicative effects of various risk-factors on age-specific death rates are often highlighted in the news media.

In an insurance setting, categorical variables for risky life-styles, occupations, or exposures might be used in *risk-rating*, i.e., in individualizing insurance premiums. While risk-rating is used routinely in casualty and property insurance underwriting, for example by increasing premiums in response to recent claims or by taking location into account, it can be politically sensitive in a life-insurance and pension context. In particular, while gender differences in mortality can be used in calculating insurance and annuity premiums, as can certain life-style factors like smoking, it is currently illegal to use racial and genetic differences in this way.

All life insurers must be conscious of the extent to which their policyholders as a group differ from the general population with respect to mortality. Insurers can collect special mortality tables on special groups, such as employee groups or voluntary organizations, and regression-type models like the Cox proportional-hazards model may be useful in quantifying group mortality differences when the special-group mortality tables are not based upon large enough cohorts for long enough times to be fully reliable. See Chapter 6, Section 6.3, for discussion about the modification of insurance premiums for select groups.

2.3 *Exercise Set 2*

(1). The sum of the present value of \$1 paid at the end of n years and \$1 paid at the end of $2n$ years is \$1. Find $(1+r)^{2n}$, where $r =$ annual interest rate, compounded annually.

(2). Suppose that an individual aged 20 has random lifetime Z with continuous density function

$$f_Z(t) \;=\; \frac{1}{360}\left(1+\frac{t}{10}\right) \;, \qquad \text{for} \qquad 20 \le t \le 80$$

and 0 for other values of t.

(a) If this individual has a contract with your company that you must pay his heirs $\$10^6 \cdot (1.4 - Z/50)$ at the exact date of his death if this occurs between ages 20 and 70, then what is the expected payment ?

(b) If the value of the death-payment described in (a) should properly be discounted by the factor $\exp(-0.08 \cdot (Z-20))$, i.e. by the nominal interest rate of $e^{0.08} - 1$ per year) to calculate the present value of the payment, then what is the expected present value of the payment under the insurance contract ?

(3). Suppose that a continuous random variable T has hazard rate function (= force of mortality)

$$h(t) \;=\; 10^{-3} \cdot \left[7.0 - 0.5t + 2e^{t/20}\right] \;, \qquad t > 0$$

This **is** a legitimate hazard rate of Gompertz-Makeham type since its minimum, which occurs at $t = 20\ln(5)$, is $(17 - 10\ln(5)) \cdot 10^{-4} = 9.1 \cdot 10^{-5} > 0$.

(a) Construct a cohort life-table with $h(t)$ as "force of mortality", based on integer ages up to 70 and cohort-size (= "radix") $l_0 = 10^5$. (Give the numerical entries, preferably by means of a little computer program. If you do the arithmetic using hand-calculators and/or tables, stop at age 20.)

(b) What is the probability that the random variable T exceeds 30, given that it exceeds 3 ? **Hint:** find a closed-form formula for $S(t) = P(T \geq t)$.

(4). Do the Mortgage-Refinancing exercise given in the Illustrative on mortgage refinancing at the end of Section 2.1.

(5). (a) The mortality pattern of a certain population may be described as follows: out of every 98 lives born together, one dies annually until there are no survivors. Find a simple function that can be used as $S(x)$ for this population, and find the probability that a life aged 30 will survive to attain age 35.

(b) Suppose that for x between ages 12 and 40 in a certain population, 10% of the lives aged x die before reaching age $x+1$. Find a simple function that can be used as $S(x)$ for this population, and find the probability that a life aged 30 will survive to attain age 35.

(6). Suppose that a survival distribution (i.e., survival function based on a cohort life table) has the property that ${}_1p_x = \gamma \cdot (\gamma^2)^x$ for some fixed γ between 0 and 1, for every real ≥ 0. What does this imply about S(x) ? (*Give as much information about S as you can.*)

(7). If the instantaneous interest rate is $r(t) = 0.01\,t$ for $0 \leq t \leq 3$, then find the equivalent single *effective rate of interest* or APR for money invested at interest over the interval $0 \leq t \leq 3$.

(8). Find the accumulated value of \$100 at the end of 15 years if the nominal interest rate compounded quarterly (i.e., $i^{(4)}$) is 8% for the first 5 years, if the effective rate of discount is 7% for the second 5 years, and if the nominal rate of discount compounded semiannually ($m = 2$) is 6% for the third 5 years.

(9). Suppose that you borrow \$1000 for 3 years at 6% APR, to be repaid in level payments every six months (twice yearly).

(a) Find the level payment amount P.

(b) What is the present value of the payments you will make if you skip the 2nd and 4th payments ? (You may express your answer in terms of P.)

(10). A survival function has the form $S(x) = \frac{c-x}{c+x}$. If a mortality table is derived from this survival function with a radix l_0 of 100,000 at age 0, and if $l_{35} = 44,000$:

(i) What is the terminal age of the table ?

(ii) What is the probability of surviving from birth to age 60 ?

(iii) What is the probability of a person at exact age 10 dying between exact ages 30 and 45 ?

(11). A separate life table has been constructed for each calendar year of birth, Y, beginning with $Y = 1950$. The mortality functions for the various tables are denoted by the appropriate superscript ^{Y}. For each Y and for all ages x

$$\mu_x^Y = A \cdot k(Y) + B\,c^x \quad , \qquad p_x^{Y+1} = (1+r)\,p_x^Y$$

where k is a function of Y alone and A, B, r are constants (with $r > 0$). If $k(1950) = 1$, then derive a general expression for $k(Y)$.

(12). A standard mortality table follows Makeham's Law with force of mortality

$$\mu_x = A + B\,c^x \qquad \text{at all ages } x$$

A separate, higher-risk mortality table also follows Makeham's Law with force of mortality

$$\mu_x^* = A^* + B^*\,c^x \qquad \text{at all ages } x$$

with the same constant c. If for all starting ages the probability of surviving 6 years according to the higher-risk table is equal to the probability of surviving 9 years according to the standard table, then express each of A^* and B^* in terms of A, B, c.

(13). A homeowner borrows $\$100,000$ at 7% APR from a bank, agreeing to repay by 30 equal yearly payments beginning one year from the time of the loan.

(a) How much is each payment ?

(b) Suppose that after paying the first 3 yearly payments, the homeowner misses the next two (i.e. pays nothing on the 4^{th} and 5^{th} anniversaries of the loan). Find the outstanding balance at the 6^{th} anniversary of the loan, figured at 7%). This is the amount which, if paid as a lump sum at time

6, has present value together with the amounts already paid of $\$100,000$ at time 0.

(14). A deposit of 300 is made into a fund at time $t = 0$. The fund pays interest for the first three years at a nominal monthly rate $d^{(12)}$ of discount. From $t = 3$ to $t = 7$, interest is credited according to the force of interest $\delta_t = 1/(3t + 3)$. As of time $t = 7$, the accumulated value of the fund is 574. Calculate $d^{(12)}$.

(15). Calculate the price at which you would sell a $\$10,000$ 30-year coupon bond with nominal 6% semi-annual coupon ($n = 30$, $m - 2$, $i^{(m)} = 0.06$), 15 years after issue, if for the next 15 years, the effective interest rate for valuation is $i_{\text{APR}} = 0.07$.

(16). Calculate the price at which you would sell a 30-year zero-coupon bond with face amount $\$10,000$ initially issued 15 years ago with $i = i_{\text{APR}} = 0.06$, if for the next 15 years, the effective interest rate for valuation is $i_{\text{APR}} = 0.07$.

2.4 Worked Examples

Example 1. How large must a half-yearly payment be in order that the stream of payments starting immediately be equivalent (in present value terms) at 6% *interest to a lump-sum payment of* $\$5000$, *if the payment-stream is to last (a) 10 years, (b) 20 years, or (c) forever ?*

If the payment size is P, then the balance equation is

$$5000 \;=\; 2\,P \cdot \ddot{a}^{(2)}_{\overline{n}|} \;=\; 2\,P\,\frac{1 - 1.06^{-n}}{d^{(2)}}$$

Since $d^{(2)} = 2(1 - 1/\sqrt{1.06}) = 2 \cdot 0.02871$, the result is

$$P \;=\; (5000 \cdot 0.02871)/(1 - 1.06^{-n}) \;=\; 143.57/(1 - 1.06^{-n})$$

So the answer to part (c), in which $n = \infty$, is $\$143.57$. For parts (a) and (b), respectively with $n = 10$ and 20, the answers are $\$325.11$, $\$208.62$.

Example 2. Assume m *is divisible by* 2. *Express in two different ways the present value of the perpetuity of payments* $1/m$ *at times* $1/m, 3/m, 5/m, \ldots$, *and use either one to give a simple formula.*

This example illustrates the general methods enunciated at the beginning of Section 2.1. Observe first of all that the specified payment-stream is exactly the same as a stream of payments of $1/m$ at times $0, 2/m, 4/m, \ldots$ forever, deferred by a time $1/m$. Since this payment-stream starting at 0 is exactly one-half that of the stream whose present value is $\ddot{a}^{(m/2)}_{\overline{\infty}|}$, a first present value expression is

$$v^{1/m}\,\frac{1}{2}\,\ddot{a}^{(m/2)}_{\overline{\infty}|}$$

A second way of looking at the payment-stream at odd multiples of $1/m$ is as the perpetuity-due payment stream ($1/m$ at times k/m for all $k \geq 0$) **minus** the payment-stream discussed above of amounts $1/m$ at times $2k/m$ for all nonnegative integers k. Thus the present value has the second expression

$$\ddot{a}^{(m)}_{\overline{\infty}|} \;-\; \frac{1}{2}\,\ddot{a}^{(m/2)}_{\overline{\infty}|}$$

Equating the two expressions allows us to conclude that

$$\frac{1}{2}\,\ddot{a}^{(m/2)}_{\overline{\infty}|} \;=\; \ddot{a}^{(m)}_{\overline{\infty}|} \Big/ (1 \,+\, v^{1/m})$$

Substituting this into the first of the displayed present-value expressions, and using the simple expression $1/d^{(m)}$ for the present value of the perpetuity-due, shows that that the present value requested in the Example is

$$\frac{1}{d^{(m)}} \cdot \frac{v^{1/m}}{1 \,+\, v^{1/m}} \;=\; \frac{1}{d^{(m)}\,(v^{-1/m} \,+\, 1)} \;=\; \frac{1}{d^{(m)}\,(2 + i^{(m)}/m)}$$

and this answer is valid whether or not m is even.

Example 3. Suppose that you are negotiating a car-loan of $10,000. *Would you rather have an interest rate of* 4% *for 4 years,* 3% *for 3 years,* 2% *for 2 years, or a cash discount of* $500 *? Show how the answer depends upon the interest rate with respect to which you calculate present values, and give numerical answers for present values calculated at 6% and 8%. Assume that all loans have monthly payments paid at the beginning of the month (e.g., the 4 year loan has 48 monthly payments paid at time 0 and at the ends of 47 succeeding months).*

The monthly payments for an n-year loan at interest-rate i is $10000/(12\,\ddot{a}^{(12)}_{\overline{n}|}) \;=\; (10000/12)\,d^{(12)}/(1-(1+i)^{-n})$. Therefore, the present value at interest-rate r of the n-year monthly payment-stream is

$$10000 \cdot \frac{1-(1+i)^{-1/12}}{1-(1+r)^{-1/12}} \cdot \frac{1-(1+r)^{-n}}{1-(1+i)^{-n}}$$

Using interest-rate $r = 0.06$, the present values are calculated as follows:

For 4-year 4% loan: \$9645.77

For 3-year 3% loan: \$9599.02

For 2-year 2% loan: \$9642.89

so that the most attractive option is the cash discount (which would make the present value of the debt owed to be \$9500). Next, using interest-rate $r = 0.08$, the present values of the various options are:

For 4-year 4% loan: \$9314.72

For 3-year 3% loan: \$9349.73

For 2-year 2% loan: \$9475.68

so that the most attractive option in this case is the 4-year loan. (The cash discount is now the least attractive option.)

Example 4. Suppose that the force of mortality $\mu(y)$ is specified for exact ages y ranging from 5 to 55 as

$$\mu(y) \;=\; 10^{-4} \cdot (20 - 0.5|30-y|)$$

Then find analytical expressions for the survival probabilities $S(y)$ for exact ages y in the same range, and for the (one-year) death-rates q_x for integer ages $x = 5, \ldots, 54$, assuming that $S(5) = 0.97$.

The key formulas connecting force of mortality and survival function are here applied separately on the age-intervals $[5, 30]$ and $[30, 55]$, as follows. First for $5 \le y \le 30$,

$$S(y) \;=\; S(5)\,\exp(-\int_5^y \mu(z)\,dz) \;=\; 0.97\,\exp\Big(-10^{-4}(5(y-5)+0.25(y^2-25))\Big)$$

so that $S(30) \;=\; 0.97\,e^{-0.034375} \;=\; 0.93722$, and for $30 \le y \le 55$

$$S(y) \;=\; S(30)\,\exp\Big(-10^{-4}\int_{30}^y (20+0.5(30-z))\,dz\Big)$$

$$= \; 0.9372 \; \exp\Big(- .002(y-30) + 2.5 \cdot 10^{-5}(y-30)^2 \Big)$$

The death-rates q_x therefore have two different analytical forms: first, in the case $x = 5, \ldots, 29$,

$$q_x \; = \; S(x+1)/S(x) \; = \; \exp\Big(-5 \cdot 10^{-5}\,(10.5+x) \Big)$$

and second, in the case $x = 30, \ldots, 54$,

$$q_x \; = \; \exp\Big(- .002 + 2.5 \cdot 10^{-5}(2(x-30)+1) \Big)$$

2.5 Useful Formulas from Chapter 2

$$v = 1/(1+i)$$

p. 24

$$a^{(m)}_{\overline{n}|} = \frac{1-v^n}{i^{(m)}} \quad , \qquad \ddot{a}^{(m)}_{\overline{n}|} = \frac{1-v^n}{d^{(m)}}$$

pp. 25–25

$$a_{\overline{n}|}m = v^{1/m}\, \ddot{a}_{\overline{n}|}m$$

p. 25

$$\ddot{a}^{(\infty)}_{\overline{n}|} = a_{\overline{n}|}\infty = \overline{a}_n = \frac{1-v^n}{\delta}$$

p. 25

$$a^{(m)}_{\overline{\infty}|} = \frac{1}{i^{(m)}} \quad , \qquad \ddot{a}_{\overline{\infty}|}m = \frac{1}{d^{(m)}}$$

p. 26

$$(I^{(m)}\ddot{a})^{(m)}_{\overline{n}|} = \ddot{a}^{(m)}_{\overline{\infty}|} \left(\ddot{a}^{(m)}_{\overline{n}|} - n\, v^n \right)$$

p. 28

$$(D^{(m)}\ddot{a})^{(m)}_{\overline{n}|} = (n + \frac{1}{m})\, \ddot{a}^{(m)}_{\overline{n}|} - (I^{(m)}\ddot{a})^{(m)}_{\overline{n}|}$$

p. 28

n-yr m'thly Mortgage Paymt : $\dfrac{Loan\ Amt}{m\, \ddot{a}^{(m)}_{\overline{n}|}}$

p. 29

n-yr Mortgage Bal. at $\dfrac{k}{m}+$: $\quad B_{n,k/m} \;=\; \dfrac{1-v^{n-k/m}}{1-v^{n}}$

p. 30

$$ {}_tp_x \;=\; \frac{S(x+t)}{S(x)} \;=\; \exp\left(-\int_0^t \mu(x+s)\,ds\right) $$

p. 38

$$ {}_tp_x \;=\; 1 \;-\; {}_tp_x $$

p. 38

$$ q_x \;=\; {}_1q_x \;=\; \frac{d_x}{l_x} \quad , \qquad p_x \;=\; {}_1p_x \;=\; 1 \;-\; q_x $$

p. 38

$$ \mu(x+t) \;=\; \frac{f(x+t)}{S(x+t)} \;=\; -\frac{\partial}{\partial t}\ln S(x+t) \;=\; -\frac{\partial}{\partial t}\ln l_{x+t} $$

p. 39

$$ S(x) \;=\; \exp(-\int_0^x \mu(y)\,dy) $$

p. 42

Unif. Failure Dist.: $\quad S(x) \;=\; \dfrac{\omega-x}{\omega} \;,\quad f(x) \;=\; \dfrac{1}{\omega} \;,\quad 0 \le x \le \omega$

p. 42

Expon. Dist.: $\quad S(x) \;=\; e^{-\mu x} \;,\quad f(x) \;=\; \mu e^{-\mu x} \;,\quad \mu(x) \;=\; \mu \;,\quad x > 0$

p. 42

Weibull. Dist.: $S(x) \; = \; e^{-\lambda x^\gamma} \quad , \quad \mu(x) \; = \; \lambda \gamma x^{\gamma - 1} \quad , \quad x > 0$

p. 42

Makeham: $\mu(x) \; = \; A + Bc^x \qquad , \quad x \geq 0$

Gompertz: $\mu(x) \; = \; Bc^x \qquad , \quad x \geq 0$

$$S(x) \; = \; \exp\left(-Ax - \frac{B}{\ln c} \left(c^x - 1 \right) \right)$$

p. 44

Chapter 3

More Probability Theory for Life Tables

3.1 Interpreting Force of Mortality

This Section consists of remarks, relating the force of mortality for a continuously distributed lifetime random variable T (with continuous density function f) to conditional probabilities for discrete random variables. Indeed, for m large (e.g. as large as 4 or 12), the discrete random variable $[Tm]/m$ gives a close approximation to T and represents the attained age at death measured in whole-number multiples of fractions $h =$ one m^{th} of a year. (Here $[\cdot]$ denotes the greatest integer less than or equal to its real argument.) Since surviving an additional time $t = nh$ can be viewed as successively surviving to reach times $h, 2h, 3h, \ldots, nh$, and since (by the definition of conditional probability)

$$P(A_1 \cap \cdots \cap A_n) \;=\; P(A_1) \cdot P(A_2|A_1) \cdots P(A_n|A_1 \cap \cdots \cap A_{n-1})$$

we have (with the interpretation $A_k = \{T \geq x + kh\}$)

$$_{nh}p_x \;=\; {}_hp_x \,\cdot\, {}_hp_{x+h} \,\cdot\, {}_hp_{x+2h} \cdots {}_hp_{x+(n-1)h}$$

The form in which this formula is most often useful is the case $h = 1$: for integers $k \geq 2$,

$$_kp_x \;=\; p_x \,\cdot\, p_{x+1} \,\cdot\, p_{x+2} \cdots p_{x+k-1} \tag{3.1}$$

Every continuous waiting-time random variable can be approximated by a discrete random variable with possible values which are multiples of a fixed small unit h of time, and therefore the random survival time can be viewed as the (first failure among a) succession of results of a sequence of independent coin-flips with successive probabilities ${}_hp_{kh}$ of heads. By the Mean Value Theorem applied up to second-degree terms on the function $S(x+h)$ expanded about $h = 0$,

$$S(x+h) \;=\; S(x) + hS'(x) + \frac{h^2}{2}\,S''(x+\tau h) \;=\; S(x) - hf(x) - \frac{h^2}{2}\,f'(x+\tau h)$$

for some $0 < \tau < 1$, if f is continuously differentiable. Therefore, using the definition of $\mu(x)$ as $f(x)/S(x)$ given on page 39,

$${}_hp_x = 1 \,-\, h \cdot \Big[\frac{S(x) - S(x+h)}{hS(x)}\Big] \;=\; 1 \,-\, h\,\Big(\mu(x) \,+\, \frac{h}{2}\,\frac{f'(x+\tau h)}{S(x)}\Big)$$

Going in the other direction, the previously derived formula

$${}_hp_x = \exp\left(-\int_x^{x+h} \mu(y)\,dy\right)$$

can be interpreted by considering the fraction of individuals observed to reach age x who thereafter experience hazard of mortality $\mu(y)\,dy$ on successive infinitesimal intervals $[y, y+dy]$ within $[x, x+h)$. The lives aged x survive to age $x+h$ with probability equal to a limiting product of infinitesimal terms $(1-\mu(y)\,dy) \sim \exp(-\mu(y)\,dy)$, yielding an overall conditional survival probability equal to the negative exponential of accumulated hazard over $[x, x+h)$.

3.2 Interpolation Between Integer Ages

There is a Taylor-series justification of "actuarial approximations" for life-table related functions. Let $g(x)$ be a smooth function with small $|g''(x)|$, and let T be the lifetime random variable (for a randomly selected member of the population represented by the life-table), with $[T]$ denoting its integer part, i.e., the largest integer k for which $k \le T$. Then by the Mean Value

Theorem, applied up to second-degree terms for the function $g(t) = g(k+u)$ (with $t = k + u,\ k = [t]$) expanded in $u \in (0,1)$ about 0,

$$E(g(T)) = E(g([T]) \ + \ (T - [T])\, g'([T]) \ + \ \frac{1}{2}\,(T - [T])^2\, g''(T_*)) \qquad (3.2)$$

where T_* lies between $[T]$ and T. Now in case the rate-of-change of g' is very small, the third term may be dropped, providing the approximate formula

$$E(g(T)) \approx Eg([T]) \ + \ E\Big((T - [T])\, g'([T])\Big) \qquad (3.3)$$

Simplifications will result from this formula especially if the behavior of conditional probabilities concerning $T - [T]$ given $[T] = k$ turns out not to depend upon the value of k. (This property can be expressed by saying that the random integer $[T]$ and random fractional part $T - [T]$ of the age at death are *independent* random variables.) This is true in particular if it is also true that $P(T - [T] \ge s \mid k \le T < k + 1)$ is approximately $1 - s$ for all k and for $0 < s < 1$, as would be the case if the density of T were constant on each interval $[k, k + 1)$ (i.e., if the distribution of T were *conditionally uniform* given $[T]$): then $T - [T]$ would be uniformly distributed on $[0, 1)$, with density $f(s) = 1$ for $0 \le s < 1$. Then $E((T - [T])\, g'([T])) \ = \ E(g'([T]))/2$, implying by (3.3) that

$$E(g(T)) \ \approx \ E(g([T]) + \frac{1}{2}\, g'([T])) \ \approx E\Big(g([T] + \frac{1}{2}\,)\Big)$$

where the last step follows by the first-order Taylor approximation

$$g(k + 1/2) \ \approx \ g(k) \ + \ \frac{1}{2}\, g'(k)$$

One particular application of the ideas of the previous paragraph concern so-called *expected residual lifetimes.* Demographers tabulate, for all integer ages x in a specified population, what is the average number e_x of remaining years of life to individuals who have just attained exact age x. This is a quantity which, when compared across national or generational boundaries, can give some insight into the way societies differ and change over time. In the setting of the previous paragraph, we are considering the function $g(t) = t - x$ for a fixed x, and calculating expectations $E(\cdot)$ conditionally for a life aged x, i.e. conditionally given $T \ge x$. In this setting, the approximation described above says that if we can treat the density of T as constant within each whole year of attained integer age, then

$$\text{Mean residual lifetime} \;=\; \mathrm{e}_x \;\approx\; \mathring{\mathrm{e}}_x \,+\, \frac{1}{2}$$

where $\mathring{\mathrm{e}}_x$ denotes the so-called *curtate mean residual life* which measures the expectation of $[T]-x$ given $T \geq x$, i.e., the expected number of additional birthdays or whole complete years of life to a life aged exactly x.

"Actuarial approximations" often involve an assumption that a life-table time until death is conditionally uniformly distributed, i.e., its density is piecewise-constant, over intervals $[k, k+1)$ of age. The following paragraphs explore this and other possibilities for survival-function interpolation *between* integer ages.

One approach to approximating survival and force-of-mortality functions at non-integer values is to use analytical or what statisticians call *parametric* models $S(x;\vartheta)$ arising in Examples (i)-(v) above, where ϑ denotes in each case the vector of real parameters needed to specify the model. Data on survival at integer ages x can be used to estimate or *fit* the value of the scalar or vector parameter ϑ, after which the model $S(x;\vartheta)$ can be used at *all* real x values. We will see some instances of this in the exercises. The disadvantage of this approach is that actuaries do not really believe that any of the simple models outlined above ought to fit the whole of a human life table. Nevertheless they can and do make assumptions on the shape of $S(x)$ in order to justify interpolation-formulas between integer ages.

Now assume that values $S(k)$ for $k = 0, 1, 2, \ldots$ have been specified or estimated. Approximations to $S(x)$, $f(x)$ and $\mu(x)$ between integers are usually based on one of the following assumptions:

(i) *(Piecewise-uniform density)* $f(k+t)$ is constant for $0 \leq t < 1$;

(ii) *(Piecewise-constant hazard)* $\mu(k+t)$ is constant for $0 \leq t < 1$;

(iii) *(Balducci hypothesis)* $1/S(k+t)$ is linear for $0 \leq t < 1$.

Note that for integers k and $0 \leq t \leq 1$,

$$\left.\begin{array}{c} S(k+t) \\ -\ln S(k+t) \\ 1/S(k+t) \end{array}\right\} \text{ is linear in } \quad t \quad \text{under } \quad \left\{\begin{array}{r} \text{assumption (i)} \\ \text{assumption (ii)} \\ \text{assumption (iii)} \end{array}\right. \tag{3.4}$$

Under assumption (i), the slope of the linear function $S(k+t)$ at $t=0$ is $-f(k)$, which implies easily that $S(k+t) = S(k) - tf(k)$, i.e.,

$$f(k) = S(k) - S(k+1) \ , \quad \text{and} \qquad \mu(k+t) = \frac{f(k)}{S(k) - tf(k)}$$

so that under (i),

$$\mu(k+\frac{1}{2}) \ = \ f_T(k+\frac{1}{2}) \Big/ S_T(k+\frac{1}{2}) \tag{3.5}$$

Under (ii), where $\mu(k+t) = \mu(k)$, (3.5) also holds, and

$$S(k+t) = S(k)\, e^{-t\,\mu(k)} \ , \quad \text{and} \qquad p_k = \frac{S(k+1)}{S(k)} = e^{-\mu(k)}$$

Under (iii), for $0 \le t < 1$,

$$\frac{1}{S(k+t)} = \frac{1}{S(k)} + t\left(\frac{1}{S(k+1)} - \frac{1}{S(k)}\right) \tag{3.6}$$

When equation (3.6) is multiplied through by $S(k+1)$ and terms are rearranged, the result is

$$\frac{S(k+1)}{S(k+t)} \ = \ t \ + \ (1-t)\,\frac{S(k+1)}{S(k)} \ = \ 1 \ - \ (1-t)\,q_k \tag{3.7}$$

Recalling that ${}_tq_k = 1 - (S(k+t)/S(k))$, reveals assumption (iii) to be equivalent to

$${}_{1-t}q_{k+t} = 1 - \frac{S(k+1)}{S(k+t)} = (1-t)\left(1 - \frac{S(k+1)}{S(k)}\right) \ = \ (1-t)\,q_k \tag{3.8}$$

Next differentiate the logarithm of the formula (3.7) with respect to t, to show (still under (iii)) that

$$\mu(k+t) \ = \ -\frac{\partial}{\partial t}\ \ln S(k+t) \ = \ \frac{q_k}{1-(1-t)q_k} \tag{3.9}$$

The most frequent insurance application for the interpolation assumptions (i)-(iii) and associated survival-probability formulas is to express probabilities of survival for fractional years in terms of probabilities of whole-year survival. In terms of the notations ${}_tp_k$ and q_k for integers k and $0 < t < 1$, the formulas are:

$$ {}_tp_k \;=\; 1 - \frac{(S(k) - t(S(k+1) - S(k))}{S(k)} \;=\; 1 - t\,q_k \qquad \text{under (i)} \tag{3.10} $$

$$ {}_tp_k \;=\; \frac{S(k+t)}{S(x)} \;=\; \left(e^{-\mu(k)}\right)^t \;=\; (1-q_k)^t \qquad \text{under (ii)} \tag{3.11} $$

$$ {}_tp_k \;=\; \frac{S(k+t)}{S(k+1)}\,\frac{S(k+1)}{S(k)} \;=\; \frac{1-q_k}{1-(1-t)q_k} \qquad \text{under (iii)} \tag{3.12} $$

The application of all of these formulas can be understood in terms of the formula for expectation of a function $g(T)$ of the lifetime random variable T. (For a concrete example, think of $g(T) = (1+i)^{-T}$ as the present value to an insurer of the payment of \$1 which it will make instantaneously at the future time T of death of a newborn which it undertakes to insure.) Then assumptions (i), (ii), or (iii) via respective formulas (3.10), (3.11), and (3.12) are used to substitute into the final expression of the following formulas:

$$ \begin{aligned} E\Big(g(T)\Big) \;&=\; \int_0^\infty g(t)\,f(t)\,dt \;=\; \sum_{k=0}^{\omega-1} \int_0^1 g(t+k)\,f(t+k)\,dt \\ &=\; \sum_{k=0}^{\omega-1} S(k) \int_0^1 g(t+k)\,\Big(-\frac{\partial}{\partial t}\,{}_tp_k\Big)\,dt \end{aligned} $$

3.3 Binomial Variables & Law of Large Numbers

This Section develops just enough machinery for the student to understand the probability theory for random variables which count numbers of successes in large numbers of independent biased coin-tosses. The motivation is that in large life-table populations, the number l_{x+t} who survive t time-units after age x can be regarded as the number of successes or heads in a large number l_x of independent coin-toss trials corresponding to the further survival of each of the l_x lives aged x , which for each such life has probability ${}_tp_x$. The one preliminary piece of mathematical machinery which the student is assumed to know is the **Binomial Theorem** stating that (for positive integers N and arbitrary real numbers x, y, z),

$$(1+x)^N = \sum_{k=0}^{N} \binom{N}{k} x^k \, , \qquad (y+z)^N = \sum_{k=0}^{N} \binom{N}{k} y^k \, z^{N-k}$$

Recall that the first of these assertions follows by equating the k^{th} derivatives of both sides at $x=0$, where $k=0,\ldots,N$. The second assertion follows immediately, in the nontrivial case when $z \neq 0$, by applying the first assertion with $x = y/z$ and multiplying both sides by z^N. This Theorem also has a direct combinatorial consequence. Consider the two-variable polynomial

$$(y+z)^N \; = \; (y+z)\cdot(y+z)\cdots(y+z) \qquad\qquad N \;\; \textit{factors}$$

expanded by making all of the different choices of y or z from each of the N factors $(y+z)$, multiplying each combination of choices out to get a monomial $y^j \, z^{N-j}$, and adding all of the monomials together. Each combined choice of y or z from the N factors $(y+z)$ can be represented as a sequence $(a_1, \ldots, a_n) \in \{0,1\}^N$, where $a_i = 1$ would mean that y is chosen $a_i = 0$ would mean that z is chosen in the i^{th} factor. Now a combinatorial fact can be deduced from the Binomial Theorem: since the coefficient $\binom{N}{k}$ is the total number of monomial terms $y^k \, z^{N-k}$ which are collected when $(y+z)^N$ is expanded as described, and since these monomial terms arise only from the combinations $(a_1,\ldots,a_N)$ of $\{y, z\}$ choices in which precisely k of the values a_j are 1's and the rest are 0's,

> The number of symbol-sequences $(a_1,\ldots,a_N) \in \{0,1\}^N$ such that $\sum_{j=1}^N a_j = k$ is given by $\binom{N}{k}$, for $k = 0,1,\ldots,N$. This number
>
> $$\binom{N}{k} \; = \; \frac{N(N-1)\cdots(N-k+1)}{k!}$$
>
> spoken as 'N choose k', therefore counts all of the ways of choosing k element subsets (the positions j from 1 to N where 1's occur) out of N objects.

The random experiment of interest in this Section consists of a large number N of independent tosses of a coin, with probability p of coming up heads each time. Such coin-tossing experiments — independently replicated two-outcome experiments with probability p of one of the outcomes, designated 'success' — are called *Bernoulli(p) trials*. The space of possible heads-and-tails configurations, or *sample space* for this experiment, consists of the strings of N zeroes and ones, with each string $\underline{\mathbf{a}} = (a_1, \ldots, a_N) \in \{0,1\}^N$ being assigned probability $p^a\,(1-p)^{N-a}$, where $a \equiv \sum_{j=1}^N a_j$. The rule by which probabilities are assigned to sets or *events* A of more than one string $\underline{\mathbf{a}} \in \{0,1\}^N$ is to add the probabilities of all individual strings $\underline{\mathbf{a}} \in A$. We are particularly interested in the event (denoted $[X = k]$) that precisely k of the coin-tosses are heads, i.e., in the subset $[X = k] \subset \{0,1\}^N$ consisting of all strings $\underline{\mathbf{a}}$ such that $\sum_{j=1}^N a_j = k$. Since each such string has the same probability $p^k\,(1-p)^{N-k}$, and since, according to the discussion following the Binomial Theorem above, there are $\binom{N}{k}$ such strings, the probability which is necessarily assigned to the event of k successes is

$$\text{P(k successes in N } \textit{Bernoulli(p)} \text{ trials)} = P(X = k) = \binom{N}{k} p^k\,(1-p)^{N-k}$$

By virtue of this result, the random variable X equal to the number of successes in N *Bernoulli(p)* trials, is said to have the **Binomial distribution** with **probability mass function** $p_X(k) = \binom{N}{k}\,p^k\,(1-p)^{N-k}$.

With the notion of Bernoulli trials and the binomial distribution in hand, we now begin to regard the ideal probabilities $S(x+t)/S(x)$ as true but unobservable probabilities ${}_tp_x = p$ with which each of the l_x lives aged x will survive to age $x + t$. Since the mechanisms which cause those lives to survive or die can ordinarily be assumed to be acting independently in a probabilistic sense, we can regard the number l_{x+t} of lives surviving to the (possibly fractional) age $x{+}t$ as a Binomial random variable with parameters $N = l_x,\ p = {}_tp_x$. From this point of view, the observed life-table counts l_x should be treated as *random data* which reflect but do not define the underlying probabilities ${}_xp_0 = S(x)$ of survival to age x. However, common sense and experience suggest that, when l_0 is large, and therefore the other life-counts l_x for moderate values x are also large, the observed ratios l_{x+t}/l_x should reliably be very close to the 'true' probability ${}_tp_x$. In other words, the ratio l_{x+t}/l_x is a *statistical estimator* of the unknown constant ${}_tp_x$. The good property, called *consistency*, of this estimator to be close with very large probability (based upon large life-table size) to the probability it estimates, is established in the famous **Law of Large Numbers**. The precise quantitative inequality proved here concerning binomial probabilities is called a *Large Deviation Inequality* and is very important in its own right.

Theorem 3.3.1 *Suppose that X is a Binomial(N, p) random variable, denoting the number of successes in N Bernoulli(p) trials.*

(a) Large Deviation Inequalities. If $1 > b > p > c > 0$, then

$$P(X \geq Nb) \leq \exp\Big\{ -N \Big[b \ln\Big(\frac{b}{p}\Big) \; + \; (1-b) \ln\Big(\frac{1-b}{1-p}\Big) \Big] \Big\}$$

$$P(X \leq Nc) \leq \exp\Big\{ -N \Big[c \ln\Big(\frac{c}{p}\Big) \; + \; (1-c) \ln\Big(\frac{1-c}{1-p}\Big) \Big] \Big\}$$

(b) Law of Large Numbers. For arbitrarily small fixed $\delta > 0$, not depending upon N, the number N of Bernoulli trials can be chosen so large that

$$P\Big(\Big| \frac{X}{N} - p \Big| \geq \delta \Big) \; \leq \; \delta$$

Proof. After the first inequality in (a) is proved, the second inequality will be derived from it, and part (b) will follow from part (a). Since the event $[X \geq Nb]$ is the union of the disjoint events $[X = k]$ for $k \geq Nb$, which in turn consist of all outcome-strings $(a_1, \ldots, a_N) \in \{0,1\}^N$ for which $\sum_{j=1}^N a_j = k \geq Nb$, a suitable subset of the binomial probability mass function values $p_X(k)$ are summed to provide

$$P(X \geq Nb) = \sum_{k: Nb \leq k \leq N} P(X = k) = \sum_{k \geq Nb} \binom{N}{k} p^k \, (1-p)^{N-k}$$

For every $s > 1$, this probability is

$$\leq \sum_{k \geq Nb} \binom{N}{k} p^k \, (1-p)^{N-k} \, s^{k-Nb} \; = \; s^{-Nb} \sum_{k \geq Nb} \binom{N}{k} (ps)^k \, (1-p)^{N-k}$$

$$\leq \; s^{-Nb} \sum_{k=0}^{N} \binom{N}{k} (ps)^k \, (1-p)^{N-k} \; = \; s^{-Nb} \, (1 - p + ps)^N$$

Here extra terms (corresponding to $k < Nb$) have been added in the next-to-last step, and the binomial theorem was applied in the last step. The trick in the proof comes now: since the left-hand side of the inequality does not involve s while the right-hand side does, and since the inequality must be valid for every $s > 1$, it remains valid if the right-hand side is minimized over s. The calculus minimum does exist and is unique, as you can check by calculating that the second derivative in s is always positive. The minimum occurs where the first derivative of the logarithm of the last expression is 0, i.e., at $s = b(1-p)/(p(1-b))$. Substituting this value for s yields

$$\begin{aligned} P(X \ge Nb) \ &\le \ \left(\frac{b\,(1-p)}{p\,(1-b)}\right)^{-Nb} \left(\frac{1-p}{1-b}\right)^{N} \\ &= \ \exp\left(-N\Big[b\,\ln\Big(\frac{b}{p}\Big) + (1-b)\,\ln\Big(\frac{1-b}{1-p}\Big)\Big]\right) \end{aligned}$$

as desired.

The second part of assertion (a) follows from the first. Replace X by $Y = N - X$. Since Y also is a count of 'successes' in *Bernoulli*$(1-p)$ trials, where the 'successes' counted by Y are precisely the 'failures' in the *Bernoulli* trials defining X, it follows that Y also has a *Binomial*(N, q) distribution, where $q = 1 - p$. Note also that $c < p$ implies $b = 1 - c > 1 - p = q$. Therefore, the first inequality applied to Y instead of X with $q = 1 - p$ replacing p and $b = 1 - c$, gives the second inequality for $P(Y \ge Nb) = P(X \le Nc)$.

Note that for all r between $0, 1$, the quantity $r \ln \frac{r}{p} + (1-r) \ln \frac{1-r}{1-p}$ as a function of r is convex and has a unique minimum of 0 at $r = p$. Therefore when $b > p > c$, the upper bound given in part (a) for $N^{-1} \ln P([X \ge bN] \cup [X \le cN])$ is strictly negative and does not involve N. For part (b), let $\delta \in (0, \min(p, 1-p))$ be arbitrarily small, choose $b = p + \delta$, $c = p - \delta$, and combine the inequalities of part (a) to give the precise estimate (b).

$$P(|\frac{X}{N} - p| \ge \delta) \le 2 \cdot \exp(-Na) \tag{3.13}$$

where

$$\begin{aligned} a \ &= \ \min\Big((p+\delta)\,\ln(1 + \tfrac{\delta}{p}) + (1-p-\delta)\,\ln(1 - \tfrac{\delta}{1-p})\,, \\ &\qquad (p-\delta)\ln(1 - \tfrac{\delta}{p}) + (1-p+\delta)\ln(1 + \tfrac{\delta}{1-p})\Big) \ > \ 0 \end{aligned} \tag{3.14}$$

This last inequality proves (b), and in fact gives a much stronger and numerically more useful upper bound on the probability with which the so-called *relative frequency of success* X/N differs from the true probability p of success by as much as δ. The probabilities of such *large deviations* between X/N and δ are in fact exponentially small as a function of the number N of repeated $Bernoulli(p)$ trials, and the upper bounds given in (a) on the log-probabilities divided by N turn out to be the correct limits for large N of these normalized log-probabilities. □

3.3.1 Exact Probabilities, Bounds & Approximations

Suppose first that you are flipping $20,000$ times a coin which is supposed to be fair (i.e., to have $p = 1/2$). The probability that the observed number of heads falls outside the range $[9800, 10200]$ is, according to the inequalities above,

$$\leq 2 \cdot \exp\Big[-9800 \ln(0.98) - 10200 \ln(1.02)\Big] = 2\, e^{-4.00} = 0.037$$

The inequalities (3.13)-(3.14) give only an upper bound for the actual binomial probability, and 0.0046 is the exact probability with which the relative frequency of heads based on 20000 fair coin-tosses lies outside the range (0.98, 1.02). The ratio of the upper bound to the actual probability is rather large (about 8), but the absolute errors are small.

To give a feeling for the probabilities with which observed life-table ratios reflect the true underlying survival-rates, we have collected in Table 3.3.1 various exact binomial probabilities and their counterparts from the inequalities of Theorem 3.3.1(a). The illustration concerns cohorts of lives aged x of various sizes l_x, together with 'theoretical' probabilities ${}_kp_x$ with which these lives will survive for a period of $k = 1,\, 5,$ or 10 years. The probability experiment determining the size of the surviving cohort l_{x+k} is modelled as the tossing of l_x independent coins with common heads-probability ${}_kp_x$: then the surviving cohort-size l_{x+k} is viewed as the $Binomial(l_x,\, {}_kp_x)$ random variable equal to the number of heads in those coin-tosses. In Table 3.3.1 are given various combinations of $x,\ l_x,\ k,\ {}_kp_x$ which might realistically arise in an insurance-company life-table, together, with the true and estimated (from Theorem 3.3.1) probabilities with which the ratios l_{x+k}/l_x agree with ${}_kp_x$

to within a fraction δ of the latter. The formulas used to compute columns 6 and 7 of the table are (for $n = l_x, \; p = {}_kp_x$):

$$\text{True binomial probability} \quad = \sum_{j: j/(np) \in [1-\delta, 1+\delta]} \binom{n}{j} \, p^j \, (1-p)^{n-j}$$

$$\text{Lower bound for probability} \;=\; 1 - (1+\delta)^{-np(1+\delta)} \left(1 - \frac{p\delta}{1-p}\right)^{-n(1-p-p\delta)}$$
$$- \; (1-\delta)^{-np(1-\delta)} \left(1 + \frac{p\delta}{1-p}\right)^{-n(1-p+p\delta)}$$

Columns 6 and 7 in the Table show how likely the life-table ratios are to be close to the 'theoretical' values, but also show that the lower bounds, while also often close to 1, are still noticeably smaller than the actual values. .

Much closer approximations to the exact probabilities for Binomial(n, p) random variables given in column 6 of Table 3.3.1 are obtained from the *Normal distribution approximation*

$$P(a \le X \le b) \;\approx\; \Phi\left(\frac{b - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{a - np}{\sqrt{np(1-p)}}\right) \qquad (3.15)$$

where Φ is the *standard normal distribution function* given explicitly in integral form in formula (3.20) below. This approximation is the **DeMoivre-Laplace Central Limit Theorem** (Feller vol. 1, 1957, pp. 168-73), which says precisely that the difference between the left- and right-hand sides of (3.15) converges to 0 when p remains fixed, $n \to \infty$. Moreover, the refined form of the DeMoivre-Laplace Theorem given in the Feller (1957, p. 172) reference says that each of the *ratios* of probabilities

$$P(X < a) \Big/ \Phi\Big(\frac{a - np}{\sqrt{np(1-p)}}\Big) \quad , \qquad P(X > b) \Big/ \Big[1 - \Phi\Big(\frac{b - np}{\sqrt{np(1-p)}}\Big)\Big]$$

converges to 1 if the 'deviation' ratios $(b - np)/\sqrt{np(1-p)}$ and $(a - np)/\sqrt{np(1-p)}$ are of smaller order than $n^{-1/6}$ when n gets large. This result suggests the approximation

$$\text{Normal approximation} \quad = \quad \Phi\left(\frac{np\delta}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{-np\delta}{\sqrt{np(1-p)}}\right)$$

Table 3.1: Probabilities (in col. 6) with which various Binomial(l_x, ${}_kp_x$) random variables lie within a factor $1 \pm \delta$ of their expectations, together with lower bounds for these probabilities derived from the large-deviation inequalities (3.13)-(3.14). The final column contains the normal-distribution (Central-Limit) approximations to the exact probabilities in column 6.

Cohort $n = l_x$	Age x	Time k	Prob. $p = {}_k p_x$	Toler. frac. δ	Pr. within within $1 \pm \delta$	Lower bound	Normal approx.
10000	40	3	0.99	.003	.9969	.9760	.9972
10000	40	5	0.98	.004	.9952	.9600	.9949
10000	40	10	0.94	.008	.9985	.9866	.9985
1000	40	10	0.94	.020	.9863	.9120	.9877
10000	70	5	0.75	.020	.9995	.9950	.9995
1000	70	5	0.75	.050	.9938	.9531	.9938
10000	70	10	0.50	.030	.9973	.9778	.9973
1000	70	10	0.50	.080	.9886	.9188	.9886

for the true binomial probability $P(|X - np| \le np\delta)$, the formula of which is displayed above. Although the deviation-ratios in this setting are actually close to $n^{-1/6}$, not smaller as they should be for applicability of the cited result of Feller, the normal approximations in the final column of Table 3.3.1 below are sensationally close to the correct binomial probabilities in column 6. A still more refined theorem which justifies this is given by Feller (1972, section XVI.7 leading up to formula 7.28, p. 553).

If the probabilities in Theorem 3.3.1(a) are generally much smaller than the upper bounds given for them, then why are those bounds of interest ? (These are 1 minus the probabilities illustrated in Table 3.3.1.) First, they provide relatively quick hand-calculated estimates showing that large batches of independent coin-tosses are extremely unlikely to yield relative frequencies of heads much different from the true probability or limiting relative frequency of heads. Another, more operational, way to render this conclusion of Theorem 3.3.1(b) is that two very large insured cohorts with the same true survival probabilities are very unlikely to have materially different survival experience. However, as the Table illustrates, for practical purposes the normal approximation to the binomial probabilities of large discrepancies from the expectation is generally much more precise than the large deviation bounds of Theorem 3.3.1(a).

The bounds given in Theorem 3.3.1(a) get small with large N *much* more rapidly than simpler bounds based on Chebychev's inequality (*cf.* Hogg and Tanis 1997, Larsen and Marx 1985, or Larson 1982). We can tolerate the apparent looseness in the bounds because it can be shown that the exponential rate of decay as a function of N in the true tail-probabilities $P_N = P(X \geq Nb)$ or $P(X \leq Nc)$ in Theorem 3.3.1(a) (i.e., the constants appearing in square brackets in the exponents on the right-hand sides of the bounds) are exactly the right ones: no larger constants replacing them could give correct bounds.

3.4 Simulation of Life Table Data

We began by regarding life-table ratios l_x/l_0 in large cohort life-tables as *defining* integer-age survival probabilities $S(x) = {}_xp_0$. We said that if the life-table was representative of a larger population of prospective insureds, then we could imagine a newly presented life aged x as being randomly chosen from the life-table cohort itself. We motivated the conditional probability ratios in this way, and similarly expectations of functions of life-table death-times were averages over the entire cohort. Although we found the calculus-based formulas for life-table conditional probabilities and expectations to be useful, at that stage they were only ideal approximations of the more detailed but still exact life-table ratios and sums. At the next stage of sophistication, we began to describe the (conditional) probabilities ${}_tp_x \equiv S(x+t)/S(x)$ based upon a smooth survival function $S(x)$ as a true but unknown survival distribution, hypothesized to be of one of a number of possible theoretical forms, governing each member of the life-table cohort and of further prospective insureds. Finally, we have come to view the life-table itself as *data*, with each ratio l_{x+t}/l_x equal to the relative frequency of success among a set of l_x *Bernoulli*$({}_tp_x)$ trials which Nature performs upon the set of lives aged x . With the mathematical justification of the Law of Large Numbers, we come full circle: these relative frequencies are random variables which are not very random. That is, they are extremely likely to lie within a very small tolerance of the otherwise unknown probabilities ${}_tp_x$. Accordingly, the life-table ratios are, at least for very large-radix life tables, highly accurate *statistical estimators* of the life-table probabilities which we earlier tried to define by them.

To make this discussion more concrete, we illustrate the difference between the entries in a life-table and the entries one would observe as data in a randomly generated life-table of the same size using the initial life-table ratios as *exact* survival probabilities. We used as a source of life-table counts the Mortality Table for U.S. White Males 1959-61 reproduced as Table 2 on page 11 of C. W. Jordan's (1967) book on *Life Contingencies*. That is, using this Table with radix $l_0 = 10^5$, with counts l_x given for integer ages x from 1 through 80, we treated the probabilities $p_x = l_{x+1}/l_x$ for $x = 0, \ldots, 79$ as the correct one-year survival probabilities for a second, computer-simulated cohort life-table with radix $l_0^* = 10^5$. Using simulated random variables generated in **Splus**, we successively generated, as x runs from 1 to 79, random variables $l_{x+1}^* \sim Binomial(l_x^*, p_x)$. In other words, the mechanism of simulation of the sequence $l_0^*, \ldots, l_{79}^*$ was to make the variable l_{x+1}^* depend on previously generated $l_1^*, \ldots, l_x^*$ only through l_x^*, and then to generate l_{x+1}^* as though it counted the heads in l_x^* independent coin-tosses with heads-probability p_x. A comparison of the actual and simulated life-table counts for ages 9 to 79 in 10-year intervals, is given below. The complete simulated life-table was given earlier as Table 1.1.

The implication of the Table is unsurprising: with radix as high as 10^5, the agreement between the initial and randomly generated life-table counts is quite good. The Law of Large Numbers guarantees good agreement, with very high probability, between the ratios l_{x+10}/l_x (which here play the role of the probability ${}_{10}p_x$ of success in l_x^* *Bernoulli* trials) and the corresponding simulated random relative frequencies of success l_{x+10}^*/l_x^*. For example, with $x = 69$, the final simulated count of 28657 lives aged 79 is the success-count in 56186 *Bernoulli* trials with success-probability $28814/56384 = .51103$. With this success-probability, assertion (a) and the final inequality proved in (b) of the Theorem show that the resulting count will differ from $.51103 \cdot 56186 = 28712.8$ by 300 or more (in either direction) with probability at most 0.08. (Verify this by substituting in the formulas with $300 = \delta \cdot 56186$).

Table 3.2: **Illustrative Real and Simulated Life-Table Data**

Age x	1959-61 Actual Life-Table	Simulated l_x
9	96801	96753
19	96051	95989
29	94542	94428
39	92705	92576
49	88178	87901
59	77083	76793
69	56384	56186
79	28814	28657

3.4.1 Expectation for Discrete Random Variables

The Binomial random variables which have just been discussed are examples of so-called *discrete random variables*, that is, random variables Z with a discrete (usually finite) list of possible outcomes z, with a corresponding list of probabilities or *probability mass function* values $p_Z(z)$ with which each of those possible outcomes occur. (These probabilities $p_Z(z)$ must be positive numbers which summed over all possible values z add to 1.) In an insurance context, think for example of Z as the unforeseeable future damage or liability upon the basis of which an insurer has to pay some scheduled *claim amount* $c(Z)$ to fulfill a specific property or liability insurance policy. The Law of Large Numbers says that we can have a *frequentist* operational interpretation of each of the probabilities $p_Z(z)$ with which a claim of size $c(z)$ is presented. In a large population of N independent policyholders, each governed by the same probabilities $p_Z(\cdot)$ of liability occurrences, for each fixed damage-amount z we can imagine a series of N *Bernoulli*$(p_Z(z))$ trials, in which the j^{th} policyholder is said to result in a 'success' if he sustains a damage amount equal to z, and to result in a 'failure' otherwise. The Law of Large Numbers for these Bernoulli trials says that the number out of these N policyholders who do sustain damage z is for large N extremely likely to differ by no more than δN from $N\, p_Z(z)$.

Returning to a general discussion, suppose that Z is a discrete random variable with a finite list of possible values $z_1, \ldots, z_m$, and let $c(\cdot)$ be a real-valued (nonrandom) cost function such that $c(Z)$ represents an economically meaningful cost incurred when the random variable value Z is given. Suppose that a large number N of independent individuals give rise to respective values Z_j, $j = 1, \ldots, N$ and costs $c(Z_1), \ldots, c(Z_N)$. Here *independent* means that the mechanism causing different individual Z_j values is such that information about the values $Z_1, \ldots, Z_{j-1}$ allows no change in the (conditional) probabilities with which Z_j takes on its values, so that for all j, i, and $b_1, \ldots, b_{j-1}$,

$$P(Z_j = z_i \,|\, Z_1 = b_1, \ldots, Z_{j-1} = b_{j-1}\,) = p_Z(z_i)$$

Then the Law of Large Numbers, applied as above, says that out of the large number N of individuals it is extremely likely that approximately $p_Z(k) \cdot N$ will have their Z variable values equal to k, where k ranges over $\{z_1, \ldots, z_m\}$. It follows that the average costs $c(Z_j)$ over the N independent individuals — which can be expressed exactly as

$$N^{-1} \sum_{j=1}^{N} c(Z_j) \;=\; N^{-1} \sum_{i=1}^{m} c(z_i) \cdot \#\{j = 1, \ldots, N : \; Z_j = z_i\}$$

— is approximately given by

$$N^{-1} \sum_{i=1}^{m} c(z_i) \cdot (N\, p_Z(z_i)) = \sum_{i=1}^{m} c(z_i)\, p_Z(z_i)$$

In other words, the Law of Large Numbers implies that the **average cost per trial** among the N independent trials resulting in random variable values Z_j and corresponding costs $c(Z_j)$ has a well-defined approximate (actually, a limiting) value for very large N

$$\textbf{Expectation of cost} \;=\; E(c(Z)) = \sum_{i=1}^{m} c(z_i)\, p_Z(z_i) \tag{3.16}$$

As an application of the formula for expectation of a discrete random variable, consider the expected value of a cost-function $g(T)$ of a lifetime random variable which is assumed to depend on T only through the function

$g([T])$ of the integer part of T. This expectation was interpreted earlier as the average cost over all members of the specified life-table cohort. Now the expectation can be verified to coincide with the life-table average previously given, if the probabilities $S(j)$ in the following expression are replaced by the life-table estimators l_j/l_0. Since $P([T]=k) = S(k) - S(k+1)$, the general expectation formula (3.16) yields

$$E(g(T)) = E(g([T]) = \sum_{k=0}^{\omega-1} g(k)\,(S(k) - S(k+1))$$

agreeing precisely with formula (1.2).

Just as we did in the context of expectations of functions of the life-table waiting-time random variable T , we can interpret the *Expectation* as a weighted average of values (costs, in this discussion) which can be incurred in each trial, weighted by the probabilities with which they occur. There is an analogy in the continuous-variable case, where Z would be a random variable whose approximate probabilities of falling in tiny intervals $[z, z+dz]$ are given by $f_Z(z)dz$, where $f_Z(z)$ is a nonnegative density function integrating to 1. In this case, the weighted average of cost-function values $c(z)$ which arise when $Z \in [z, z+dz]$, with approximate probability-weights $f_Z(z)dz$, is written as a limit of sums or an integral, namely $\int c(z)\, f(z)\, dz$.

3.4.2 Rules for Manipulating Expectations

We have separately defined *expectation* for continuous and discrete random variables. In the continuous case, we treated the expectation of a specified function $g(T)$ of a lifetime random variable governed by the survival function $S(x)$ of a cohort life-table, as the approximate numerical average of the values $g(T_i)$ over all individuals i with data represented through observed lifetime T_i in the life-table. The discrete case was handled more conventionally, along the lines of a 'frequentist' approach to the mathematical theory of probability. First, we observed that our calculations with $Binomial(n,p)$ random variables justified us in saying that the sum $X = X_n$ of a large number n of independent coin-toss variables $\epsilon_1, \ldots, , \epsilon_n$, each of which is 1 with probability p and 0 otherwise, has a value which with very high probability differs from $n \cdot p$ by an amount smaller than δn, where $\delta > 0$ is an arbitrarily small number not depending upon n. The *Expectation* p of each of the variables ϵ_i is recovered approximately as the numerical average $X/n = n^{-1}\sum_{i=1}^{n} \epsilon_i$ of the independent outcomes ϵ_i of independent trials. This Law of Large Numbers extends to arbitrary sequences of independent and identical finite-valued discrete random variables, saying that

> if $Z_1, Z_2, \ldots$ are independent random variables, in the sense that for all $k \geq 2$ and all numbers r,
>
> $$P(Z_k \leq r \,|\, Z_1 = z_1, \ldots, Z_{k-1} = z_{k-1}) = P(Z_1 \leq r)$$
>
> regardless of the precise values $z_1, \ldots, z_{k-1}$, then for each $\delta > 0$, as n gets large
>
> $$P\Big(|n^{-1}\sum_{i=1}^{n} c(Z_i) - E(c(Z_1))| \geq \delta\Big) \longrightarrow 0 \qquad (3.17)$$
>
> where, in terms of the finite set S of possible values of Z,
>
> $$E(c(Z_1)) = \sum_{z\in S} c(z)\, P(Z_1 = z) \qquad (3.18)$$

Although we do not give any further proof here, it is a fact that the same **Law of Large Numbers** given in equation (3.17) continues to hold if the definition of *independent* sequences of random variables Z_i is suitably generalized, as long as either

> Z_i are discrete with infinitely many possible values defining a set S, and the expectation is as given in equation (3.18) above whenever the function $c(z)$ is such that
>
> $$\sum_{z\in S} |c(z)|\, P(Z_1 = z) \;<\; \infty$$

or

> the independent random variables Z_i are *continuous*, all with the same density $f(t)$ such that $P(q \leq Z_1 \leq r) = \int_q^r f(t)\,dt$, and expectation is defined by
>
> $$E(c(Z_1)) \;=\; \int_{-\infty}^{\infty} c(t)\, f(t)\, dt \qquad (3.19)$$

whenever the function $c(t)$ is such that

$$\int_{-\infty}^{\infty} |c(t)|\, f(t)\, dt \;<\; \infty$$

All of this serves to indicate that there really is no choice in coming up with an appropriate definition of expectations of cost-functions defined in terms of random variables Z, whether discrete or continuous. For the rest of these lectures, and more generally in applications of probability within actuarial science, we are interested in evaluating expectations of various functions of random variables related to the contingencies and uncertain duration of life. Many of these expectations concern superpositions of random amounts to be paid out after random durations. The following rules for the manipulation of expectations arising in such superpositions considerably simplify the calculations. Assume throughout the following that all payments and times which are not certain are functions of a single lifetime random variable T.

(1). If a payment consists of a nonrandom multiple (e.g., face-amount F) times a random amount $c(T)$, then the expectation of the payment is the product of F and the expectation of $c(T)$:

$$\textit{Discrete case:}\quad E(Fc(T)) = \sum_t F\, c(t)\, P(T=t)$$

$$=\; F \sum_t c(t)\, P(T=t) \;=\; F\cdot E(c(T))$$

$$\textit{Continuous case:}\quad E(Fc(T)) = \int F\, c(t) f(t)\, dt = F \int c(t) f(t)\, dt = F\cdot E(c(T))$$

(2). If a payment consists of the sum of two separate random payments $c_1(T)$, $c_2(T)$ (which may occur at different times, taken into account by treating both terms $c_k(T)$ as present values as of the same time), then the overall payment has expectation which is the sum of the expectations of the separate payments:

$$\textit{Discrete case:}\quad E(c_1(T)+c_2(T)) = \sum_t (c_1(t)+c_2(t))\, P(T=t)$$

$$= \sum_t c_1(t)\, P(T=t) + \sum_t c_2(t)\, P(T=t) = E(c_1(T)) + E(c_2(T))$$

$$\textit{Continuous case:} \quad E(c_1(T) + c_2(T)) = \int (c_1(t) + c_2(t))\, f(t)\, dt$$

$$= \int c_1(t)\, f(t)\, dt + \int c_2(t)\, f(t)\, dt = E(c_1(T)) + E(c_2(T))$$

Thus, if an uncertain payment under an insurance-related contract, based upon a continuous lifetime variable T with density f_T, occurs only if $a \le T < b$ and in that case consists of a payment of a fixed amount F occurring at a fixed time h, then the expected present value under a fixed nonrandom interest-rate i with $v = (1+i)^{-1}$, becomes by rule **(1)** above,

$$E(v^h\, F\, I_{[a \le T < b]}) = v^h\, F\, E(I_{[a \le T < b]})$$

where the indicator-notation $I_{[a \le T < b]}$ denotes a random quantity which is 1 when the condition $[a \le T < b]$ is satisfied and is 0 otherwise. Since an indicator random variable has the two possible outcomes $\{0, 1\}$ like the coin-toss variables ϵ_i above, we conclude that $E(I_{[a \le T < b]}) = P(a \le T < b) = \int_a^b f_T(t)\, dt$, and the expected present value above is

$$E(v^h\, F\, I_{[a \le T < b]}) = v^h\, F \int_a^b f_T(t)\, dt$$

3.5 Some Special Integrals

While actuaries ordinarily do not allow themselves to represent real life-table survival distributions by simple finite-parameter families of theoretical distributions (for the good reason that they never approximate the real large-sample life-table data well enough), it is important for the student to be conversant with several integrals which would arise by substituting some of the theoretical models into formulas for various net single premiums and expected lifetimes.

Consider first the *Gamma* functions and integrals arising in connection with Gamma survival distributions. The *Gamma* function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) \;=\; \int_0^\infty x^{\alpha-1}\, e^{-x}\, dx \;\;, \qquad \alpha > 0$$

This integral is easily checked to be equal to 1 when $\alpha = 1$, giving the total probability for an exponentially distributed random variable, i.e., a lifetime with constant force-of-mortality 1. For $\alpha = 2$, the integral is the expected value of such a unit-exponential random variable, and it is a standard integration-by-parts exercise to check that it too is 1. More generally, integration by parts in the *Gamma* integral with $u = x^\alpha$ and $dv = e^{-x}\,dx$ immediately yields the famous *recursion relation* for the *Gamma* integral, first derived by Euler, and valid for all $\alpha > 0$:

$$\Gamma(\alpha+1) = \int_0^\infty x^\alpha\, e^{-x}\, dx = \left(-x^\alpha\, e^{-x}\right)\Big|_0^\infty + \int_0^\infty \alpha\, x^{\alpha-1}\, e^{-x}\, dx \;=\; \alpha\cdot\Gamma(\alpha)$$

This relation, applied inductively, shows that for all positive integers n,

$$\Gamma(n+1) \;=\; n\cdot(n-1)\cdots 2\cdot\Gamma(1) \;=\; n!$$

The only other simple-to-derive formula explicitly giving values for (non-integer) values of the *Gamma* function is $\Gamma(\,\frac{1}{2}\,) = \sqrt{\pi}$, obtained as follows:

$$\Gamma(\,\frac{1}{2}\,) \;=\; \int_0^\infty x^{-1/2}\, e^{-x} dx \;=\; \int_0^\infty e^{-z^2/2}\,\sqrt{2}\, dz$$

Here we have made the integral substitution $x = z^2/2$, $x^{-1/2}\, dx = \sqrt{2}\, dz$. The last integral can be given by symmetry as

$$\frac{1}{\sqrt{2}}\int_{-\infty}^\infty e^{-z^2/2}\, dz \;=\; \sqrt{\pi}$$

where the last equality is equivalent to the fact (proved in most calculus texts as an exercise in double integration using change of variable to polar coordinates) that the *standard normal distribution*

$$\Phi(x) \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-z^2/2}\, dz \tag{3.20}$$

is a *bona-fide* distribution function with limit equal to 1 as $x \to \infty$.

One of the integrals which arises in calculating expected remaining lifetimes for Weibull-distributed variables is a *Gamma* integral, after integration-by-parts and a change-of-variable. Recall that the *Weibull* density with parameters λ, γ is

$$f(t) \;=\; \lambda\,\gamma\, t^{\gamma-1}\, e^{-\lambda\, t^\gamma} \quad , \qquad t > 0$$

so that $S(x) = \exp(-\lambda\, x^\gamma)$. The expected remaining life for a Weibull-distributed life aged x is calculated, via an integration by parts with $u = t - x$ and $dv = f(t)dt = -S'(t)dt$, as

$$\int_x^\infty (t-x)\,\frac{f(t)}{S(x)}\,dt = \frac{1}{S(x)}\Big[-(t-x)\,e^{-\lambda t^\gamma}\Big|_x^\infty + \int_x^\infty e^{-\lambda t^\gamma}\,dt \Big]$$

The first term in square brackets evaluates to 0 at the endpoints, and the second term can be re-expressed via the change-of-variable $w = \lambda\, t^\gamma$, to give, in the Weibull example,

$$E(T-x\,|\,T\ge x) = e^{\lambda x^\gamma}\,\frac{1}{\gamma}\,\lambda^{-1/\gamma}\int_{\lambda\, x^\gamma}^\infty w^{(1/\gamma)-1}\,e^{-w}\,dw$$

$$= \Gamma(\frac{1}{\gamma})\,e^{\lambda\, x^\gamma}\,\frac{1}{\gamma}\,\lambda^{-1/\gamma}\Big(1 - G_{1/\gamma}(\lambda\, x^\gamma)\Big)$$

where we denote by $G_\alpha(z)$ the *Gamma distribution function* with shape parameter α,

$$G_\alpha(z) \;=\; \frac{1}{\Gamma(\alpha)}\int_0^z v^{\alpha-1}\,e^{-v}\,dv$$

and the integral on the right-hand side is called the *incomplete Gamma function*. Values of $G_\alpha(z)$ can be found either in published tables which are now quite dated, or among the standard functions of many mathematical/statistical computer packages, such as **Mathematica**, **Matlab**, or **Splus**. One particular case of these integrals, the case $\alpha = 1/2$, can be recast in terms of the standard normal distribution function $\Phi(\cdot)$. We change variables by $v = y^2/2$ to obtain for $z \ge 0$,

$$G_{1/2}(z) \;=\; \frac{1}{\Gamma(1/2)}\int_0^z v^{-1/2}\,e^{-v}\,dv \;=\; \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{2z}}\sqrt{2}\,e^{-y^2/2}\,dy$$

$$= \sqrt{\frac{2}{\pi}}\cdot\sqrt{2\pi}\cdot(\Phi(\sqrt{2z}) - \Phi(0)) \;=\; 2\Phi(\sqrt{2z}) - 1$$

One further expected-lifetime calculation with a common type of distribution gives results which simplify dramatically and become amenable to numerical calculation. Suppose that the lifetime random variable T is assumed *lognormally distributed* with parameters m, σ^2. Then the expected remaining lifetime of a life aged x is

$$E(\,T-x\,|\,T\ge x\,) = \frac{1}{S(x)}\int_x^\infty t\,\frac{d}{dt}\Phi(\frac{\log(t)-\log(m)}{\sigma})\,dt \,-\, x$$

Now change variables by $y = (\log(t) - \log(m))/\sigma = \log(t/m)/\sigma$, so that $t = m\, e^{\sigma y}$, and define in particular

$$x' = \frac{\log(x) - \log(m)}{\sigma}$$

Recalling that $\Phi'(z) = \exp(-z^2/2)/\sqrt{2\pi}$, we find

$$E(\, T - x \,|\, T \geq x \,) \;=\; \frac{1}{1 - \Phi(x')} \int_{x'}^{\infty} \frac{m}{\sqrt{2\pi}}\, e^{\sigma y - y^2/2}\, dy$$

The integral simplifies after completing the square $\sigma y - y^2/2 = \sigma^2/2 - (y - \sigma)^2/2$ in the exponent of the integrand and changing variables by $z = y - \sigma$. The result is:

$$E(\, T - x \,|\, T \geq x \,) \;=\; \frac{m e^{\sigma^2/2}}{1 - \Phi(x')}\, (1 - \Phi(x' - \sigma))$$

3.6 *Exercise Set 3*

(1). Show that: $\quad \frac{\partial}{\partial x}\, {}_tp_x \;=\; {}_tp_x \cdot (\mu_x - \mu_{x+t})$.

(2). For a certain value of x, it is known that ${}_tq_x = kt$ over the time-interval $t \in [0, 3]$, where k is a constant. Express μ_{x+2} as a function of k.

(3). Suppose that an individual aged 20 has random lifetime Z with continuous density function

$$f_Z(t) \;=\; 0.02\,(t - 20)\, e^{-(t-20)^2/100} \quad , \qquad t > 20$$

(a) If this individual has a contract with your company that you must pay his heirs $\$10^6 \cdot (1.4 - Z/50)$ on the date of his death between ages 20 and 70, then what is the expected payment ?

(b) If the value of the death-payment described in (a) should properly be discounted by the factor $\exp(-0.08(Z - 20))$ (i.e. by the effective interest rate of $e^{.08} - 1$ per year) to calculate the present value of the payment, then what is the expected present value of the insurance contract ?

Hint for both parts: After a change of variables, the integral in (a) can be evaluated in terms of incomplete Gamma integrals $\int_c^\infty s^{\alpha-1}\, e^{-s}\, ds$, where the complete Gamma integrals (for c=0) are known to yield the **Gamma function** $\Gamma(\alpha) = (\alpha-1)!$, for integer $\alpha > 0$. **Also:** $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ for all real > 0, and $\Gamma(1/2) = \sqrt{\pi}$.

(4). Suppose that a life-table mortality pattern is this: from ages 20 through 60, twice as many lives die in each 5-year period as in the previous five-year period. Find the probability that a life aged 20 will die between exact ages 40 and 50. If the force of mortality can be assumed constant over each five-year age period (20-24, 25-29, etc.), and if you are told that $l_{60}/l_{20} = 0.8$, then find the probability that a life aged 20 will survive at least until exact age 48.0 .

(5). Obtain an expression for μ_x if $l_x = k\, s^x\, w^{x^2}\, g^{c^x}$, where $k,\ s,\ w,\ g,\ c$ are positive constants.

(6). Show that: $\int_0^\infty l_{x+t}\, \mu_{x+t}\, dt \;=\; l_x$.

(7). A man wishes to accumulate $\$50,000$ in a fund at the end of 20 years. If he deposits $\$1000$ in the fund at the end of each of the first 10 years and $\$1000+x$ in the fund at the end of each of the second 10 years, then find x to the nearest dollar, where the fund earns an effective interest rate of 6% .

(8). Express in terms of annuity-functions $a^{(m)}_{\overline{N}|}$ the present value of an annuity of $100 per month paid the first year, $200 per month for the second year, up to $1000 per month the tenth year. Find the numerical value of the present value if the effective annual interest rate is 7% .

(9). Find upper bounds for the following Binomial probabilities, and compare them with the exact values calculated via computer (e.g., using a spreadsheet or exact mathematical function such as **pbinom** in **Splus**) :

(a). The probability that in *Bernoulli* trials with success-probability 0.4, the number of successes lies outside the (inclusive) range [364, 446].

(b). The probability that of 1650 lives aged exactly 45, for whom ${}_{20}p_{45} = 0.72$, no more than 1075 survive to retire at age 65.

(10). If the force of mortality governing a cohort life-table is such that

$$\mu_t \;=\; \frac{2}{1+t} + \frac{2}{100-t} \quad \text{for real } t\,, \qquad 0 < t < 100$$

then find the number of deaths which will be expected to occur between ages 1 and 4, given that the radix l_0 of the life-table is $10,000$.

(11). Find the expected present value at 5% APR of an investment whose proceeds will with probability $1/2$ be a payment of \$10,000 in exactly 5 years, and with the remaining probability $1/2$ will be a payment of \$20,000 in exactly 10 years.
Hint: calculate the respective present values V_1, V_2 *of the payments in each of the two events with probability* 0.5, *and find the expected value of a discrete random variable which has values* V_1 *or* V_2 *with probabilities* 0.5 *each.*

3.7 Worked Examples

Example 1. Assume that a cohort life-table population satisfies $l_0 = 10^4$ *and*

$$d_x \;=\; \begin{cases} 200 & \textit{for} \quad 0 \le x \le 14 \\ 100 & \textit{for} \quad 15 \le x \le 48 \\ 240 & \textit{for} \quad 49 \le x \le 63 \end{cases}$$

(a) Suppose that an insurer is to pay an amount $\$100 \cdot (64 - X)$ *(without regard to interest or present values related to the time-deferral of the payment) for a newborn in the life-table population, if* X *denotes the attained integer age at death. What is the expected amount to be paid ?*

(b) Find the expectation requested in (a) if the insurance is purchased for a life currently aged exactly 10 *.*

(c) Find the expected present value at 4% *interest of a payment of* \$1000 *to be made at the end of the year of death of a life currently aged exactly* 20.

The first task is to develop an expression for survival function and density governing the cohort life-table population. Since the numbers of deaths are constant over intervals of years, the survival function is piecewise linear, and the life-distribution is *piecewise uniform* because the the density is piecewise constant. Specifically for this example, at integer values y,

$$l_y \;=\; \begin{cases} 10000 - 200y & \text{for} \;\; 0 \le y \le 15 \\ 7000 - 100(y - 15) & \text{for} \;\; 16 \le y \le 49 \\ 3600 - 240(y - 49) & \text{for} \;\; 50 \le y \le 64 \end{cases}$$

It follows that the terminal age for this population is $\omega = 64$ for this population, and $S(y) = 1 - 0.02\, y$ for $0 \le y \le 15$, $0.85 - 0.01\, y$ for $15 \le y \le 49$, and $1.536 - .024\, y$ for $49 \le y \le 64$. Alternatively, extending the function S linearly, we have the survival density $f(y) = -S'(y) = 0.02$ on $[0, 15)$, $= 0.01$ on $[15, 49)$, and $= 0.024$ on $[49, 64]$.

Now the expectation in (a) can be written in terms of the random lifetime variable with density f as

$$\int_0^{15} 0.02 \cdot 100 \cdot (64 - [y])\, dy \;+\; \int_{15}^{49} 0.01 \cdot 100 \cdot (64 - [y])\, dy$$

$$+ \int_{49}^{64} 0.024 \cdot 100 \cdot (64 - [y])\, dy$$

The integral has been written as a sum of three integrals over different ranges because the analytical form of the density f in the expectation-formula $\int g(y) f(y) dy$ is different on the three different intervals. In addition, observe that the integrand (the function $g(y) = 100(64 - [y])$ of the random lifetime Y whose expectation we are seeking) itself takes a different analytical form on successive one-year age intervals. Therefore the integral just displayed can immediately be seen to agree with the summation formula for the expectation of the function $100(64 - X)$ for the integer-valued random variable X whose probability mass function is given by

$$P(X = k) \;=\; d_k / l_0$$

The formula is

$$E(g(Y)) \;=\; E(100(64 - X)) \;=\; \sum_{k=0}^{14} 0.02 \cdot 100 \cdot (64 - k) \; +$$

$$\sum_{k=15}^{48} 0.01 \cdot 100 \cdot (64 - k) \; + \; \sum_{k=49}^{63} 0.024 \cdot 100 \cdot (64 - k)$$

Thus the solution to (a) is given (after the change-of-variable $j = 64 - k$), by

$$2.4 \sum_{j=1}^{15} j \; + \; \sum_{j=16}^{49} j \; + \; 2 \sum_{j=50}^{64} j$$

The displayed expressions can be summed either by a calculator program or by means of the easily-checked formula $\sum_{j=1}^{n} j \;=\; j(j+1)/2$ to give the numerical answer \$3103 .

The method in part (b) is very similar to that in part (a), except that we are dealing with conditional probabilities of lifetimes given to be at least 10 years long. So the summations now begin with $k = 10$, or alternatively end with $j = 64 - k = 54$, and the denominators of the conditional probabilities $P(X = k | X \geq 10)$ are $l_{10} = 8000$. The expectation in (b) then becomes

$$\sum_{k=10}^{14} \frac{200}{8000} \cdot 100 \cdot (64 - k) + \sum_{k=15}^{48} \frac{100}{8000} \cdot 100 \cdot (64 - k) + \sum_{k=49}^{63} \frac{240}{8000} \cdot 100 \cdot (64 - k)$$

which works out to the numerical value

$$3.0 \sum_{1}^{15} j \,+\, 1.25 \sum_{16}^{49} j \,+\, 2.5 \sum_{50}^{54} j \;=\; \$2391.25$$

Finally, we find the expectation in (c) as a summation beginning at $k = 20$ for a function $1000 \cdot (1.04)^{-X+19}$ of the random variable X with conditional probability distribution $P(X = k | X \geq 20) = d_k / l_{20}$ for $k \geq 20$. (Note that the function 1.04^{-X+19} is the present value of a payment of 1 at the end of the year of death, because the end of the age- X year for an individual currently at the 20^{th} birthday is $X - 19$ years away.) Since $l_{20} = 6500$, the answer to part (c) is

$$1000 \Big\{ \sum_{k=20}^{48} \frac{100}{6500} (1.04)^{19-k} \,+\, \sum_{k=49}^{63} \frac{240}{6500} (1.04)^{19-k} \Big\}$$

$$=\; 1000 \Big(\frac{1}{65} \frac{1 - 1.04^{-29}}{0.04} \,+\, \frac{24}{650} 1.04^{-29} \frac{1 - (1.04)^{-15}}{0.04} \Big) \;=\; 392.92$$

Example 2. Find the change in the expected lifetime of a cohort life-table population governed by survival function $S(x) = 1 - (x/\omega)$ *for* $0 \leq x \leq \omega$ *if* $\omega = 80$ *and*

(a) the force of mortality $\mu(y)$ *is multiplied by* 0.9 *at all exact ages* $y \geq 40$, *or*

(b) the force of mortality $\mu(y)$ *is decreased by the constant amount* 0.1 *at all ages* $y \geq 40$.

The force of mortality here is

$$\mu(y) \;=\; -\frac{d}{dy} \ln(1 - y/80) \;=\; \frac{1}{80 - y}$$

So multiplying it by 0.9 at ages over 40 changes leaves unaffected the density of $1/80$ for ages less than 40, and for ages y over 40 changes the density from $f(y) = 1/80$ to

$$f^*(y) \;=\; -\frac{d}{dy} \Big(S(40) \exp(-0.9 \int_{40}^{y} (80 - z)^{-1} \, dz) \Big)$$

$$= -\frac{d}{dy}\Big(0.5\, e^{0.9\,\ln((80-y)/40)}\Big) \;=\; -0.5\,\frac{d}{dy}\left(\frac{80-y}{40}\right)^{0.9}$$

$$= \frac{0.9}{80}\,(2-y/40)^{-0.1}$$

Thus the expected lifetime changes from $\int_0^{80} (y/80)\, dy = 40$ to

$$\int_0^{40} (y/80)\, dy \;+\; \int_{40}^{80} y\,\frac{0.9}{80}\,(2-y/40)^{-0.1}\, dy$$

Using the change of variable $z = 2 - y/40$ in the last integral gives the expected lifetime $= 10 + .45(80/.9 - 40/1.9) = 40.53$.

Example 3. Suppose that you have available to you two investment possibilities, into each of which you are being asked to commit \$5000. *The first investment is a risk-free bond (or bank savings-account) which returns compound interest of* 5% *for a* 10*-year period. The second is a 'junk bond' which has probability* 0.6 *of paying* 11% *compound interest and returning your principal after 10 years, probability* 0.3 *of paying yearly interest at* 11% *for* 5 *years and then returning your principal of* \$5000 *at the end of the* 10^{th} *year with no further interest payments, and probability* 0.1 *of paying yearly interest for* 3 *years at* 11% *and then defaulting, paying no more interest and not returning the principal. Suppose further that the going rate of interest with respect to which present values should properly be calculated for the next* 10 *years will either be* 4.5% *or* 7.5%, *each with probability* 0.5. *Also assume that the events governing the junk bond's paying or defaulting are independent of the true interest rate's being* 4.5% *versus* 7.5% *for the next* 10 *years. Which investment provides the better expected return in terms of current (time-*0*) dollars ?*

There are six relevant events, named and displayed along with their probabilities in the following table, corresponding to the possible combinations of true interest rate (Low versus High) and payment scenarios for the junk bond (Full payment, Partial interest payments with return of principal, and Default after 3 years' interest payments):

Event Name	Description	Probability
A_1	Low $\cap$ Full	0.30
A_2	Low $\cap$ Partial	0.15
A_3	Low $\cap$ Default	0.05
A_4	High $\cap$ Full	0.30
A_5	High $\cap$ Partial	0.15
A_6	High $\cap$ Default	0.05

Note that because of independence (first defined in Section 1.1), the probabilities of intersected events are calculated as the products of the separate probabilities, e.g.,

$$P(A_2) \;=\; P(Low) \cdot P(Partial) \;=\; (0.5) \cdot (0.30) \;=\; 0.15$$

Now, under each of the events A_1, A_2, A_3, the present value of the first investment (the risk-free bond) is

$$5000 \left\{ \sum_{k=1}^{10} 0.05\,(1.045)^{-k} \;+\; (1.045)^{-10} \right\} \;=\; 5197.82$$

On each of the events A_4, A_5, A_6, the present value of the first investment is

$$5000 \left\{ \sum_{k=1}^{10} 0.05\,(1.075)^{-k} \;+\; (1.075)^{-10} \right\} \;=\; 4141.99$$

Thus, since

$$P(Low) \;=\; P(A_1 \cup A_2 \cup A_3) \;=\; P(A_1) + P(A_2) + P(A_3) \;=\; 0.5$$

the overall expected present value of the first investment is

$$0.5 \cdot (5197.82 + 4141.99) \;=\; 4669.90$$

Turning to the second investment (the junk bond), denoting by PV the present value considered as a random variable, we have

$$E(PV \,|\, A_1)/5000 \;\;=\;\; 0.11 \sum_{k=1}^{10} (1.045)^{-k} \;+\; (1.045)^{-10} \;=\; 1.51433$$

$$\begin{aligned}
E(PV \,|\, A_4)/5000 &= 0.11 \sum_{k=1}^{10} (1.075)^{-k} + (1.075)^{-10} = 1.24024 \\
E(PV \,|\, A_2)/5000 &= 0.11 \sum_{k=1}^{5} (1.045)^{-k} + (1.045)^{-10} = 1.12683 \\
E(PV \,|\, A_5)/5000 &= 0.11 \sum_{k=1}^{5} (1.075)^{-k} + (1.075)^{-10} = 0.93024 \\
E(PV \,|\, A_3)/5000 &= 0.11 \sum_{k=1}^{3} (1.045)^{-k} = 0.302386 \\
E(PV \,|\, A_6)/5000 &= 0.11 \sum_{k=1}^{3} (1.075)^{-k} = 0.286058
\end{aligned}$$

Therefore, we conclude that the overall expected present value $E(PV)$ of the second investment is

$$\sum_{i=1}^{6} E(PV \cdot I_{A_i}) = \sum_{i=1}^{6} E(PV|A_i)\, P(A_i) = 5000 \cdot (1.16435) = 5821.77$$

So, although the first-investment is 'risk-free', it does not keep up with inflation in the sense that its present value is not even as large as its starting value. The second investment, risky as it is, nevertheless beats inflation (i.e., the expected present value of the accumulation after 10 years is greater than the initial face value of \$5000) although with probability $P(\mathit{Default}) = 0.10$ the investor may be so unfortunate as to emerge (in present value terms) with only 30% of his initial capital.

3.8 Useful Formulas from Chapter 3

$$ {}_kp_x \;=\; p_x\, p_{x+1}\, p_{x+2}\, \cdots\, p_{x+k-1} \quad , \qquad k \ge 1 \quad \text{integer} $$

p. 61

$$ {}_{k/m}p_x \;=\; \prod_{j=0}^{k-1} {}_{1/m}p_{x+j/m} \quad , \qquad k \ge 1 \quad \text{integer} $$

p. 61

(i) Piecewise Unif.. $S(k{+}t) \;=\; tS(k{+}1){+}(1{-}t)S(k)$, k integer , $t \in [0,1]$

p. 64

(ii) Piecewise Const. $\mu(y)$ $\ln S(k{+}t) \;=\; t\ln S(k{+}1){+}(1{-}t)\ln S(k)$, k integer

p. 64

(iii) Balducci assump. $\dfrac{1}{S(k+t)} \;=\; \dfrac{t}{S(k+1)} \;+\; \dfrac{1-t}{S(k)}$, k integer

p. 64

$$ {}_tp_k \;=\; \frac{S(k) - t(S(k+1) - S(k))}{S(k)} \;=\; 1 - t\, q_k \qquad \text{under (i)} $$

p. 66

$$ {}_tp_k \;=\; \frac{S(k+t)}{S(x)} \;=\; \left(e^{-\mu(k)}\right)^t \;=\; (1 - q_k)^t \qquad \text{under (ii)} $$

p. 66

$$ {}_tp_k \;=\; \frac{S(k+t)}{S(k+1)}\, \frac{S(k+1)}{S(k)} \;=\; \frac{1 - q_k}{1 - (1-t)q_k} \qquad \text{under (iii)} $$

p. 66

Binomial(N, p) probability $$P(X = k) = \binom{N}{k} p^k \, (1-p)^{N-k}$$

p. 68

Discrete r.v. Expectation $$E(c(Z)) = \sum_{i=1}^{m} c(z_i) \, p_Z(z_i)$$

p. 77

$$\Gamma(\alpha) \;=\; \int_0^\infty x^{\alpha-1} \, e^{-x} \, dx$$

p. 82

$$\Phi(x) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} \, dz$$

p. 82

Chapter 4

Expected Present Values of Insurance Contracts

We are now ready to draw together the main strands of the development so far: (i) expectations of discrete and continuous random variables defined as functions of a life-table waiting time T until death, and (ii) discounting of future payment (streams) based on interest-rate assumptions. The approach is first to define the contractual terms of and discuss relations between the major sorts of insurance, endowment and life annuity contracts, and next to use interest theory to define the present value of the contractual payment stream *by the insurer* as a nonrandom function of the random individual lifetime T. In each case, this leads to a formula for the *expected* present value of the payout by the insurer, an amount called the **net single premium** or **net single risk premium** of the contract because it is the single cash payment by the insured at the beginning of the insurance period which would exactly compensate for the average of the future payments which the insurer will have to make.

The details of the further mathematical discussion fall into two parts: first, the specification of formulas in terms of cohort life-table quantities for net single premiums of insurances and annuities which pay only at whole-year intervals; and second, the application of the various survival assumptions concerning interpolation between whole years of age, to obtain the corresponding formulas for insurances and annuities which have m payment times per year. We close this Chapter with a discussion of instantaneous-payment insurance, continuous-payment annuity, and mean-residual-life formulas, all of which involve continuous-time expectation integrals. We also relate these expectations with their m-payment-per-year discrete analogues, and compare the corresponding integral and summation formulas.

Similar and parallel discussions can be found in the *Life Contingencies* book of Jordan (1967) and the *Actuarial Mathematics* book of Bowers et al. (1986). The approach here differs in unifying concepts by discussing together all of the different contracts, first in the whole-year case, next under interpolation assumptions in the m-times-per-year case, and finally in the instantaneous case.

4.1 Expected Present Values of Payments

Throughout the Chapter and from now on, it is helpful to distinguish notationally the expectations relating to present values of life insurances and annuities *for a life aged* x. Instead of the notation $E(g(T) \,|\, T \geq x)$ for expectations of functions of life-length random variables, we define

$$\mathcal{E}_x \, (\, g(T) \,) \;=\; E(g(T) \mid T \geq x)$$

The expectations formulas can then be written in terms of the residual-lifetime variable $S = T - x$ (or the change-of-variable $s = t - x$) as follows:

$$\mathcal{E}_x \, (\, g(T) \,) \;=\; \int_x^\infty g(t) \, \frac{f(t)}{S(x)} \, dt \;=\; \int_x^\infty g(t) \, \frac{\partial}{\partial t} \left(- \frac{S(t)}{S(x)} \right) dt$$

$$= \int_0^\infty g(s+x) \, \frac{\partial}{\partial s} \, (- \, {}_s p_x) \, ds \;=\; \int_0^\infty g(s+x) \, \mu(s+x) \, {}_s p_x \, ds$$

4.1.1 Types of Insurance & Life Annuity Contracts

There are three types of contracts to consider: insurance, life annuities, and endowments. More complicated kinds of contracts — which we do not discuss in detail — can be obtained by combining (superposing or subtracting) these in various ways. A further possibility, which we address in Chapter 10, is to restrict payments to some further contingency (e.g., death-benefits only under specified cause-of-death).

In what follows, we adopt several uniform notations and assumptions. Let x denote the initial age of the holder of the insurance, life annuity, or endowment contract, and assume for convenience that the contract is initiated on the holder's birthday. Fix a nonrandom effective (i.e., APR) interest rate i, and retain the notation $v = (1+i)^{-1}$, together with the other notations previously discussed for annuities of nonrandom duration. Next, denote by m the number of payment-periods per year, all times being measured from the date of policy initiation. Thus, for given m, insurance will pay off at the end of the fraction $1/m$ of a year during which death occurs, and life-annuities pay regularly m times per year until the annuitant dies. The *term* or *duration* n of the contract will always be assumed to be an integer multiple of $1/m$. Note that policy durations are all measured from policy initiation, and therefore are exactly x smaller than the exact age of the policyholder at termination.

The random exact age at which the policyholder dies is denoted by T, and all of the contracts under discussion have the property that T is the only random variable upon which either the amount or time of payment can depend. *We assume further that the payment amount depends on the time T of death only through the attained age T_m measured in multiples of $1/m$ year.* As before, the survival function of T is denoted $S(t)$, and the density either $f(t)$. The probabilities of the various possible occurrences under the policy are therefore calculated using the conditional probability distribution of T given that $T \geq x$, which has density $f(t)/S(x)$ at all times $t \geq x$. Define from the random variable T the related discrete random variable

$$T_m \;=\; \frac{[Tm]}{m} \;=\; \text{age at beginning of } \quad \frac{1}{m}th \quad \text{of year of death}$$

which for integer initial age x is equal to $x + k/m$ whenever $x + k/m \leq T < x + (k+1)/m$. Observe that the probability mass function of this random variable is given by

$$P(T_m = x + \frac{k}{m} \,\Big|\, T \geq x) \;=\; P(\frac{k}{m} \leq T - x < \frac{k+1}{m} \,\Big|\, T \geq x)$$

$$=\; \frac{1}{S(x)} \left[S(x + \frac{k}{m}) - S(x + \frac{k+1}{m}) \right] \;=\; {}_{k/m}p_x \;-\; {}_{(k+1)/m}p_x$$

$$= \; P(T \geq x + \frac{k}{m} \Big| T \geq x) \cdot P(T < x + \frac{k+1}{m} \Big| T \geq x + \frac{k}{m}) \qquad (4.1)$$

$$= \; {}_{k/m}p_x \; \cdot \; {}_{1/m}q_{x+k/m}$$

As has been mentioned previously, a key issue in understanding the special nature of life insurances and annuities with multiple payment periods is to understand how to calculate or interpolate these probabilities from the probabilities ${}_jp_y$ (for integers j, y) which can be deduced or estimated from life-tables.

An **Insurance** contract is an agreement to pay a *face amount* — perhaps modified by a specified function of the time until death — if the insured, a life aged x, dies at any time during a specified period, the *term* of the policy, with payment to be made at the end of the $1/m$ year within which the death occurs. Usually the payment will simply be the face amount $F(0)$, but for example in *decreasing term* policies the payment will be $F(0) \cdot (1 - \frac{k-1}{nm})$ if death occurs within the k^{th} successive fraction $1/m$ year of the policy, where n is the term. (The insurance is said to be a *whole-life* policy if $n = \infty$, and a *term insurance* otherwise.) The general form of this contract, for a specified term $n \leq \infty$, payment-amount function $F(\cdot)$, and number m of possible payment-periods per year, is to

> pay $F(T - x)$ at time $T_m - x + \frac{1}{m}$ following policy initiation,
> if death occurs at T between x and $x + n$.

The present value of the insurance company's payment under the contract is evidently

$$\begin{cases} F(T-x)\, v^{T_m - x + 1/m} & \text{if} \quad x \leq T < x + n \\ 0 & \text{otherwise} \end{cases} \qquad (4.2)$$

The simplest and most common case of this contract and formula arise when the face-amount $F(0)$ is the constant amount paid whenever a death within the term occurs. Then the payment is $F(0)$, with present value $F(0)\, v^{-x+([mT]+1)/m}$, if $x \leq T < x + n$, and both the payment and present value are 0 otherwise. In this case, with $F(0) \equiv 1$, the *net single premium* has the standard notation $A^{(m)1}_{\;\,x:\overline{n|}}$. In the further special case where $m = 1$, the superscript m is dropped, and the net single premium is denoted $A^{1}_{x:\overline{n}|}$. Similarly, when the insurance is whole-life ($n = \infty$), the subscript n and bracket $_{\overline{n}|}$ are dropped.

A **Life Annuity** contract is an agreement to pay a scheduled payment to the policyholder at every interval $1/m$ of a year while the annuitant is alive, up to a maximum number of nm payments. Again the payment amounts are ordinarily constant, but in principle any nonrandom time-dependent schedule of payments $F(k/m)$ can be used, where $F(s)$ is a fixed function and s ranges over multiples of $1/m$. In this general setting, the life annuity contract requires the insurer to

> pay an amount $F(k/m)$ at each time $k/m \le T - x$, up to a maximum of nm payments.

To avoid ambiguity, we adopt the convention that in the finite-term life annuities, *either* $F(0) = 0$ *or* $F(n) = 0$. As in the case of annuities certain (i.e., the nonrandom annuities discussed within the theory of interest), we refer to life annuities with first payment at time 0 as (life) *annuities-due* and to those with first payment at time $1/m$ (and therefore last payment at time n in the case of a finite term n over which the annuitant survives) as (life) *annuities-immediate*. The present value of the insurance company's payment under the life annuity contract is

$$\sum_{k=0}^{(T_m - x)m} F(k/m)\, v^{k/m} \tag{4.3}$$

Here the situation is definitely simpler in the case where the payment amounts $F(k/m)$ are *level* or constant, for then the life-annuity-due payment stream becomes an annuity-due certain (the kind discussed previously under the Theory of Interest) as soon as the random variable T is fixed. Indeed, if we replace $F(k/m)$ by $1/m$ for $k = 0, 1, \ldots, nm - 1$, and by 0 for larger indices k, then the present value in equation (4.3) is $\ddot{a}^{(m)}_{\overline{\min(T_m + 1/m,\, n)}|}$, and its expected present value (= net single premium) is denoted $\ddot{a}^{(m)}_{x:\overline{n}|}$.

In the case of temporary life annuities-immediate, which have payments commencing at time $1/m$ and continuing at intervals $1/m$ either until death or for a total of nm payments, the expected-present value notation is $a^{(m)}_{x:\overline{n}|}$. However, unlike the case of annuities-certain (i.e., nonrandom-duration annuities), one cannot simply multiply the present value of the life annuity-due for fixed T by the discount-factor $v^{1/m}$ in order to obtain the corresponding present value for the life annuity-immediate with the same term n. The difference arises because the payment streams (for the life annuity-due deferred $1/m$ year and the life-annuity immediate) end at the same *time* rather than with the same number of payments when death occurs before time n. The correct conversion-formula is obtained by treating the life annuity-immediate of term n as paying, in all circumstances, a present value of $1/m$ (equal to the cash payment at policy initiation) less than the life annuity-due with term $n + 1/m$. Taking expectations leads to the formula

$$a^{(m)}_{x:\overline{n}|} \;=\; \ddot{a}^{(m)}_{x:\overline{n+1/m}|} \;-\; 1/m \tag{4.4}$$

In both types of life annuities, the superscripts $^{(m)}$ are dropped from the net single premium notations when $m = 1$, and the subscript n is dropped when $n = \infty$.

The third major type of insurance contract is the **Endowment**, which pays a contractual face amount $F(0)$ at the end of n policy years if the policyholder initially aged x survives to age $x + n$. This contract is the simplest, since neither the amount nor the time of payment is uncertain. The pure endowment contract commits the insurer to

pay an amount $F(0)$ at time n if $T \geq x + n$

The present value of the pure endowment contract payment is

$$F(0)\, v^n \qquad \text{if} \quad T \geq x + n, \quad 0 \qquad \text{otherwise} \tag{4.5}$$

The net single premium or expected present value for a pure endowment contract with face amount $F(0) = 1$ is denoted $A_{x:\overline{n}|}^{\;\;\;1}$ or ${}_nE_x$ and is evidently equal to

$$A_{x:\overline{n}|}^{\;\;\;1} \;=\; {}_nE_x \;=\; v^n\, {}_np_x \tag{4.6}$$

The other contract frequently referred to in beginning actuarial texts is the **Endowment Insurance**, which for a life aged x and term n is simply the sum of the pure endowment and the term insurance, both with term n and the same face amount 1. Here the contract calls for the insurer to pay \$1 at time $T_m + \frac{1}{m}$ if $T < n$, and at time n if $T \geq n$

The present value of this contract has the form v^n on the event $[T \geq n]$ and the form $v^{T_m - x + 1/m}$ on the complementary event $[T < n]$. Note that $T_m + 1/m \leq n$ whenever $T < n$. Thus, in both cases, the present value is given by

$$v^{\min(T_m - x + 1/m,\, n)} \tag{4.7}$$

The expected present value of the unit endowment insurance is denoted $A^{(m)}_{x:\overline{n}|}$. Observe (for example in equation (4.10) below) that the notations for the net single premium of the term insurance and of the pure endowment are intended to be mnemonic, respectively denoting the parts of the endowment insurance determined by the expiration of life — and therefore positioning the superscript 1 above the x — and by the expiration of the fixed term, with the superscript 1 in the latter case positioned above the n.

Another example of an insurance contract which does not need separate treatment, because it is built up simply from the contracts already described, is the *n-year deferred insurance.* This policy pays a constant face amount at the end of the time-interval $1/m$ of death, but only if death occurs after time n , i.e., after age $x + n$ for a new policyholder aged precisely x. When the face amount is 1, the contractual payout is precisely the difference between the unit whole-life insurance and the n-year unit term insurance, and the formula for the net single premium is

$$A^{(m)}_x \;-\; A^{(m)1}_{\overline{x:n}|} \tag{4.8}$$

Since this insurance pays a benefit only if the insured survives at least n years, it can alternatively be viewed as an endowment with benefit equal to a whole life insurance to the insured after n years (then aged $x + n$) if the insured lives that long. With this interpretation, the n-year deferred insurance has net single premium $= {}_nE_x \cdot A_{x+n}$. This expected present value must therefore be equal to (4.8), providing the identity:

$$A^{(m)}_x \;-\; A^{(m)1}_{\overline{x:n}|} \;=\; v^n \; {}_np_x \;\cdot\; A_{x+n} \tag{4.9}$$

4.1.2 Formal Relations among Net Single Premiums

In this subsection, we collect a few useful identities connecting the different types of contracts, which hold without regard to particular life-table interpolation assumptions. The first, which we have already seen, is the definition of endowment insurance as the superposition of a constant-face-amount term insurance with a pure endowment of the same face amount and term. In terms of net single premiums, this identity is

$$A^{(m)}_{x:\overline{n}|} \;=\; A^{(m)}{}^{\;1}_{x:\overline{n}|} + A^{(m)}{}_{x:\overline{n}|}^{\quad\;1} \tag{4.10}$$

The other important identity concerns the relation between expected present values of endowment insurances and life annuities. The great generality of the identity arises from the fact that, for a fixed value of the random lifetime T, the present value of the life annuity-due payout coincides with the annuity-due certain. The unit term-n life annuity-due payout is then given by

$$\ddot{a}^{(m)}_{\overline{\min(T_m-x+1/m,\,n)}|} \;=\; \frac{1-v^{\min(T_m-x+1/m,\,n)}}{d^{(m)}}$$

The key idea is that the unit life annuity-due has present value which is a simple linear function of the present value $v^{\min(T_m-x+1/m,n)}$ of the unit endowment insurance. Taking expectations (over values of the random variable T, conditionally given $T \geq x$) in the present value formula, and substituting $A^{(m)}_{x:\overline{n}|}$ as expectation of (4.7), then yields:

$$\ddot{a}^{(m)}_{x:\overline{n}|} \;=\; \mathcal{E}_x\Big(\frac{1-v^{\min(T_m-x+1/m,n)}}{d^{(m)}}\Big) = \frac{1-A^{(m)}_{x:\overline{n}|}}{d^{(m)}} \tag{4.11}$$

where recall that $\mathcal{E}_x(\,\cdot\,)$ denotes the conditional expectation $E(\,\cdot\,|\,T \geq x)$. A more common and algebraically equivalent form of the identity (4.11) is

$$d^{(m)}\,\ddot{a}^{(m)}_{x:\overline{n}|} \;+\; A^{(m)}_{x:\overline{n}|} \;=\; 1 \tag{4.12}$$

To obtain a corresponding identity relating net single premiums for life annuities-immediate to those of endowment insurances, we appeal to the conversion-formula (4.4), yielding

$$a^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x:\overline{n+1/m}|} - \frac{1}{m} = \frac{1-A^{(m)}_{x:\overline{n+1/m}|}}{d^{(m)}} - \frac{1}{m} = \frac{1}{i^{(m)}} - \frac{1}{d^{(m)}}\,A^{(m)}_{x:\overline{n+1/m}|} \tag{4.13}$$

and

$$d^{(m)}\, \ddot{a}^{(m)}_{x:\overline{n}|} + A^{(m)}_{x:\overline{n+1/m}|} \;=\; \frac{d^{(m)}}{i^{(m)}} \;=\; v^{1/m} \tag{4.14}$$

In these formulas, we have made use of the definition

$$\frac{m}{d^{(m)}} \;=\; (1 + \frac{i^{(m)}}{m}) \Big/ (\frac{i^{(m)}}{m})$$

leading to the simplifications

$$\frac{m}{d^{(m)}} \;=\; \frac{m}{i^{(m)}} + 1 \quad , \qquad \frac{i^{(m)}}{d^{(m)}} \;=\; 1 + \frac{i^{(m)}}{m} \;= v^{-1/m}$$

4.1.3 Formulas for Net Single Premiums

This subsection collects the expectation-formulas for the insurance, annuity, and endowment contracts defined above. Throughout this Section, the same conventions as before are in force (integer x and n, fixed m, i, and conditional survival function ${}_tp_x$) .

First, the expectation of the present value (4.2) of the random term insurance payment (with level face value $F(0) \equiv 1$) is

$$A^{1}_{x:\overline{n}|} \;=\; \mathcal{E}_x \left(v^{T_m - x + 1/m} \right) \;=\; \sum_{k=0}^{nm-1} v^{(k+1)/m} \;{}_{k/m}p_x \;{}_{1/m}q_{x+k/m} \tag{4.15}$$

The index k in the summation formula given here denotes the multiple of $1/m$ beginning the interval $[k/m, (k+1)/m)$ within which the policy age $T - x$ at death is to lie. The summation itself is simply the weighted sum, over all indices k such that $k/m < n$, of the present values $v^{(k+1)/m}$ to be paid by the insurer in the event that the policy age at death falls in $[k/m, (k+1)/m)$ multiplied by the probability, given in formula (4.1), that this event occurs.

Next, to figure the expected present value of the life annuity-due with term n, note that payments of $1/m$ occur at all policy ages k/m, $k = 0, \ldots, nm-1$, for which $T - x \geq k/m$. Therefore, since the present values of these payments are $(1/m)\, v^{k/m}$ and the payment at k/m is made with probability ${}_{k/m}p_x$,

$$\ddot{a}^{(m)}_{x:\overline{n}|} \;=\; \mathcal{E}_x\left(\sum_{k=0}^{nm-1} \frac{1}{m}\, v^{k/m}\, I_{[T-x\ge k/m]}\right) \;=\; \frac{1}{m}\sum_{k=0}^{nm-1} v^{k/m}\; {}_{k/m}p_x \tag{4.16}$$

Finally the pure endowment has present value

$${}_nE_x \;=\; \mathcal{E}_x\left(v^n\, I_{[T-x\ge n]}\right) \;=\; v^n\; {}_xp_n \tag{4.17}$$

4.1.4 Expected Present Values for $m = 1$

It is clear that for the general insurance and life annuity payable at whole-year intervals ($m = 1$), with payment amounts determined solely by the whole-year age $[T]$ at death, the net single premiums are given by discrete-random-variable expectation formulas based upon the present values (4.2) and (4.3). Indeed, since the events $\{[T] \ge x\}$ and $\{T \ge x\}$ are identical for integers x, the discrete random variable $[T]$ for a life aged x has conditional probabilities given by

$$P([T] = x + k \,|\, T \ge x) \;=\; {}_kp_x \;-\; {}_{k+1}p_x \;=\; {}_kp_x \cdot q_{x+k}$$

Therefore the expected present value of the term-n insurance paying $F(k)$ at time $k+1$ whenever death occurs at age T between $x+k$ and $x+k+1$ (with $k < n$) is

$$E\Big(v^{[T]-x+1}\, F([T]-x)\, I_{[T\le x+n]}\,\Big|\, T \ge x\Big) = \sum_{k=0}^{n-1} F(k)\, v^{k+1}\, {}_kp_x\, q_{x+k}$$

Here and from now on, for an event B depending on the random lifetime T, the notation I_B denotes the so-called *indicator random variable* which is equal to 1 whenever T has a value such that the condition B is satisfied and is equal to 0 otherwise. The corresponding life annuity which pays $F(k)$ at each $k = 0, \ldots, n$ at which the annuitant is alive has expected present value

$$\mathcal{E}_x\Big(\sum_{k=0}^{\min(n,\,[T]-x)} v^k\, F(k)\Big) \;=\; \mathcal{E}_x\Big(\sum_{k=0}^{n} v^k\, F(k)\;\, I_{[T\ge x+k]}\Big) \;=\; \sum_{k=0}^{n} v^k\, F(k)\, {}_kp_x$$

In other words, the payment of $F(k)$ at time k is received only if the annuitant is alive at that time and so contributes expected present value equal to $v^k\, F(k)\, {}_kp_x$. This makes the annuity equal to the superposition of pure endowments of terms $k = 0, 1, 2, \ldots, n$ and respective face-amounts $F(k)$.

In the most important special case, where the non-zero face-amounts $F(k)$ are taken as constant, and for convenience are taken equal to 1 for $k = 0, \ldots, n-1$ and equal to 0 otherwise, we obtain the useful formulas

$$A^{1}_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x\, q_{x+k} \tag{4.18}$$

$$\ddot{a}_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^{k}\, {}_kp_x \tag{4.19}$$

$$A_{x:\overset{\;1}{\overline{n|}}} = \mathcal{E}_x\Big(v^n\, I_{[T-x\geq n]}\Big) = v^n\, {}_np_x \tag{4.20}$$

$$A_{x:\overline{n|}} = \sum_{k=0}^{\infty} v^{\min(n,k+1)}\, {}_kp_x\, q_{x+k}$$

$$= \sum_{k=0}^{n-1} v^{k+1}\, ({}_kp_x - {}_{k+1}p_x) + v^n\, {}_np_x \tag{4.21}$$

Two further manipulations which will complement this circle of ideas are left as exercises for the interested reader: (i) first, to verify that formula (4.19) gives the same answer as the formula $\mathcal{E}_x(\ddot{a}_{\overline{x:\min([T]-x+1,\,n)|}})$; and
(ii) second, to sum by parts (collecting terms according to like subscripts k of ${}_kp_x$ in formula (4.21)) to obtain the equivalent expression

$$1 + \sum_{k=0}^{n-1} (v^{k+1} - v^k)\, {}_kp_x = 1 - (1-v) \sum_{k=0}^{n-1} v^k\, {}_kp_x$$

The reader will observe that this final expression together with formula (4.19) gives an alternative proof, for the case $m = 1$, of the identity (4.12).

Let us work out these formulas analytically in the special case where $[T]$ has the *Geometric*$(1-\gamma)$ distribution, i.e., where

$$P([T] = k) = P(k \leq T < k+1) = \gamma^k\, (1-\gamma) \qquad \text{for} \qquad k = 0, 1, \ldots$$

with γ a fixed constant parameter between 0 and 1. This would be true if the force of mortality μ were constant at all ages, i.e., if T were exponentially distributed with parameter μ, with $f(t) = \mu\, e^{-\mu t}$ for $t \geq 0$. In that case, $P(T \geq k) = e^{-\mu k}$, and $\gamma = P(T = k | T \geq k) = 1 - e^{-\mu}$. Then

$$ {}_k p_x \, q_{x+k} = P([T] = x + k \,|\, T \geq x) = \gamma^k \, (1 - \gamma) \quad , \qquad {}_n p_x = \gamma^n $$

so that

$$ A_{x:\overline{n}|}^{\;\;1} = (\gamma v)^n \quad , \qquad A^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1}\, \gamma^k \, (1-\gamma) = v(1-\gamma)\, \frac{1 - (\gamma v)^n}{1 - \gamma v} $$

Thus, for the case of interest rate $i = 0.05$ and $\gamma = 0.97$, corresponding to expected lifetime $= \gamma/(1-\gamma) = 32.33$ years,

$$ A_{x:\overline{20}|} \;=\; (0.97/1.05)^{20} \;+\; \frac{.03}{1.05} \cdot \frac{1 - (.97/1.05)^{20}}{(1 - (.97/1.05)} \;=\; .503 $$

which can be compared with $A_x \equiv A^1_{x:\overline{\infty}|} = \frac{.03}{.08} = .375$.

The formulas (4.18)-(4.21) are benchmarks in the sense that they represent a complete solution to the problem of determining net single premiums without the need for interpolation of the life-table survival function between integer ages. However the insurance, life-annuity, and endowment-insurance contracts payable only at whole-year intervals are all slightly impractical as insurance vehicles. In the next chapter, we approach the calculation of net single premiums for the more realistic context of m-period-per-year insurances and life annuities, using only the standard cohort life-table data collected by integer attained ages.

4.2 Continuous-Time Expectations

So far in this Chapter, all of the expectations considered have been associated with the discretized random lifetime variables $[T]$ and $T_m = [mT]/m$. However, Insurance and Annuity contracts can also be defined with respectively instantaneous and continuous payments, as follows. First, an **instantaneous-payment** or **continuous insurance** with face-value F

is a contract which pays an amount F at the instant of death of the insured. (In practice, this means that when the actual payment is made at some later time, the amount paid is F together with interest compounded from the instant of death.) As a function of the random lifetime T for the insured life initially with exact integer age x, the present value of the amount paid is $F \cdot v^{T-x}$ for a whole-life insurance and $F \cdot v^{T-x} \cdot I_{[T<x+n]}$ for an n-year term insurance. The expected present values or net single premiums on a life aged x are respectively denoted $\overline{\mathrm{A}}_x$ for a whole-life contract and $\overline{\mathrm{A}}^1_{x:\overline{n}|}$ for an n-year temporary insurance. The **continuous life annuity** is a contract which provides continuous payments at rate 1 per unit time for duration equal to the smaller of the remaining lifetime of the annuitant or the term of n years. Here the present value of the contractual payments, as a function of the exact age T at death for an annuitant initially of exact integer age x, is $\overline{\mathrm{a}}_{\overline{\min(T-x,\,n)}|}$ where n is the (possibly infinite) duration of the life annuity. Recall that

$$\overline{\mathrm{a}}_{\overline{K}|} \;=\; \int_0^\infty v^t \, I_{[t \le K]} \, dt \;=\; \int_0^K v^t \, dt \;=\; (1 - v^K)/\delta$$

is the present value of a continuous payment stream of 1 per unit time of duration K units, where $v = (1+i)^{-1}$ and $\delta = \ln(1+i)$.

The objective of this section is to develop and interpret formulas for these continuous-time net single premiums, along with one further quantity which has been defined as a continuous-time expectation of the lifetime variable T, namely the **mean residual life** (also called **complete life expectancy**) $\mathrm{e}_x = \mathcal{E}_x(T-x)$ for a life aged x. The underlying general conditional expectation formula (1.3) was already derived in Chapter 1, and we reproduce it here in the form

$$\mathcal{E}_x\{\, g(T) \,\} \;=\; \frac{1}{S(x)} \int_x^\infty g(y)\, f(y)\, dy = \int_0^\infty g(x+t)\, \mu(x+t)\, {}_tp_x \, dt \qquad (4.22)$$

We apply this formula directly for the three choices

$$g(y) \quad = \quad y - x \quad , \quad v^{y-x} \quad , \quad \text{or} \qquad v^{y-x} \cdot I_{[y-x<n]}$$

which respectively have the conditional $\mathcal{E}_x(\cdot)$ expectations

$$\mathrm{e}_x \quad , \qquad \overline{\mathrm{A}}_x \quad , \qquad \overline{\mathrm{A}}^1_{x:\overline{n}|}$$

For easy reference, the integral formulas for these three cases are:

$$\mathrm{e}_x \;=\; \mathcal{E}_x(T-x) \;=\; \int_0^\infty t\,\mu(x+t)\,{}_tp_x\,dt \tag{4.23}$$

$$\overline{A}_x \;=\; \mathcal{E}_x(v^{T-x}) \;=\; \int_0^\infty v^t\,\mu(x+t)\,{}_tp_x\,dt \tag{4.24}$$

$$\overline{\mathrm{A}}^{\,1}_{\overline{x:n}|} \;=\; E_x\left(v^{T-x}\,I_{[T-x\le n]}\right) \;=\; \int_0^n v^{t-x}\,\mu(x+t)\,{}_tp_x\,dt \tag{4.25}$$

Next, we obtain two additional formulas, for continuous life annuities-due

$$\overline{\mathrm{a}}_x \quad \text{and} \quad \overline{\mathrm{a}}_{\overline{x:n}|}$$

which correspond to $\mathcal{E}_x\{g(T)\}$ for the two choices

$$g(t) \quad = \quad \int_0^\infty v^t\,I_{[t\le y-x]}\,dt \quad \text{or} \qquad \int_0^n v^t\,I_{[t\le y-x]}\,dt$$

After switching the order of the integrals and the conditional expectations, and evaluating the conditional expectation of an indicator as a conditional probability, in the form

$$\mathcal{E}_x\left(\,I_{[t\le T-x]}\,\right) \;=\; P(T\ge x+t\,|\,T\ge x) \;=\; {}_tp_x$$

the resulting two equations become

$$\overline{\mathrm{a}}_x \;=\; \mathcal{E}_x\left(\int_0^\infty v^t\,I_{[t\le T-x]}\,dt\right) \;=\; \int_0^\infty v^t\,{}_tp_x\,dt \tag{4.26}$$

$$\overline{\mathrm{a}}_{\overline{x:n}|} \;=\; \mathcal{E}_x\left(\int_0^n v^t\,I_{[t\le T-x]}\,dt\right) \;=\; \int_0^n v^t\,{}_tp_x\,dt \tag{4.27}$$

As might be expected, the continuous insurance and annuity contracts have a close relationship to the corresponding contracts with m payment periods per year for large m. Indeed, it is easy to see that the term insurance net single premiums

$$A^{(m)\,1}_{\overline{x:n}|} \;=\; \mathcal{E}_x\left(\,v^{T_m-x+1/m}\right)$$

approach the continuous insurance value (4.24) as a limit when $m\to\infty$. A simple proof can be given because the payments at the end of the fraction

$1/m$ of year of death are at most $1/m$ years later than the continuous-insurance payment at the instant of death, so that the following obvious inequalities hold:

$$\overline{\mathrm{A}}^{\,1}_{x:\overline{n}|} \;\le\; A^{(m)\,1}_{x:\overline{n}|} \;\le\; v^{1/m}\,\overline{\mathrm{A}}^{\,1}_{x:\overline{n}|} \tag{4.28}$$

Since the right-hand term in the inequality (4.28) obviously converges for large m to the leftmost term, the middle term which is sandwiched in between must converge to the same limit (4.25).

For the continuous annuity, (4.27) can be obtained as a limit of formulas (4.16) using Riemann sums, as the number m of payments per year goes to infinity, i.e.,

$$\overline{\mathrm{a}}_{x:\overline{n}|} \;=\; \lim_{m\to\infty}\, \ddot{\mathrm{a}}^{(m)}_{x:\overline{n}|} \;=\; \lim_{m\to\infty} \sum_{k=0}^{nm-1} \frac{1}{m}\, v^{k/m}\;{}_{k/m}p_x \;=\; \int_0^n v^t\;{}_tp_x\,ds$$

The final formula coincides with (4.27), according with the intuition that the limit as $m \to \infty$ of the payment-stream which pays $1/m$ at intervals of time $1/m$ between 0 and $T_m - x$ inclusive is the continuous payment-stream which pays 1 per unit time throughout the policy-age interval $[0, T-x)$.

Each of the expressions in formulas (4.23), (4.24), and (4.27) can be contrasted with a related approximate expectation for a function of the integer-valued random variable $[T]$ (taking $m = 1$). First, alternative general formulas are developed for the integrals by breaking the formulas down into sums of integrals over integer-endpoint intervals and substituting the definition ${}_kp_x/S(x+k) = 1/S(x)$:

$$\begin{aligned} \mathcal{E}_x(g(T)) &= \sum_{k=0}^{\infty} \int_{x+k}^{x+k+1} g(y)\frac{f(y)}{S(x)}\,dy \qquad \text{changing to } z = y-x-k \\ &= \sum_{k=0}^{\infty} {}_kp_x \int_0^1 g(x+k+z)\,\frac{f(x+k+z)}{S(x+k)}\,dz \end{aligned} \tag{4.29}$$

Substituting into (4.29) the special function $g(y) = y - x$, leads to

$$\mathrm{e}_x = \sum_{k=0}^{\infty} {}_kp_x \left\{ k\,\frac{S(x+k)-S(x+k+1)}{S(x+k)} \;+\; \int_0^1 z\,\frac{f(x+k+z)}{S(x+k)}\,dz \right\} \tag{4.30}$$

Either of two assumptions between integer ages can be applied to simplify the integrals:

(a) *(Uniform distribution of failures)* $f(y) = f(x+k) = S(x+k) - S(x+k+1)$ for all y between $x+k$, $x+k+1$;

(b) *(Constant force of mortality)* $\mu(y) = \mu(x+k)$ for $x+k \le y < x+k+1$, in which case $1 - q_{x+k} = \exp(-\mu(x+k))$.

In case (a), the last integral in (4.30) becomes

$$\int_0^1 z\, \frac{f(x+k+z)}{S(x+k)}\, dz \;=\; \int_0^1 z\, \frac{S(x+k) - S(x+k+1)}{S(x+k)}\, dz \;=\; \frac{1}{2}\, q_{x+k}$$

and in case (b), we obtain within (4.30)

$$\int_0^1 z\, \frac{f(x+k+z)}{S(x+k)}\, dz \;=\; \int_0^1 z\, \mu(x+k)\, e^{-z\,\mu(x+k)}\, dz$$

which in turn is equal (after integration by parts) to

$$-e^{-\mu(x+k)} \;+\; \frac{1 - e^{-\mu(x+k)}}{\mu(x+k)} \;\approx\; \frac{1}{2}\,\mu(x+k) \;\approx\; \frac{1}{2}\, q_{x+k}$$

where the last approximate equalities hold if the death rates are small. It follows, exactly in the case (a) where failures are uniformly distributed within integer-age intervals or approximately in case (b) when death rates are small, that

$$\mathrm{e}_x \;=\; \sum_{k=0}^{\infty} (k + \frac{1}{2})\, {}_kp_x\, q_{x+k} \;=\; \sum_{k=0}^{\infty} k\, {}_kp_x\, q_{x+k} \;+\; \frac{1}{2} \tag{4.31}$$

The final summation in (4.31), called the **curtate life expectancy**

$$\mathring{\mathrm{e}}_x \;=\; \sum_{k=0}^{\infty} k\, {}_kp_x\, q_{x+k} \tag{4.32}$$

has an exact interpretation as the expected number of whole years of life remaining to a life aged x. The behavior of and comparison between complete and curtate life expectancies is explored numerically in subsection 4.2.1 below.

Return now to the general expression for $\mathcal{E}_x(g(T))$, substituting $g(y) = v^{y-x}$ but restricting attention to case (a):

$$\overline{\mathrm{A}}^{1}_{x:\overline{n}|} \;=\; \mathcal{E}\left\{ v^{T-x}\, I_{[T<x+n]} \right\} \;=\; \sum_{k=0}^{n} \int_{x+k}^{x+k+1} v^{y-x}\, \frac{f(x+k)}{S(x)}\, dy$$

$$= \sum_{k=0}^{n} \int_{x+k}^{x+k+1} v^{y-x} \, \frac{S(x+k)-S(x+k+1)}{S(x+k)} \, {}_kp_x \, dy$$

$$= \sum_{k=0}^{n} {}_kp_x \, q_{x+k} \int_0^1 v^{k+t} \, dt \;=\; \sum_{k=0}^{n} {}_kp_x \, q_{x+k} \, v^{k+1} \, \frac{1-e^{-\delta}}{v\delta}$$

where $v = 1/(1+i) = e^{-\delta}$, and δ is the *force of interest*. Since $1-e^{-\delta} = iv$, we have found in case (a) that

$$\overline{\mathrm{A}}^{1}_{x:\overline{n}|} \;=\; A^{1}_{x:\overline{n}|} \;\cdot\; (i/\delta) \tag{4.33}$$

Finally, return to the formula (4.27) under case (b) to find

$$\overline{\mathrm{a}}_{x:\overline{n}|} \;=\; \sum_{k=0}^{n-1} \int_k^{k+1} v^t \; {}_tp_x \, dt \;=\; \sum_{k=0}^{n-1} \int_k^{k+1} e^{-(\delta+\mu)t} \, dt$$

$$= \sum_{k=0}^{n-1} \frac{e^{-(\delta+\mu)k} \;-\; e^{-(\delta+\mu)(k+1)}}{\delta+\mu} \;=\; \sum_{k=0}^{n-1} v^k \; {}_kp_x \;\cdot\; \frac{1-e^{-(\delta+\mu)}}{\delta+\mu}$$

Thus, in case (b) we have shown

$$\overline{\mathrm{a}}_{x:\overline{n}|} \;=\; \frac{1-e^{-(\delta+\mu)n}}{\delta+\mu} \;=\; \ddot{\mathrm{a}}_{x:\overline{n}|} \;\cdot\; \frac{1-e^{-(\delta+\mu)}}{\delta+\mu} \tag{4.34}$$

In the last two paragraphs, we have obtained formulas (4.33) and (4.34) respectively under cases (a) and (b) relating net single premiums for continuous contracts to those of the corresponding single-payment-per-year contracts. More elaborate relations will be given in the next Chapter between net single premium formulas which do require interpolation-assumptions for probabilities of survival to times between integer ages to formulas for $m = 1$, which do not require such interpolation.

4.2.1 Numerical Calculations of Life Expectancies

Formulas (4.23) or (4.30) and (4.32) above respectively provide the complete and curtate age-specific life expectancies, in terms respectively of survival densities and life-table data. Formula (4.31) provides the actuarial approximation for complete life expectancy in terms of life-table data, based upon interpolation-assumption (i) (Uniform mortality within year of age). In this Section, we illustrate these formulas using the Illustrative simulated and extrapolated life-table data of Table 1.1.

Life expectancy formulas necessarily involve life table data and/or survival distributions specified out to arbitrarily large ages. While life tables may be based on large cohorts of insured for ages up to the seventies and even eighties, beyond that they will be very sparse and very dependent on the particular small group(s) of aged individuals used in constructing the particular table(s). On the other hand, the fraction of the cohort at moderate ages who will survive past 90, say, is extremely small, so a reasonable extrapolation of a well-established table out to age 80 or so may give sufficiently accurate life-expectancy values at ages not exceeding 80. Life expectancies are in any case forecasts based upon an implicit assumption of future mortality following exactly the same pattern as recent past mortality. Life-expectancy calculations necessarily ignore likely changes in living conditions and medical technology which many who are currently alive will experience. Thus an assertion of great accuracy for a particular method of calculation would be misplaced.

All of the numerical life-expectancy calculations produced for the Figure of this Section are based on the extrapolation (2.9) of the illustrative life table data from Table 1.1. According to that extrapolation, death-rates q_x for all ages 78 and greater are taken to grow exponentially, with $log(q_x/q_{78}) = (x-78)\ln(1.0885)$. This exponential behavior is approximately but not precisely compatible with a Gompertz-form force-of-mortality function

$$\mu(78+t) \;=\; \mu(78)\; c^t$$

in light of the approximate equality $\mu(x) \approx q_x$, an approximation which progressively becomes less valid as the force of mortality gets larger. To see this, note that under a Gompertz survival model,

$$\mu(x) = Bc^x \quad , \qquad q_x = 1 - \exp\left(-Bc^x\,\frac{c-1}{\ln c}\right)$$

and with $c = 1.0885$ in our setting, $(c-1)/\ln c = 1.0436$.

Since curtate life expectancy (4.32) relies directly on (extrapolated) life-table data, its calculation is simplest and most easily interpreted. Figure 4.1

presents, as plotted points, the age-specific curtate life expectancies for integer ages $x = 0, 1, \ldots, 78$. Since the complete life expectancy at each age is larger than the curtate by exactly $1/2$ under interpolation assumption (a), we calculated for comparison the complete life expectancy at all (real-number) ages, under assumption (b) of piecewise-constant force of mortality within years of age. Under this assumption, by formula (3.11), mortality within year of age $(0 < t < 1)$ is ${}_tp_x = (p_x)^t$. Using formula (4.31) and interpolation assumption (b), the exact formula for complete life expectancy becomes

$$\mathrm{e}_x - \mathring{\mathrm{e}}_x = \sum_{k=0}^{\infty} {}_kp_x \left\{ \frac{q_{x+k} + p_{x+k} \ln(p_{x+k})}{- \ln(p_{x+k})} \right\}$$

The complete life expectancies calculated from this formula were found to exceed the curtate life expectancy by amounts ranging from 0.493 at ages 40 and below, down to 0.485 at age 78 and 0.348 at age 99. Thus there is essentially no new information in the calculated complete life expectancies, and they are not plotted.

The aspect of Figure 4.1 which is most startling to the intuition is the large expected numbers of additional birthdays for individuals of advanced ages. Moreover, the large life expectancies shown are comparable to actual US male mortality circa 1959, so would be still larger today.

4.3 *Exercise Set 4*

(1). For each of the following three lifetime distributions, find (a) the expected remaining lifetime for an individual aged 20, and (b) ${}_{7/12}q_{40}/q_{40}$.

(i) *Weibull*$(.00634, 1.2)$, with $S(t) = \exp(-0.00634\, t^{1.2})$,

(ii) *Lognormal*$(\log(50), 0.325^2)$, with $S(t) = 1 - \Phi((\log(t) - \log(50))/0.325)$,

(iii) *Piecewise exponential* with force of mortality given the constant value $\mu_t = 0.015$ for $20 < t \le 50$, and $\mu_t = 0.03$ for $t \ge 50$. *In these integrals, you should be prepared to use integrations by parts, gamma function values, tables of the normal distribution function* $\Phi(x)$, *and/or numerical integrations via calculators or software.*

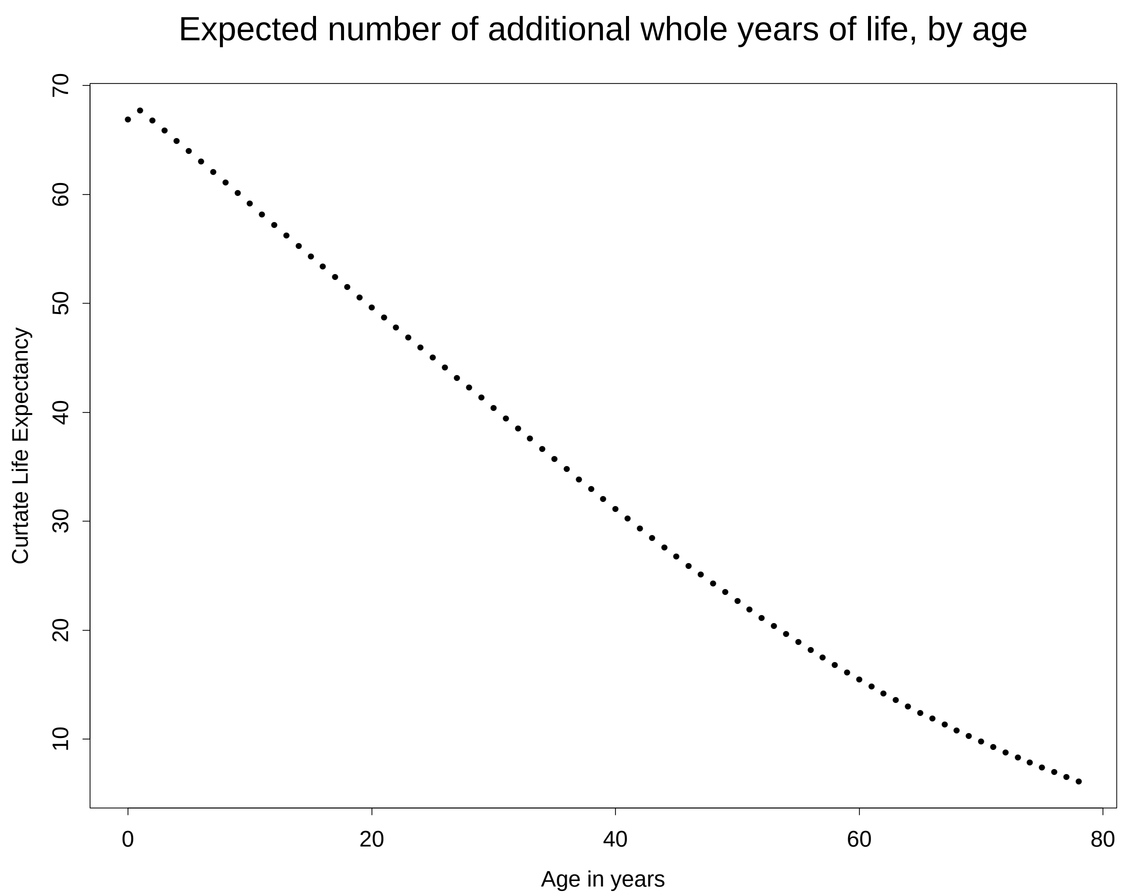

Figure 4.1: Curtate life expectancy $\mathring{e}_x$ as a function of age, calculated from the simulated illustrative life table data of Table 1.1, with age-specific death-rates q_x extrapolated as indicated in formula (2.9).

(2). (a) Find the expected present value, with respect to the constant effective interest rate $r = 0.07$, of an insurance payment of \$1000 to be made at the instant of death of an individual who has just turned 40 and whose remaining lifetime $T - 40 = S$ is a continuous random variable with density $f(s) = 0.05\, e^{-0.05\, s}$, $s > 0$.

(b) Find the expected present value of the insurance payment in (a) if the insurer is allowed to delay the payment to the end of the year in which the individual dies. Should this answer be larger or smaller than the answer in (a) ?

(3). If the individual in Problem 2 pays a life insurance premium P at the **beginning** of each remaining year of his life (including this one), then what is the expected total present value of all the premiums he pays before his death ?

(4). Suppose that an individual has equal probability of dying within each of the next 40 years, and is certain to die within this time, i.e., his age is x and

$$ {}_kp_x - {}_{k+1}p_x = 0.025 \qquad \text{for} \qquad k = 0, 1, \ldots, 39 $$

Assume the fixed interest rate $r = 0.06$.

(a) Find the net single whole-life insurance premium A_x for this individual.

(b) Find the net single premium for the term and endowment insurances $A^1_{x:\overline{20}|}$ and $A_{x:\overline{30}|}$.

(5). Show that the expected whole number of years of remaining life for a life aged x is given by

$$ c_x \;=\; E([T] - x \,|\, T \geq x) \;=\; \sum_{k=0}^{\omega - x - 1} k \; {}_kp_x \, q_{x+k} $$

and prove that this quantity as a function of integer age x satisfies the recursion equation

$$ c_x \;=\; p_x \;(1 \,+\, c_{x+1}) $$

(6). Show that the expected present value b_x of an insurance of 1 payable at the beginning of the year of death (or equivalently, payable at the end of the year of death along with interest from the beginning of that same year) satisfies the recursion relation (4.35) above.

(7). Prove the identity (4.9) algebraically.

For the next two problems, consider a cohort life-table population for which you know only that $l_{70} = 10,000$, $l_{75} = 7000$, $l_{80} = 3000$, and $l_{85} = 0$, and that the distribution of death-times within 5-year age intervals is uniform.

(8). Find (a) $\mathring{e}_{75}$ and (b) the probability of an individual aged 70 in this life-table population dying between ages 72.0 and 78.0.

(9). Find the probability of an individual aged 72 in this life-table population dying between ages 75.0 and 83.0, if the assumption of uniform death-times within 5-year intervals is replaced by:

(a) an assumption of constant force of mortality within 5-year age-intervals;

(b) the Balducci assumption (of linearity of 1/S(t)) within 5-year age intervals.

(10). Suppose that a population has survival probabilities governed at all ages by the force of mortality

$$\mu_t = \begin{cases} .01 & \text{for} \quad 0 \le t < 1 \\ .002 & \text{for} \quad 1 \le t < 5 \\ .001 & \text{for} \quad 5 \le t < 20 \\ .004 & \text{for} \quad 20 \le t < 40 \\ .0001 \cdot t & \text{for} \quad 40 \le t \end{cases}$$

Then (a) find ${}_{30}p_{10}$, and (b) find $\mathring{e}_{50}$.

(11). Suppose that a population has survival probabilities governed at all ages by the force of mortality

$$\mu_t = \begin{cases} .01 & \text{for} \quad 0 \le t < 10 \\ .1 & \text{for} \quad 10 \le t < 30 \\ 3/t & \text{for} \quad 30 \le t \end{cases}$$

Then (a) find ${}_{30}p_{20} =$ the probability that an individual aged 20 survives for at least 30 more years, and (b) find $\mathring{e}_{30}$.

(12). Assuming the same force of mortality as in the previous problem, find $\mathring{e}_{70}$ and $\overline{A}_{60}$ if $i = 0.09$.

(13). The force of mortality for impaired lives is three times the standard force of mortality at all ages. The standard rates q_x of mortality at ages 95, 96, and 97 are respectively 0.3, 0.4, and 0.5 . What is the probability that an impaired life age 95 will live to age 98 ?

(14). You are given a survival function $S(x) = (10-x)^2/100$, $0 \le x \le 10$.

(a) Calculate the average number of future years of life for an individual who survives to age 1.

(b) Calculate the difference between the force of mortality at age 1, and the probability that a life aged 1 dies before age 2.

(15). An n-year term life insurance policy to a life aged x provides that if the insured dies within the n-year period an annuity-certain of yearly payments of 10 will be paid to the beneficiary, with the first annuity payment made on the policy-anniversary following death, and the last payment made on the N^{th} policy anniversary. Here $1 < n \le N$ are fixed integers. If $B(x, n, N)$ denotes the net single premium (= expected present value) for this policy, and if mortality follows the law $l_x = C(\omega - x)/\omega$ for some terminal integer age ω and constant C, then find a simplified expression for $B(x, n, N)$ in terms of interest-rate functions, ω, and the integers x, n, N. Assume $x + n \le \omega$.

(16). The father of a newborn child purchases an endowment and insurance contract with the following combination of benefits. The child is to receive \$100,000 for college at her 18^{th} birthday if she lives that long and \$500,000 at her 60^{th} birthday if she lives that long, and the father as beneficiary is to receive \$200,000 at the end of the year of the child's death if the child dies before age 18. Find expressions, **both** in actuarial notations and in terms of $v = 1/(1+i)$ and of the survival probabilities ${}_kp_0$ for the child, for the net single premium for this contract.

4.4 Worked Examples

Example 1. Toy Life-Table *(assuming uniform failures)*

Consider the following life-table with only six equally-spaced ages. (That is, assume $l_6 = 0$.) Assume that the rate of interest $i = .09$, so that $v = 1/(1+i) = 0.9174$ and $(1 - e^{-\delta})/\delta = (1-v)/\delta = 0.9582$.

x	Age-range	l_x	d_x	e_x	$\overline{A}_x$
0	0 – 0.99	1000	60	4.2	0.704
1	1 – 1.99	940	80	3.436	0.749
2	2 – 2.99	860	100	2.709	0.795
3	3 – 3.99	760	120	2.0	0.844
4	4 – 4.99	640	140	1.281	0.896
5	5 – 5.99	500	500	0.5	0.958

Using the data in this Table, and interest rate $i = .09$, we begin by calculating the expected present values for simple contracts for term insurance, annuity, and endowment. First, for a life aged 0, a term insurance with payoff amount \$1000 to age 3 has present value given by formula (4.18) as

$$1000\, A^1_{0:\overline{3}|} \;=\; 1000 \left\{ 0.917\, \frac{60}{1000} \;+\; (0.917)^2\, \frac{80}{1000} \;+\; (0.917)^3\, \frac{100}{1000} \right\} \;=\; 199.60$$

Second, for a life aged 2, a term annuity-due of \$700 per year up to age 5 has present value computed from (4.19) to be

$$700\, \ddot{a}_{2:\overline{3}|} \;=\; 700 \left\{ 1 \;+\; 0.917\, \frac{760}{860} \;+\; (0.917)^2\, \frac{640}{860} \right\} \;=\; 1705.98$$

For the same life aged 2, the 3-year Endowment for \$700 has present value

$$700\, A_{0:\overline{3}|}^{\;\;1} \;=\; 700 \cdot (0.9174)^3\, \frac{500}{860} \;=\; 314.26$$

Thus we can also calculate (for the life aged 2) the present value of the 3-year annuity-immediate of \$700 per year as

$$700 \cdot \left(\ddot{a}_{2:\overline{3}|} \;-\; 1 \;+\; A_{0:\overline{3}|}^{\;\;1} \right) \;=\; 1705.98 - 700 + 314.26 \;=\; 1320.24$$

We next apply and interpret the formulas of Section 4.2, together with the observation that

$$ {}_jp_x \cdot q_{x+j} \;=\; \frac{l_{x+j}}{l_x} \cdot \frac{d_{x+j}}{l_{x+j}} \;=\; \frac{d_{x+j}}{l_x} $$

to show how the last two columns of the Table were computed. In particular, by (4.31)

$$ \mathrm{e}_2 = \frac{100}{860}\cdot 0 + \frac{120}{860}\cdot 1 + \frac{140}{860}\cdot 2 + \frac{500}{860}\cdot 3 + \frac{1}{2} = \frac{1900}{860} + 0.5 = 2.709 $$

Moreover: observe that $c_x = \sum_{k=0}^{5-x} k\, {}_kp_x q_{x+k}$ satisfies the "recursion equation" $c_x = p_x\,(1 + c_{x+1})$ (*cf.* Exercise 5 above), with $c_5 = 0$, from which the e_x column is easily computed by: $\mathrm{e}_x = c_x + 0.5$.

Now apply the present value formula for conitunous insurance to find

$$ \overline{A}_x \;=\; \sum_{k=0}^{5-x} {}_kp_x\, q_x\, v^k\, \frac{1-e^{-\delta}}{\delta} \;=\; 0.9582 \sum_{k=0}^{5-x} {}_kp_x\, q_x\, v^k \;=\; 0.9582\, b_x $$

where b_x is the expected present value of an insurance of 1 payable at the *beginning* of the year of death (so that $A_x = v\, b_x$) and satisfies $b_5 = 1$ together with the recursion-relation

$$ b_x \;=\; \sum_{k=0}^{5-x} {}_kp_x\, q_x\, v^k \;=\; p_x\, v\, b_{x+1} \;+\; q_x \tag{4.35} $$

(Proof of this recursion is Exercise 6 above.)

Example 2. Find a simplified expression in terms of actuarial exprected present value notations for the net single premium of an insurance on a life aged x, *which pays* $F(k) = C\ \ddot{a}_{\overline{n-k}|}$ *if death occurs at any exact ages between* $x+k$ *and* $x+k+1$, *for* $k = 0, 1, \ldots, n-1$, *and interpret the result.*

Let us begin with the interpretation: the beneficiary receives at the end of the year of death a lump-sum equal in present value to a payment stream of $\$C$ annually beginning at the end of the year of death and terminating at the end of the n^{th} policy year. This payment stream, if superposed upon an n-year life annuity-immediate with annual payments $\$C$, would result in a certain payment of $\$C$ at the end of policy years $1, 2, \ldots, n$. Thus the expected present value in this example is given by

$$C\, a_{\overline{n}|} \; - \; C\, a_{\overline{x:n}|} \tag{4.36}$$

Next we re-work this example purely in terms of analytical formulas. By formula (4.36), the net single premium in the example is equal to

$$\sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x\, q_{x+k}\, C\, \ddot{a}_{\overline{n-k+1}|} \; = \; C \sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x\, q_{x+k}\, \frac{1 - v^{n-k}}{d}$$

$$= \; \frac{C}{d} \left\{ \sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x\, q_{x+k} \; - \; v^{n+1} \sum_{k=0}^{n-1} \left({}_kp_x \; - \; {}_{k+1}p_x \right) \right\}$$

$$= \; \frac{C}{d} \left\{ A^{1}_{\overline{x:n}|} \; - \; v^{n+1}\, (1 - \, {}_np_x) \right\}$$

$$= \; \frac{C}{d} \left\{ \mathrm{A}_{\overline{x:n}|} \; - \; v^n\, {}_np_x \; - \; v^{n+1}\, (1 - \, {}_np_x) \right\}$$

and finally, by substituting expression (4.14) with $m = 1$ for $\mathrm{A}_{\overline{x:n}|}$, we have

$$\frac{C}{d} \left\{ 1 - \, d\, \ddot{\mathrm{a}}_{\overline{x:n}|} \; - \; (1 - v)\, v^n\, {}_np_x \; - \; v^{n+1} \right\}$$

$$= \; \frac{C}{d} \left\{ 1 - \, d\, (1 + \mathrm{a}_{\overline{x:n}|} \; - \; v^n\, {}_np_x) \; - \; d\, v^n\, {}_np_x \; - \; v^{n+1} \right\}$$

$$= \; \frac{C}{d} \left\{ v \; - \; d\, \mathrm{a}_{\overline{x:n}|} \; - \; v^{n+1} \right\} \; = \; C \left\{ \frac{1 - v^n}{i} \; - \; \mathrm{a}_{\overline{x:n}|} \right\}$$

$$= \; C\, \{ \mathrm{a}_{\overline{n}|} \; - \; \mathrm{a}_{\overline{x:n}|} \}$$

So the analytically derived answer agrees with the one intuitively arrived at in formula (4.36).

4.5 Useful Formulas from Chapter 4

$$T_m \;=\; [Tm]/m$$

p. 97

$$P(T_m = x + \frac{k}{m} \mid T \geq x) \;=\; {}_{k/m}p_x \;-\; {}_{(k+1)/m}p_x \;=\; {}_{k/m}p_x \;\cdot\; {}_{1/m}q_{x+k/m}$$

p. 98

$$\text{Term life annuity} \qquad \text{a}^{(m)}_{x:\overline{n}|} \;=\; \ddot{\text{a}}^{(m)}_{x:\overline{n+1/m}|} \;-\; 1/m$$

p. 100

$$\text{Endowment} \qquad A_{x:\overline{n}|}^{\;\;1} \;=\; {}_nE_x \;= v^n\, {}_np_x$$

p. 100

$$A^{(m)}_x \;-\; A^{(m)1}_{\;x:\overline{n}|} \;=\; v^n\, {}_np_x \;\cdot\; A_{x+n}$$

p. 101

$$A^{(m)}_{x:\overline{n}|} \;=\; A^{(m)1}_{\;x:\overline{n}|} + A^{(m)}{}_{x:\overline{n}|}^{\;\;1} \;=\; A^{(m)1}_{\;x:\overline{n}|} +_n E_x$$

p. 102

$$\ddot{\text{a}}^{(m)}_{x:\overline{n}|} \;=\; \mathcal{E}_x\Big(\frac{1 - v^{\min(T_m - x + 1/m,\, n)}}{d^{(m)}}\Big) = \frac{1 - A^{(m)}_{x:\overline{n}|}}{d^{(m)}}$$

p. 102

$$d^{(m)}\, \ddot{\text{a}}^{(m)}_{x:\overline{n}|} \;+\; A^{(m)}_{x:\overline{n}|} \;=\; 1$$

p. 102

$$A^1_{x:\overline{n}|} \;=\; \mathcal{E}_x\left(v^{T_m - x + 1/m}\right) \;=\; \sum_{k=0}^{nm-1} v^{(k+1)/m}\; {}_{k/m}p_x\; {}_{1/m}q_{x+k/m}$$

p. 103

$$A^1_{x:\overline{n}|} \;=\; \sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x\, q_{x+k}$$

p. 105

$$\ddot{a}_{x:\overline{n}|} \;=\; \sum_{k=0}^{n-1} v^{k}\, {}_kp_x$$

p. 105

$$A_{x:\overline{n}|}^{\;\;\;1} \;=\; \mathcal{E}_x\Big(v^n\, I_{[T-x\ge n]}\Big) \;=\; v^n\, {}_np_x$$

p. 105

$$A_{x:\overline{n}|} \;=\; \sum_{k=0}^{n-1} v^{k+1}\, ({}_kp_x - {}_{k+1}p_x) +\; v^n\, {}_np_x$$

p. 105

Chapter 5

Premium Calculation

This Chapter treats the most important topics related to the calculation of (risk) premiums for realistic insurance and annuity contracts. We begin by considering at length net single premium formulas for insurance and annuities, under each of three standard assumptions on interpolation of the survival function between integer ages, when there are multiple payments per year. One topic covered more rigorously here than elsewhere is the calculus-based and numerical comparison between premiums under these slightly different interpolation assumptions, justifying the standard use of the simplest of the interpolation assumptions, that deaths occur uniformly within whole years of attained age. Next we introduce the idea of calculating *level* premiums, setting up equations balancing the stream of level premium payments coming in to an insurer with the payout under an insurance, endowment, or annuity contract. Finally, we discuss single and level premium calculation for insurance contracts where the death benefit is modified by (fractional) premium amounts, either as refunds or as amounts still due. Here the issue is first of all to write an exact balance equation, then load it appropriately to take account of administrative expenses and the cushion required for insurance-company profitability, and only then to approximate and obtain the usual formulas.

5.1 m-Payment Net Single Premiums

The objective in this section is to relate the formulas for net single premiums for life insurance, life annuities, pure endowments and endowment insurances in the case where there are multiple payment periods per year to the case where there is just one. Of course, we must now make some interpolation assumptions about within-year survival in order to do this, and we consider the three main assumptions previously introduced: piecewise uniform failure distribution (constant failure density within each year), piecewise exponential failure distribution (constant force of mortality within each year), and Balducci assumption. As a practical matter, it usually makes relatively little difference which of these is chosen, as we have seen in exercises and will illustrate further in analytical approximations and numerical tabulations. However, of the three assumptions, Balducci's is least important practically, because of the remark that the force of mortality it induces within years is actually decreasing (the reciprocal of a linear function with positive slope), since formula (3.9) gives it under that assumption as

$$\mu(x+t) \;=\; -\frac{d}{dt}\ln S(x+t) \;=\; \frac{q_x}{1-(1-t)\,q_x}$$

Thus the inclusion of the Balducci assumption here is for completeness only, since it is a recurring topic for examination questions. However, we do not give separate net single premium formulas for the Balducci case.

In order to display simple formulas, and emphasize closed-form relationships between the net single premiums with and without multiple payments per year, *we adopt a further restriction throughout this Section, namely that the duration* n *of the life insurance or annuity is an integer even though* $m > 1$. There is in principle no reason why all of the formulas cannot be extended, one by one, to the case where n is assumed only to be an integer multiple of $1/m$, but the formulas are less simple that way.

5.1.1 Dependence Between Integer & Fractional Ages at Death

One of the clearest ways to distinguish the three interpolation assumptions is through the probabilistic relationship they impose between the greatest-integer $[T]$ or attained integer age at death and the fractional age $T - [T]$ at death. The first of these is a discrete, nonnegative-integer-valued random variable, and the second is a continuous random variable with a density on the time-interval $[0, 1)$. In general, the dependence between these random variables can be summarized through the calculated joint probability

$$P([T] = x + k,\, T - [T] < t \,|\, T \geq x) \;=\; \int_{x+k}^{x+k+t} \frac{f(y)}{S(x)}\, dy \;=\; {}_tq_{x+k}\;{}_kp_x \qquad (5.1)$$

where k, x are integers and $0 \leq t < 1$. From this we deduce the following formula (for $k \geq 0$) by dividing the formula (5.1) for general t by the corresponding formula at $t = 1$:

$$P(\, T - [T] \leq t \,|\, [T] = x + k) = \frac{{}_tq_{x+k}}{q_{x+k}} \qquad (5.2)$$

where we have used the fact that $T - [T] < 1$ with certainty.

In case (i) from Section 3.2, with the density f assumed piecewise constant, we already know that ${}_tq_{x+k} = t\, q_{x+k}$, from which formula (5.2) immediately implies

$$P(\, T - [T] \leq t \,|\, [T] = x + k) = t$$

In other words, given complete information about the age at death, the fractional age at death is always uniformly distributed between 0, 1. Since the conditional probability does not involve the age at death, we say under the interpolation assumption (i) that the fractional age and whole-year age at death are *independent* as random variables.

In case (ii), with piecewise constant force of mortality, we know that

$${}_tq_{x+k} \;=\; 1 \;-\; {}_tp_{x+k} \;=\; 1 \;-\; e^{-\mu(x+k)\,t}$$

and it is no longer true that fractional and attained ages at death are independent except in the very special (completely artificial) case where $\mu(x+k)$ has the same constant value μ for all x, k. In the latter case, where T is an *exponential* random variable, it is easy to check from (5.2) that

$$P(\, T - [T] \leq t \,|\, [T] = x + k) \;=\; \frac{1 - e^{-\mu t}}{1 - e^{-\mu}}$$

In that case, $T - [T]$ is indeed independent of $[T]$ and has a *truncated exponential* distribution on $[0,1)$, while $[T]$ has the *Geometric*$(1 - e^{-\mu})$ distribution given, according to (5.1), by

$$P([T] = x + k \,|\, T \geq x) \;=\; (1 - e^{-\mu})(e^{-\mu})^k$$

In case (iii), under the Balducci assumption, formula (3.8) says that ${}_{1-t}q_{x+t} \;=\; (1-t)\, q_x$, which leads to a special formula for (5.2) but not a conclusion of conditional independence. The formula comes from the calculation

$$(1-t)\; q_{x+k} \;=\; {}_{(1-t)}q_{x+k+t} \;=\; 1 - \frac{p_{x+k}}{{}_t p_{x+k}}$$

leading to

$${}_t q_{x+k} = 1 - {}_t p_{x+k} = 1 - \frac{p_{x+k}}{1 - (1-t)\, q_{x+k}} = \frac{t\; q_{x+k}}{1 - (1-t)\, q_{x+k}}$$

Thus Balducci implies via (5.2) that

$$P(\,T - [T] \leq t \,|\, [T] = x + k) = \frac{t}{1 - (1-t)\; q_{x+k}}$$

5.1.2 Net Single Premium Formulas — Case (i)

In this setting, the formula (4.15) for insurance net single premium is simpler than (4.16) for life annuities, because

$${}_{j/m}p_{x+k} - {}_{(j+1)/m}p_{x+k} \;=\; \frac{1}{m}\, q_{x+k}$$

Here and throughout the rest of this and the following two subsections, x, k, j are integers and $0 \leq j < m$, and $k + \frac{j}{m}$ will index the possible values for the last multiple $T_m - x$ of $1/m$ year of the policy age at death. The formula for net single insurance premium becomes especially simple when n is an integer, because the double sum over j and k factors into the product of a sum of terms depending only on j and one depending only on k :

$$A^{(m)1}_{\;\overline{x:n|}} \;=\; \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} v^{k+(j+1)/m}\, \frac{1}{m}\, q_{x+k}\; {}_k p_x$$

$$= \left(\sum_{k=0}^{n-1} v^{k+1} \, q_{x+k} \; {}_k p_x \right) \frac{v^{-1+1/m}}{m} \sum_{j=0}^{m-1} v^{j/m} \;=\; A^1_{x:\overline{n}|} \, v^{-1+1/m} \, \ddot{a}^{(m)}_{\overline{1}|}$$

$$= A^1_{x:\overline{n}|} \, v^{-1+1/m} \, \frac{1-v}{d^{(m)}} \;=\; \frac{i}{i^{(m)}} \, A^1_{x:\overline{n}|} \tag{5.3}$$

The corresponding formula for the case of non-integer n can clearly be written down in a similar way, but does not bear such a simple relation to the one-payment-per-year net single premium.

The formulas for life annuities should not be re-derived in this setting but rather obtained using the general identity connecting endowment insurances with life annuities. Recall that in the case of integer n the net single premium for a pure n-year endowment does not depend upon m and is given by

$$A_{x:\overline{n}|}^{\;\;1} \;=\; {}_n p_x \, v^n$$

Thus we continue by displaying the net single premium for an endowment insurance, related in the m-payment-period-per year case to the formula with single end-of-year payments:

$$A^{(m)}_{x:\overline{n}|} \;=\; A^{(m)1}_{x:\overline{n}|} \;+\; A_{x:\overline{n}|}^{\;\;1} \;=\; \frac{i}{i^{(m)}} \, A^1_{x:\overline{n}|} \;+\; {}_n p_x \, v^n \tag{5.4}$$

As a result of (4.11), we obtain the formula for net single premium of a temporary life-annuity due:

$$\ddot{a}^{(m)}_{x:\overline{n}|} \;=\; \frac{1 - A^{(m)}_{x:\overline{n}|}}{d^{(m)}} \;=\; \frac{1}{d^{(m)}} \left[1 \;-\; \frac{i}{i^{(m)}} \, A^1_{x:\overline{n}|} \;-\; {}_n p_x \, v^n \right]$$

Re-expressing this formula in terms of annuities on the right-hand side, using $\ddot{a}_{x:\overline{n}|} \;=\; d^{-1} \, (1 - \; v^n \, {}_n p_x \;-\; A^1_{x:\overline{n}|})$, immediately yields

$$\ddot{a}^{(m)}_{x:\overline{n}|} \;=\; \frac{d\, i}{d^{(m)} \, i^{(m)}} \, \ddot{a}_{x:\overline{n}|} \;+\; \left(1 - \frac{i}{i^{(m)}} \right) \frac{1 - v^n \, {}_n p_x}{d^{(m)}} \tag{5.5}$$

The last formula has the form that the life-annuity due with m payments per year is a weighted linear combination of the life-annuity due with a single payment per year, the n-year pure endowment, and a constant, where the weights and constant depend only on interest rates and m but not on survival probabilities:

$$\begin{aligned} \ddot{a}^{(m)}_{x:\overline{n}|} \;&=\; \alpha(m) \, \ddot{a}_{x:\overline{n}|} \;-\; \beta(m) \, (1 \;-\; {}_n p_x \, v^n) \\ &=\; \alpha(m) \, \ddot{a}_{x:\overline{n}|} \;-\; \beta(m) \;+\; \beta(m) \, A_{x:\overline{n}|}^{\;\;1} \end{aligned} \tag{5.6}$$

Table 5.1: Values of $\alpha(m)$, $\beta(m)$ for Selected m, i

i	m=	2	3	4	6	12
0.03						
	$\alpha(m)$	1.0001	1.0001	1.0001	1.0001	1.0001
	$\beta(m)$	0.2537	0.3377	0.3796	0.4215	0.4633
0.05						
	$\alpha(m)$	1.0002	1.0002	1.0002	1.0002	1.0002
	$\beta(m)$	0.2562	0.3406	0.3827	0.4247	0.4665
0.07						
	$\alpha(m)$	1.0003	1.0003	1.0004	1.0004	1.0004
	$\beta(m)$	0.2586	0.3435	0.3858	0.4278	0.4697
0.08						
	$\alpha(m)$	1.0004	1.0004	1.0005	1.0005	1.0005
	$\beta(m)$	0.2598	0.3450	0.3873	0.4294	0.4713
0.10						
	$\alpha(m)$	1.0006	1.0007	1.0007	1.0007	1.0008
	$\beta(m)$	0.2622	0.3478	0.3902	0.4325	0.4745

Here the interest-rate related constants $\alpha(m)$, $\beta(m)$ are given by

$$\alpha(m) \;=\; \frac{d\,i}{d^{(m)}\,i^{(m)}} \;, \qquad \beta(m) \;=\; \frac{i \,-\, i^{(m)}}{d^{(m)}\,i^{(m)}}$$

Their values for some practically interesting values of m, i are given in Table 5.1. Note that $\alpha(1) = 1$, $\beta(1) = 0$, reflecting that $\ddot{a}^{(m)}_{x:\overline{n}|}$ coincides with $\ddot{a}_{x:\overline{n}|}$ by definition when $m = 1$. The limiting case for $i = 0$ is given in Exercises 6 and 7:

$$\text{for} \quad i \,=\, 0\,, \;\; m \ge 1\,, \qquad \alpha(m) \;=\; 1 \quad , \qquad \beta(m) \;=\; \frac{m-1}{2m}$$

Equations (5.3), (5.5), and (5.6) are useful because they summarize concisely the modification needed for one-payment-per-year formulas (which used only life-table and interest-rate-related quantities) to accommodate multiple payment-periods per year. Let us specialize them to cases where either the duration n, the number of payment-periods m, or both approach ∞. Recall that failures continue to be assumed uniformly distributed within years of age.

Consider first the case where the insurances and life-annuities are whole-life, with $n = \infty$. The net single premium formulas for insurance and life annuity due reduce to

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x \quad , \qquad \ddot{a}_x^{(m)} = \alpha(m)\, \ddot{a}_x - \beta(m)$$

Next consider the case where n is again allowed to be finite, but where m is taken to go to ∞, or in other words, the payments are taken to be instantaneous. Recall that both $i^{(m)}$ and $d^{(m)}$ tend in the limit to the force-of-interest δ, so that the limits of the constants $\alpha(m), \beta(m)$ are respectively

$$\alpha(\infty) = \frac{d\, i}{\delta^2} \quad , \qquad \beta(\infty) = \frac{i - \delta}{\delta^2}$$

Recall also that the instantaneous-payment notations replace the superscripts ${}^{(m)}$ by an overbar. The single-premium formulas for instantaneous-payment insurance and life-annuities due become:

$$\overline{A}^1_{x:\overline{n}|} = \frac{i}{\delta} A^1_{x:\overline{n}|} \quad , \qquad \overline{a}_{x:\overline{n}|} = \frac{d\, i}{\delta^2}\, \ddot{a}_{x:\overline{n}|} - \frac{i - \delta}{\delta^2} (1 - v^n \, {}_np_x)$$

5.1.3 Net Single Premium Formulas — Case (ii)

In this setting, where the force of mortality is constant within single years of age, the formula for life-annuity net single premium is simpler than the one for insurance, because for integers $j, k \geq 0$,

$${}_{k+j/m}p_x = {}_kp_x \, e^{-j\mu_{x+k}/m}$$

Again restrict attention to the case where n is a positive integer, and calculate from first principles (as in 4.16)

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} \frac{1}{m} v^{k+j/m} \; {}_{j/m}p_{x+k} \; {}_kp_x \tag{5.7}$$

$$= \sum_{k=0}^{n-1} v^k \, {}_kp_x \sum_{j=0}^{m-1} \frac{1}{m} (ve^{-\mu_{x+k}})^{j/m} = \sum_{k=0}^{n-1} v^k \, {}_kp_x \frac{1 - vp_{x+k}}{m(1 - (vp_{x+k})^{1/m})}$$

where we have used the fact that when force of mortality is constant within years, $p_{x+k} = e^{-\mu_{x+k}}$. In order to compare this formula with equation (5.5) established under the assumption of uniform distribution of deaths within years of policy age, we apply the first-order Taylor series approximation about 0 for formula (5.7) with respect to the death-rates q_{x+k} inside the denominator-expression $1 - (vp_{x+k})^{1/m} = 1 - (v - vq_{x+k})^{1/m}$. (These annual death-rates q_{x+k} *are* actually small over a large range of ages for U.S. life tables.) The final expression in (5.7) will be Taylor-approximated in a slightly modified form: the numerator and denominator are both multiplied by the factor $1 - v^{1/m}$, and the term

$$(1 - v^{1/m})/(1 - (vp_{x+k})^{1/m})$$

will be analyzed first. The first-order Taylor-series approximation about $z = 1$ for the function $(1 - v^{1/m})/(1 - (vz)^{1/m})$ is

$$\frac{1 - v^{1/m}}{1 - (vz)^{1/m}} \approx 1 - (1 - z)\left[\frac{v^{1/m}\,(1 - v^{1/m})\,z^{-1+1/m}}{m\,(1 - (vz)^{1/m})^2}\right]_{z=1}$$

$$= 1 - (1 - z)\,\frac{v^{1/m}}{m\,(1 - v^{1/m})} = 1 - \frac{1 - z}{i^{(m)}}$$

Evaluating this Taylor-series approximation at $z = p_{x+k} = 1 - q_{x+k}$ then yields

$$\frac{1 - v^{1/m}}{1 - (vp_{x+k})^{1/m}} \approx 1 - \frac{q_{x+k}}{i^{(m)}}$$

Substituting this final approximate expression into equation (5.7), with numerator and denominator both multiplied by $1 - v^{1/m}$, we find for piecewise-constant force of mortality which is assumed small

$$\ddot{a}^{(m)}_{x:\overline{n}|} \approx \sum_{k=0}^{n-1} v^k\,{}_kp_x\,\frac{1 - vp_{x+k}}{m(1 - v^{1/m})}\,(1 - q_{x+k}/i^{(m)})$$

$$\approx \sum_{k=0}^{n-1} v^k\,{}_kp_x\,\frac{1}{d^{(m)}}\left\{1 - vp_{x+k} - \frac{1 - v}{i^{(m)}}\,q_{x+k}\right\} \qquad (5.8)$$

where in the last line we have applied the identity $m(1 - v^{1/m}) = d^{(m)}$ and discarded a quadratic term in q_{x+k} within the large curly bracket.

We are now close to our final objective: proving that the formulas (5.5) and (5.6) of the previous subsection are in the present setting still valid as approximate formulas. *Indeed, we now prove that the final expression (5.8) is precisely equal to the right-hand side of formula (5.6). The interest of this result is that (5.6) applied to piecewise-uniform mortality* (Case (i)), *while we are presently operating under the assumption of piecewise-constant hazards* (Case ii). The proof of our assertion requires us to apply simple identities in several steps. First, observe that (5.8) is equal by definition to

$$\frac{1}{d^{(m)}} \left[\ddot{a}_{\overline{x:n}|} \,-\, a_{\overline{x:n}|} \,-\, v^{-1}\,\frac{1-v}{i^{(m)}}\,A^{\;1}_{\overline{x:n}|} \right] \tag{5.9}$$

Second, apply the general formula for $\ddot{a}_{\overline{x:n}|}$ as a sum to check the identity

$$\ddot{a}_{\overline{x:n}|} \;=\; \sum_{k=0}^{n-1} v^k\,{}_kp_x \;=\; 1 \,-\, v^n\,{}_np_x \,+\, a_{\overline{x:n}|} \tag{5.10}$$

and third, recall the identity

$$\ddot{a}_{\overline{x:n}|} \;=\; \frac{1}{d}\left(1 \,-\, A^{1}_{\overline{x:n}|} \,-\, v^n\,{}_np_x\right) \tag{5.11}$$

Substitute the identities (5.10) and (5.11) into expression (5.9) to re-express the latter as

$$\begin{aligned} &\frac{1}{d^{(m)}}\left[1 \,-\, v^n\,{}_np_x \,-\, \frac{i}{i^{(m)}}\,(1 - \,v^n\,{}_np_x \,-\, d\,\ddot{a}_{\overline{x:n}|})\right] \\ &= \frac{d\,i}{d^{(m)}i^{(m)}}\,\ddot{a}_{\overline{x:n}|} \,+\, \frac{1}{d^{(m)}}\,(1 \,-\, v^n\,{}_np_x)\,(1 \,-\, \frac{i}{i^{(m)}}) \end{aligned} \tag{5.12}$$

The proof is completed by remarking that (5.12) coincides with expression (5.6) in the previous subsection.

Since formulas for the insurance and life annuity net single premiums can each be used to obtain the other when there are m payments per year, and since in the case of integer n, the pure endowment single premium $A_{\overline{x:n}|}^{\;\;1}$ does not depend upon m, it follows from the result of this section that *all* of the formulas derived in the previous section for case (i) can be used as approximate formulas (to first order in the death-rates q_{x+k}) also in case (ii).

5.2 Approximate Formulas via Case(i)

The previous Section developed a Taylor-series justification for using the very convenient net-single-premium formulas derived in case (i) (of uniform distribution of deaths within whole years of age) to approximate the corresponding formulas in case (ii) (constant force of mortality within whole years of age. The approximation was derived as a first-order Taylor series, up to linear terms in q_{x+k}. However, some care is needed in interpreting the result, because for this step of the approximation to be accurate, the year-by-year death-rates q_{x+k} must be small compared to the nominal rate of interest $i^{(m)}$. While this may be roughly valid at ages 15 to 50, at least in developed countries, this is definitely not the case, even roughly, at ages larger than around 55.

Accordingly, it is interesting to compare numerically, under several assumed death- and interest- rates, the individual terms $A^{(m)1}_{\;x:\overline{k+1}|} - A^{(m)1}_{\;x:\overline{k}|}$ which arise as summands under the different interpolation assumptions. (Here and throughout this Section, k is an integer.) We first recall the formulas for cases (i) and (ii), and for completeness supply also the formula for case (iii) (the Balducci interpolation assumption). Recall that Balducci's assumption was previously faulted both for complexity of premium formulas *and* lack of realism, because of its consequence that the force of mortality decreases within whole years of age. The following three formulas are exactly valid under the interpolation assumptions of cases (i), (ii), and (iii) respectively.

$$A^{(m)1}_{\;x:\overline{k+1}|} - A^{(m)1}_{\;x:\overline{k}|} \;=\; \frac{i}{i^{(m)}}\, v^{k+1}\, {}_kp_x \,\cdot\, q_{x+k} \tag{5.13}$$

$$A^{(m)1}_{\;x:\overline{k+1}|} - A^{(m)1}_{\;x:\overline{k}|} \;=\; v^{k+1}\, {}_kp_x\,(1 - p^{1/m}_{x+k})\, \frac{i + q_{x+k}}{1 + (i^{(m)}/m) - p^{1/m}_{x+k}} \tag{5.14}$$

$$A^{(m)1}_{\;x:\overline{k+1}|} - A^{(m)1}_{\;x:\overline{k}|} \;=\; v^{k+1}\, {}_kp_x\, q_{x+k} \sum_{j=0}^{m-1} \frac{p_{x+k}\, v^{-j/m}}{m\,(1 - \frac{j+1}{m}\, q_{x+k})\,(1 - \frac{j}{m}\, q_{x+k})} \tag{5.15}$$

Formula (5.13) is an immediate consequence of the formula $A^{(m)1}_{\;x:\overline{n}|} = i\, A^{1}_{x:\overline{n}|} \,/\, i^{(m)}$ derived in the previous section. To prove (5.14), assume (ii) and calculate from first principles and the identities $v^{-1/m} = 1 + i^{(m)}/m$ and

$p_{x+k} = \exp(-\mu_{x+k})$ that

$$\begin{aligned}
&\sum_{j=0}^{m-1} v^{k+(j+1)/m} \, {}_kp_x \left({}_{j/m}p_{x+k} - {}_{(j+1)/m}p_{x+k} \right) \\
= \;& v^{k+1} \, {}_kp_x \, v^{-1+1/m} \, (1 - e^{-\mu_{x+k}/m}) \sum_{j=0}^{m-1} (v \, e^{-\mu_{x+k}})^{j/m} \\
= \;& v^{k+1} \, {}_kp_x \, (1 - e^{-\mu_{x+k}/m}) \, \frac{1 - v \, p_{x+k}}{1 - (vp_{x+k})^{1/m}} \cdot \frac{v^{-1}}{v^{-1/m}} \\
= \;& v^{k+1} \, {}_kp_x \, (1 - e^{-\mu_{x+k}/m}) \, \frac{i \, + \, q_{x+k}}{1 + i^{(m)}/m \, - \, p_{x+k}^{1/m}}
\end{aligned}$$

Finally, for the Balducci case, (5.15) is established by calculating first

$$ {}_{j/m}p_{x+k} \; = \; \frac{p_{x+k}}{1 \, - \, {}_{1-j/m}q_{x+k+j/m}} \; = \; \frac{p_{x+k}}{1 \, - \, \frac{m-j}{m} \, q_{x+k}} $$

Then the left-hand side of (5.15) is equal to

$$\begin{aligned}
&\sum_{j=0}^{m-1} v^{k+(j+1)/m} \, {}_kp_x \left({}_{j/m}p_{x+k} - {}_{(j+1)/m}p_{x+k} \right) \\
= \;& v^{k+1} \, {}_kp_x \, q_{x+k} \, v^{-1+1/m} \sum_{j=0}^{m-1} \frac{p_{x+k} \, v^{j/m}}{m \, (1 - \frac{m-j}{m} q_{x+k}) \, (1 - \frac{m-j-1}{m} q_{x+k})}
\end{aligned}$$

which is seen to be equal to the right-hand side of (5.15) after the change of summation-index $j' = m - j - 1$.

Formulas (5.13), (5.14), and (5.15) are progressively more complicated, and it would be very desirable to stop with the first one if the choice of interpolation assumption actually made no difference. In preparing the following Table, the ratios both of formulas (5.14)/(5.13) and of (5.15)/(5.13) were calculated for a range of possible death-rates $q = q_{x+k}$, interest-rates i, and payment-periods-per-year m. We do not tabulate the results for the ratios (5.14)/(5.13) because these ratios were equal to 1 to three decimal places except in the following cases: the ratio was 1.001 when i ranged from 0.05 to 0.12 and $q = 0.15$ or when i was .12 or .15 and q was .12, achieving a value of 1.002 only in the cases where $q = i = 0.15$, $m \geq 4$.

Such remarkable correspondence between the net single premium formulas in cases (i), (ii) was by no means guaranteed by the previous Taylor series calculation, and is made only somewhat less surprising by the remark that the ratio of formulas (5.14)/(5.13) is smooth in both parameters q_{x+k}, i and exactly equal to 1 when either of these parameters is 0.

The Table shows a bit more variety in the ratios of (5.15)/(5.13), showing in part why the Balducci assumption is not much used in practice, but also showing that for a large range of ages and interest rates it also gives correct answers within 1 or 2 %. Here also there are many cases where the Balducci formula (5.15) agrees extremely closely with the usual actuarial (case (i)) formula (5.13). This also can be partially justified through the observation (a small exercise for the reader) that the ratio of the right-hand sides of formulas (5.15) divided by (5.13) are identical in either of the two limiting cases where $i = 0$ or where $q_{x+k} = 0$. The Table shows that the deviations from 1 of the ratio (5.15) divided by (5.13) are controlled by the parameter m and the interest rate, with the death-rate much less important within the broad range of values commonly encountered.

5.3 Net Level (Risk) Premiums

The general principle previously enunciated regarding equivalence of two different (certain) payment-streams if their present values are equal, has the following extension to the case of uncertain (time-of-death-dependent) payment streams: two such payment streams are equivalent (in the sense of having equal 'risk premiums') if their expected present values are equal. This definition makes sense if each such equivalence is regarded as the matching of random income and payout for the insurer with respect to each of a large number of independent (and identical) policies. Then the **Law of Large Numbers** has the interpretation that the actual random net payout minus income for the aggregate of the policies *per policy* is with very high probability very close (percentagewise) to the mathematical expectation of the difference between the single-policy payout and income. That is why, from a pure-risk perspective, before allowing for administrative expenses and the 'loading' or cushion which an insurer needs to maintain a very tiny probability of going bankrupt after starting with a large but fixed fund of reserve

Table 5.2: Ratios of Values (5.15)/(5.13)

q_{x+k}	i	m= 2	m= 4	m= 12
.002	.03	1.015	1.007	1.002
.006	.03	1.015	1.007	1.002
.02	.03	1.015	1.008	1.003
.06	.03	1.015	1.008	1.003
.15	.03	1.015	1.008	1.003
.002	.05	1.025	1.012	1.004
.006	.05	1.025	1.012	1.004
.02	.05	1.025	1.012	1.004
.06	.05	1.025	1.013	1.005
.15	.05	1.026	1.014	1.005
.002	.07	1.034	1.017	1.006
.006	.07	1.034	1.017	1.006
.02	.07	1.035	1.017	1.006
.06	.07	1.035	1.018	1.006
.15	.07	1.036	1.019	1.007
.002	.10	1.049	1.024	1.008
.006	.10	1.049	1.024	1.008
.02	.10	1.049	1.024	1.008
.06	.10	1.050	1.025	1.009
.15	.10	1.051	1.027	1.011
.002	.12	1.058	1.029	1.010
.006	.12	1.058	1.029	1.010
.02	.12	1.059	1.029	1.010
.06	.12	1.059	1.030	1.011
.15	.12	1.061	1.032	1.013
.002	.15	1.072	1.036	1.012
.006	.15	1.072	1.036	1.012
.02	.15	1.073	1.036	1.012
.06	.15	1.074	1.037	1.013
.15	.15	1.075	1.039	1.016

capital, this expected difference should be set equal to 0 in figuring premiums. The resulting rule for calculation of the premium amount P which must multiply the *unit* amount in a specified payment pattern is as follows:

> P = Expected present value of life insurance, annuity, or endowment contract proceeds divided by the expected present value of a unit amount paid regularly, according to the specified payment pattern, until death or expiration of term.

5.4 Benefits Involving Fractional Premiums

The general principle for calculating risk premiums sets up a balance between expected payout by an insurer and expected payment stream received as premiums. In the simplest case of level payment streams, the insurer receives a life-annuity due with level premium P, and pays out according to the terms of the insurance product purchased, say a term insurance. If the insurance purchased pays only at the end of the year of death, but the premium payments are made m times per year, then the balance equation becomes

$$A^{1}_{x:\overline{n}|} \;=\; P \cdot m\, \ddot{a}^{(m)}_{x:\overline{n}|}$$

for which the solution P is called the *level risk premium for a term insurance*. The reader should distinguish this premium from the level premium payable m times yearly for an insurance which pays at the end of the $(1/m)^{th}$ year of death. In the latter case, where the number of payment periods per year for the premium agrees with that for the insurance, the balance equation is

$$A^{(m)}{}^{1}_{x:\overline{n}|} \;=\; P \cdot m\, \ddot{a}^{(m)}_{x:\overline{n}|}$$

In standard actuarial notations for premiums, not given here, level premiums are *annualized* (which would result in the removal of a factor m from the right-hand sides of the last two equations).

Two other applications of the balancing-equation principle can be made in calculating level premiums for insurances which either (a) deduct the additional premium payments for the remainder of the year of death from the insurance proceeds, or (b) refund a pro-rata share of the premium for the portion of the $1/m$ year of death from the instant of death to the end of the $1/m$ year of death. Insurance contracts with provision (a) are called insurances with *installment* premiums: the meaning of this term is that the insurer views the full year's premium as due at the beginning of the year, but that for the convenience of the insured, payments are allowed to be made in installments at m regularly spaced times in the year. Insurances with provision (b) are said to have *apportionable refund of premium*, with the implication that premiums are understood to cover only the period of the year during which the insured is alive . First in case (a), the expected amount paid out by the insurer, if each level premium payment is P and the face amount of the policy is $F(0)$, is equal to

$$F(0)\, A^{(m)\,1}_{x:\overline{n}|} \;-\; \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} v^{k+(j+1)/m}\; {}_{k+j/m}p_x \,\cdot\, {}_{1/m}q_{x+k+j/m}\, (m-1-j)\, P$$

and the exact balance equation is obtained by setting this equal to the expected amount paid in, which is again $P\, m\, \ddot{a}^{(m)}_{x:\overline{n}|}$. Under the interpolation assumption of case (i), using the same reasoning which previously led to the simplified formulas in that case, this balance equation becomes

$$F(0)\, A^{(m)\,1}_{x:\overline{n}|} \;-\; A^{1}_{x:\overline{n}|}\, \frac{P}{m} \sum_{j=0}^{m-1} v^{-(m-j-1)/m}\, (m-j-1) \;=\; P\, m\, \ddot{a}^{(m)}_{x:\overline{n}|} \tag{5.16}$$

Although one could base an exact calculation of P on this equation, a further standard approximation leads to a simpler formula. If the term $(m-j-1)$ is replaced in the final sum by its average over j, or by $m^{-1}\sum_{j=0}^{m-1}(m-j-1) \;=\; m^{-1}\,(m-1)m/2 \;=\; (m-1)/2$, we obtain the installment premium formula

$$P \;=\; \frac{F(0)\, A^{(m)\,1}_{x:\overline{n}|}}{m\ddot{a}^{(m)}_{x:\overline{n}|} \;+\; \frac{m-1}{2}\, A^{(m)\,1}_{x:\overline{n}|}}$$

and this formula could be related using previous formulas derived in Section 5.1 to the insurance and annuity net single premiums with only one payment period per year.

In the case of the apportionable return of premium, the only assumption usually considered is that of case (i), that the fraction of a single premium payment which will be returned is on average $1/2$ regardless of which of the $1/m$ fractions of the year contains the instant of death. The balance equation is then very simple:

$$A^{(m)1}_{\overline{x:n}|} \, (F(0) + \frac{1}{2}P) = P \, m \, \ddot{a}^{(m)}_{\overline{x:n}|} \tag{5.17}$$

and this equation has the straightforward solution

$$P = \frac{F(0) \, A^{(m)1}_{\overline{x:n}|}}{m \, \ddot{a}^{(m)}_{\overline{x:n}|} - \frac{1}{2} \, A^{(m)1}_{\overline{x:n}|}}$$

It remains only to remark what is the effect of *loading* for administrative expenses and profit on insurance premium calculation. If all amounts paid out by the insurer were equally loaded (i.e., multiplied) by the factor $1+L$, then formula (5.17) would involve the loading in the second term of the denominator, but this is apparently not the usual practice. In both the apportionable refund and installment premium contracts, as well as the insurance contracts which do not modify proceeds by premium fractions, it is apparently the practice to load the level premiums P directly by the factor $1+L$, which can easily be seen to be equivalent to inflating the face-amount $F(0)$ in the balance-formulas by this factor.

5.5 *Exercise Set 5*

(1). Show from first principles that for all integers x, n, and all fixed interest-rates and life-distributions

$$a_{\overline{x:n}|} = \ddot{a}_{\overline{x:n}|} - 1 + v^n \, {}_np_x$$

(2). Show from first principles that for all integers x, and all fixed interest-rates and life-distributions

$$A_x = v \, \ddot{a}_x - a_x$$

Show further that this relation is obtained by taking the expectation on both sides of an identity in terms of present values of payment-streams, an identity whoch holds for *each* value of (the greatest integer $[T]$ less than or equal to) the exact-age-at-death random variable T.

(3). Using the same idea as in problem **(2)**, show that (for all x, n, interest rates, and life-distributions)

$$A^{1}_{x:\overline{n}|} = v\,\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|}$$

(4). Suppose that a life aged x (precisely, where x is an integer) has the survival probabilities $p_{x+k} = 0.98$ for $k = 0, 1, , \ldots, 9$. Suppose that he wants to purchase a term insurance which will pay \$30,000 at the end of the quarter-year of death if he dies within the first five years, and will pay \$10,000 (also at the end of the quarter-year of death) if he dies between exact ages 5, 10. In both parts (a), (b) of the problem, assume that the interest rate is fixed at 5%, and assume wherever necessary that the individual's distribution of death-time is uniform within each whole year of age.

(a) Find the net single premium of the insurance contract described.

(b) Suppose that the individual purchasing the insurance described wants to pay level premiums semi-annually, beginning immediately. Find the amount of each semi-annual payment.

(5). Re-do problem **(4)** assuming *in place of the uniform distribution of age at death* that the insured individual has constant force of mortality within each whole year of age. Give your numerical answers to at least 6 significant figures so that you can compare the exact numerical answers in these two problems.

(6). Using the exact expression for the interest-rate functions $i^{(m)}$, $d^{(m)}$ respectively as functions of i and d, expand these functions in Taylor series about 0 up to quadratic terms. Use the resulting expressions to approximate the coefficients $\alpha(m)$, $\beta(m)$ which were derived in the Chapter. Hence justify the so-called *traditional approximation*

$$\ddot{a}^{(m)}_x \approx \ddot{a}_x - \frac{m-1}{2m}$$

(7). Justify the 'traditional approximation' (the displayed formula in Exercise 6) as an exact formula in the case (i) in the limit $i \to 0$, by filling in the details of the following argument.

No matter which policy-year is the year of death of the annuitant, the policy with $m = 1$ *(and expected present value* $\ddot{a}_x$*) pays* 1 *at the beginning* *of that year while the policy with $m > 1$ pays amounts $1/m$ at the beginning of each $1/m$'th year in which the annuitant is alive. Thus, the annuity with one payment per years pays more than the annuity with $m > 1$ by an absolute amount $1-(T_m-[T]+1/m)$. Under assumption (i), $T_m-[T]$ is a discrete random variable taking on the possible values $0,\ 1,\ \ldots,(m-1)/m$ each with probability $1/m$. Disregard the interest and present-value discounting on the excess amount $1-(T_m-[T])/m$ paid by the m-payment-per year annuity, and show that it is exactly $(m-1)/2m$.*

(8). Give an exact formula for the error of the 'traditional approximation' given in the previous problem, in terms of m, the constant interest rate i (or $v = (1+i)^{-1}$), and the constant force μ of mortality, when the lifetime T is assumed to be distributed precisely as an *Exponential*(μ) random variable.

(9). Show that the ratio of formulas (5.14)/(5.13) is 1 whenever either q_{x+k} or i is set equal to 0.

(10). Show that the ratio of formulas (5.15)/(5.13) is 1 whenever either q_{x+k} or i is set equal to 0.

(11). For a temporary life annuity on a life aged 57, with benefits *deferred* for three years, you are given that $\mu_x = 0.04$ is constant, $\delta = .06$, that premiums are paid continuously (with $m = \infty$) only for the first two years, at rate $\overline{P}$ per year, and that the annuity benefits are payable at beginnings of years according to the following schedule:

Year	0	1	2	3	4	5	6	7	8+
Benefit	0	0	0	10	8	6	4	2	0

(a) In terms of $\overline{P}$, calculate the expected present value of the premiums paid.

(b) Using the equivalence principle, calculate $\overline{P}$ numerically.

(12). You are given that (i) $q_{60} = 0.3$, $q_{61} = 0.4$, (ii) f denotes the probability that a life aged 60 will die between ages 60.5 and 61.5 under the assumption of uniform distribution of deaths within whole years of age, and (iii) g denotes the probability that a life aged 60 will die between ages 60.5 and 61.5 under the Balducci assumption. Calculate $10,000 \cdot (g - f)$ numerically, with accuracy to the nearest whole number.

(13). You are given that $S(40) = 0.500$, $S(41) = 0.475$, $i = 0.06$, $\overline{A}_{41} = 0.54$, and that deaths are uniformly distributed over each year of age. Find A_{40} exactly.

(14). If a mortality table follows Gompertz' law (with exponent c), prove that

$$\mu_x \;=\; \overline{A}_x \Big/ \overline{a}'_x$$

where $\overline{A}_x$ is calculated at interest rate i while $\overline{a}'_x$ is calculated at a rate of interest $i' = \frac{1+i}{c} - 1$.

(15). You are given that $i = 0.10, q_x = 0.05$, and $q_{x+1} = 0.08$, and that deaths are uniformly distributed over each year of age. Calculate $\overline{A}^{\;1}_{x:\overline{2}|}$.

(16). A special life insurance policy to a life aged x provides that if death occurs at any time within 15 years, then the only benefit is the return of premiums with interest compounded to the end of the year of death. If death occurs after 15 years, the benefit is \$10,000. In either case, the benefit is paid at the end of the year of death. If the premiums for this policy are to be paid yearly for only the first 5 years (starting at the time of issuance of the policy), then find a simplified expression for the level annual pure-risk premium for the policy, in terms of standard actuarial and interest functions.

(17). Prove that for every m, n, x, k, the net single premium for an n-year term insurance for a life aged x, with benefit deferred for k years, and payable at the end of the $1/m$ year of death is given by either side of the identity

$$A^{n+k\;m}_{\overline{x}|} - A^{k\;m}_{\overline{x}|} \;=\; {}_kE_x\; A^{n\;m}_{\overline{x+k}|}$$

First prove the identity algebraically; then give an alternative, intuitive explanation of why the right-hand side represents the expected present value of the same contingent payment stream as the left-hand side.

5.6 Worked Examples

Overview of Premium Calculation for Single-Life Insurance & Annuities

Here is a schematic overview of the calculation of net single and level premiums for life insurances and life annuities, based on life-table or theoretical survival probabilities and constant interest or discount rate. We describe the general situation and follow a specific case study/example throughout.

(I) First you will be given information about the constant assumed interest rate in any of the equivalent forms $i^{(m)}$, $d^{(m)}$, or δ, and you should immediately convert to find the effective annual interest rate (APR) i and one-year discount factor $v = 1/(1+i)$. In our case-study, assume that

$$\text{the force of interest} \quad \delta \quad \text{is constant} \quad = -\ln(0.94)$$

so that $i = \exp(\delta) - 1 = (1/0.94) - 1 = 6/94$, and $v = 0.94$. In terms of this quantity, one immediately answers a question such as "what is the present value of \$1 at the end of $7\frac{1}{2}$ years ?" by: $v^{7.5}$.

(II) Next you must be given **either** a theoretical survival function for the random age at death of a life aged x, in any of the equivalent forms $S(x+t)$, ${}_tp_x$, $f(x+t)$, or $\mu(x+t)$, **or** a cohort-form life-table, e.g.,

$l_{25} =$	10,000	${}_0p_{25} =$	1.0
$l_{26} =$	9,726	${}_1p_{25} =$	0.9726
$l_{27} =$	9,443	${}_2p_{25} =$	0.9443
$l_{28} =$	9,137	${}_3p_{25} =$	0.9137
$l_{29} =$	8,818	${}_4p_{25} =$	0.8818
$l_{30} =$	8,504	${}_5p_{25} =$	0.8504

From such data, one calculates immediately that (for example) the probability of dying at an odd attained-age between 25 and 30 inclusive is

$$(1 - 0.9726) + (0.9443 - 0.9137) + (0.8818 - 0.8504) = 0.0894$$

The generally useful additional column to compute is:

$$q_{25} = 1 - {}_1p_{25} = 0.0274, \quad {}_1p_{25} - {}_2p_{25} = 0.0283, \quad {}_2p_{25} - {}_3p_{25} = 0.0306$$

$$_3p_{25} \, - \, _4p_{25} \, = \, 0.0319, \qquad _4p_{25} \, - \, _5p_{25} \, = \, 0.0314$$

(III) In any problem, the terms of the life insurance or annuity to be purchased will be specified, and you should re-express its present value in terms of standard functions such as $a_{\overline{x:n}|}$ or $A^1_{\overline{x:n}|}$. For example, suppose a life aged x purchases an endowment/annuity according to which he receives \$10,000 once a year starting at age $x+1$ until either death occurs or n years have elapsed, and if he is alive at the end of n years he receives \$15,000. This contract is evidently a superposition of a n-year pure endowment with face value \$15,000 and a n-year temporary life annuity-immediate with yearly payments \$10,000. Thus, the expected present value (= *net single premium*) is

$$10,000 \, a_{\overline{x:n}|} \, + \, 15,000 \, _np_x \, v^n$$

In our case-study example, this expected present value is

$$= \, 10000 \, \Big(0.94(0.9726) \, + \, 0.94^2(0.9443) \, + \, 0.94^3(0.9137) \, +$$

$$+ \, 0.94^4(0.8818) \, + \, 0.94^5(0.8504)\Big) \, + \, 15000(0.94^5 \, \cdot \, 0.8504)$$

The annuity part of this net single premium is \$38,201.09 , and the pure-endowment part is \$9,361.68 , for a total net single premium of \$47,562.77

(IV) The final part of the premium computation problem is to specify the type of payment stream with which the insured life intends to pay for the contract whose expected present value has been figured in step *(III)*. If the payment is to be made at time 0 in one lump sum, then the net single premium has already been figured and we are done. If the payments are to be constant in amount (*level premiums*), once a year, to start immediately, and to terminate at death or a maximum of n payments, then we divide the net single premium by the expected present value of a unit life annuity $\ddot{a}_{\overline{x:n}|}$. In general, to find the premium we divide the net single premium of *(III)* by the expected present value of a unit amount paid according to the desired premium-payment stream.

In the case-study example, consider two cases. The first is that the purchaser aged x wishes to pay in two equal installments, one at time 0 and one after 3 years (with the second payment to be made only if he is alive at that time). The expected present value of a unit amount paid in this fashion is

$$1 \, + \, v^3 \, _3p_x \, = \, 1 \, + \, (0.94)^3 \, 0.9137 \, = \, 1.7589$$

Thus the premium amount to be paid at each payment time is

$$\$\,47,563\,/\,1.7589 \;=\; \$27,041$$

Alternatively, as a second example, suppose that the purchaser is in effect taking out his annuity/endowment in the form of a loan, and agrees to (have his estate) repay the loan unconditionally (i.e. without regard to the event of his death) over a period of 29 years, with 25 equal payments to be made every year beginning at the end of 5 years. In this case, no probabilities are involved in valuing the payment stream, and the present value of such a payment stream of unit amounts is

$$v^4\, a_{\overline{25}|} \;=\; (0.94)^4\,(.94/.06)\,(1-(0.94)^{25}) \;=\; 9.627$$

In this setting, the amount of each of the equal payments must be

$$\$\,47,563\,/\,9.627 \;=\; \$\,4941$$

(V) To complete the circle of ideas given here, let us re-do the case-study calculation of paragraphs (III) to cover the case where the insurance has quarterly instead of annual payments. Throughout, assume that deaths within years of attained age are uniformly distributed (case(i)).

First, the expected present value to find becomes

$$10,000\,\mathrm{a}^{(4)}_{\overline{x:n}|} \;+\; 15,000\,A_{\overline{x:n}|}^{\;\;1} \;=\; 10000\,\Big(\ddot{\mathrm{a}}^{(4)}_{\overline{x:n}|} \;-\; \frac{1}{4}\,(1-v^n\;{}_np_x)\Big) \;+\; 15000\,v^n\;{}_np_x$$

which by virtue of (5.6) is equal to

$$=\; 10000\,\alpha(4)\,\ddot{\mathrm{a}}_{\overline{x:n}|} \;-\; (1-v^n\;{}_np_x)\,(10000\beta(4) \;+\; 2500) \;+\; 15000\,v^n\;{}_np_x$$

In the particular case with $v = 0.94$, $x = 25$, $n = 5$, and cohort life-table given in (II), the net single premium for the endowment part of the contract has exactly the same value $9361.68 as before, while the annuity part now has the value

$$10000\,(1.0002991)\,(1 \;+\; 0.94(0.9726) \;+\; 0.94^2(0.9443) \;+\; 0.94^3(0.9137)\,+$$
$$+\,0.94^4(0.8818)) \;-\; (6348.19)\,(1-0.94^5(0.8504)) \;=\; 39586.31$$

Thus the combined present value is 48947.99: the increase of 1385 in value arises mostly from the earlier annuity payments: consider that the interest on the annuity value for one-half year is $38201(0.94^{-0.5}-1) = 1200$.

5.7 Useful Formulas from Chapter 5

$$P([T] = x+k,\, T-[T] < t \,|\, T \geq x) \;=\; \int_{x+k}^{x+k+t} \frac{f(y)}{S(x)}\, dy \;=\; {}_tq_{x+k}\; {}_kp_x$$

p. 125

$$P(\, T-[T] \leq t \,|\, [T] = x+k) = \frac{{}_tq_{x+k}}{q_{x+k}}$$

p. 125

$$A^{(m)\,1}_{x:\overline{n}|} \;=\; \frac{i}{i^{(m)}}\, A^{1}_{x:\overline{n}|} \qquad \text{under (i)}$$

p. 127

$$\ddot{a}^{(m)}_{x:\overline{n}|} \;=\; \frac{1-A^{(m)}_{x:\overline{n}|}}{d^{(m)}} \;=\; \frac{1}{d^{(m)}}\left[1 \,-\, \frac{i}{i^{(m)}}\, A^{1}_{x:\overline{n}|} \,-\, {}_np_x\, v^n\right]$$

p. 127

$$\ddot{a}^{(m)}_{x:\overline{n}|} \;=\; \frac{d\, i}{d^{(m)}\, i^{(m)}}\, \ddot{a}_{x:\overline{n}|} \,+\, \left(1-\frac{i}{i^{(m)}}\right) \frac{1-v^n\; {}_np_x}{d^{(m)}} \qquad \text{under (i)}$$

p. 127

$$\ddot{a}^{(m)}_{x:\overline{n}|} \;=\; \alpha(m)\;\, \ddot{a}_{x:\overline{n}|} \,-\, \beta(m)\,(1 \,-\, {}_np_x\, v^n) \qquad \text{under (i)}$$

p. 127

$$\alpha(m) \;=\; \frac{d\, i}{d^{(m)}\, i^{(m)}} \;\;, \qquad \beta(m) \;=\; \frac{i \,-\, i^{(m)}}{d^{(m)}\, i^{(m)}}$$

p. 128

$$\alpha(m) \;=\; 1 \quad , \qquad \beta(m) \;=\; \frac{m-1}{2m} \qquad \text{when} \quad i = 0$$

p. 128

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k \, {}_kp_x \, \frac{1 - vp_{x+k}}{m(1-(vp_{x+k})^{1/m})} \qquad \text{under (ii)}$$

p. 129

$$A^{(m)\,1}_{x:\overline{k+1}|} - A^{(m)\,1}_{x:\overline{k}|} = \frac{i}{i^{(m)}} \, v^{k+1} \, {}_kp_x \cdot q_{x+k} \qquad \text{under (i)}$$

p. 132

$$A^{(m)\,1}_{x:\overline{k+1}|} - A^{(m)\,1}_{x:\overline{k}|} = v^{k+1} \, {}_kp_x \, (1 - p^{1/m}_{x+k}) \, \frac{i + q_{x+k}}{1 + (i^{(m)}/m) - p^{1/m}_{x+k}} \qquad \text{under (ii)}$$

p. 132

$$A^{(m)\,1}_{x:\overline{k+1}|} - A^{(m)\,1}_{x:\overline{k}|} = v^{k+1} \, {}_kp_x \, q_{x+k} \sum_{j=0}^{m-1} \frac{p_{x+k} \, v^{-j/m}}{m\,(1 - \frac{j+1}{m} \, q_{x+k})\,(1 - \frac{j}{m} \, q_{x+k})}$$

under (iii) p. 132

$$\text{Level Installment Risk Premium} = \frac{F(0)\, A^{(m)\,1}_{x:\overline{n}|}}{m\ddot{a}^{(m)}_{x:\overline{n}|} + \frac{m-1}{2} A^{(m)\,1}_{x:\overline{n}|}}$$

p. 137

$$\text{Apportionable Refund Risk Premium} = \frac{F(0)\, A^{(m)\,1}_{x:\overline{n}|}}{m\,\ddot{a}^{(m)}_{x:\overline{n}|} - \frac{1}{2} A^{(m)\,1}_{x:\overline{n}|}}$$

p. 138

Chapter 6

Commutation Functions, Reserves & Select Mortality

In this Chapter, we consider first the historically important topic of *Commutation Functions* for actuarial calculations, and indicate why they lose their computational usefulness as soon as the insurer entertains the possibility (as demographers often do) that life-table survival probabilities display some slow secular trend with respect to year of birth. We continue our treatment of premiums and insurance contract valuation by treating briefly the idea of insurance reserves and policy cash values as the life-contingent analogue of mortgage amortization and refinancing. The Chapter concludes with a brief section on Select Mortality, showing how models for select-population mortality can be used to calculate whether modified premium and deferral options are sufficient protections for insurers to insure such populations.

6.1 Idea of Commutation Functions

The Commutation Functions are a computational device to ensure that net single premiums for life annuities, endowments, and insurances from the same life table and figured at the same interest rate, for lives of differing ages and for policies of differing durations, can all be obtained from a single table look-up. Historically, this idea has been very important in saving calculational labor when arriving at premium quotes. Even now, assuming that a governing life table and interest rate are chosen provisionally, company employees without quantitative training could calculate premiums in a spreadsheet format with the aid of a life table.

To fix the idea, consider first the contract with the simplest net-single-premium formula, namely the pure n-year endowment. The expected present value of $1 one year in the future *if the policyholder aged* x *is alive at that time* is denoted in older books as ${}_nE_x$ and is called the **actuarial present value** of a life-contingent n-year future payment of 1:

$$A_{x:\overline{n|}}^{\;\;1} \;=\; {}_nE_x \;=\; v^n \, {}_np_x$$

Even such a simple life-table and interest-related function would seem to require a table in the *two* integer parameters x, n, but the following expression immediately shows that it can be recovered simply from a single tabulated column:

$$A_{x:\overline{n|}}^{\;\;1} \;=\; \frac{v^{n+x}\, l_{n+x}}{v^x\, l_x} \;=\; \frac{D_{x+n}}{D_x} \;, \qquad D_y \equiv v^y\, l_y \tag{6.1}$$

In other words, at least for integer ages and durations we would simply augment the insurance-company life-table by the column D_x. The addition of just a few more columns allows the other main life-annuity and insurance quantities to be recovered with no more than simple arithmetic. Thus, if we begin by considering whole life insurances (with only one possible payment at the end of the year of death), then the net single premium is re-written

$$A_x \;=\; A_{x:\overline{\infty|}}^{1} \;=\; \sum_{k=0}^{\infty} v^{k+1}\, {}_kp_x \cdot q_{x+k} \;=\; \sum_{k=0}^{\infty} \frac{v^{x+k+1}\,(l_{x+k} - l_{x+k+1})}{v^x\, l_x}$$

$$=\; \sum_{y=x}^{\infty} v^{y+1}\, \frac{d_y}{D_x} \;=\; \frac{M_x}{D_x} \;, \qquad M_x \;\equiv\; \sum_{y=x}^{\infty} v^{y+1}\, d_y$$

The insurance of finite duration also has a simple expression in terms of the same *commutation* columns M, D :

$$A_{x:\overline{n|}}^{1} \;=\; \sum_{k=0}^{n-1} v^{k+1}\, \frac{d_{k+x}}{D_x} \;=\; \frac{M_x - M_{x+n}}{D_x} \tag{6.2}$$

Next let us pass to to life annuities. Again we begin with the life annuity-due of infinite duration:

$$\ddot{a}_x \;=\; \ddot{a}_{\overline{x:\infty|}} \;=\; \sum_{k=0}^{\infty} v^{k+x}\frac{l_{k+x}}{D_x} \;=\; \frac{N_x}{D_x} \quad , \qquad N_x \;=\; \sum_{y=x}^{\infty} v^y \, l_y \tag{6.3}$$

The commutation column N_x turns is the reverse cumulative sum of the D_x column:

$$N_x \quad = \quad \sum_{y=x}^{\infty} D_y$$

The expected present value for the finite-duration life-annuity due is obtained as a simple difference

$$\ddot{a}_{\overline{x:n|}} \;=\; \sum_{k=0}^{n-1} v^{k+x}\,\frac{l_{x+k}}{D_x} \;=\; \frac{N_x - N_{x+n}}{D_x}$$

There is no real need for a separate commutation column M_x since, as we have seen, there is an identity relating net single premiums for whole life insurances and annuities:

$$A_x \;=\; 1 \,-\, d\,\ddot{a}_x$$

Writing this identity with A_x and $\ddot{a}_x$ replaced by their respective commutation-function formulas, and then multupliying through by D_x, immediately yields

$$M_x \;=\; D_x \,-\, d\,N_x \tag{6.4}$$

Based on these tabulated **commutation columns** D, M, N, a quantitatively unskilled person could use simple formulas and tables to provide on-the-spot insurance premium quotes, a useful and desirable outcome even in these days of accessible high-powered computing. Using the case-(i) interpolation assumption, the m-payment-per-year net single premiums $A^{(m)}{}^{\,1}_{\overline{x:n|}}$ and $\ddot{a}^{(m)}_{\overline{x:n|}}$ would be related to their single-payment counterparts (whose commutation-function formulas have just been provided) through the standard formulas

$$A^{(m)}{}^{\,1}_{\overline{x:n|}} \;=\; \frac{i}{i^{(m)}}\,A^{\,1}_{\overline{x:n|}} \quad , \qquad \ddot{a}^{(m)}_{\overline{x:n|}} \;=\; \alpha(m)\,\ddot{a}_{\overline{x:n|}} \,-\, \beta(m)\left(1 -\, A_{\overline{x:n|}}^{\;\;\,1}\right)$$

Table 6.1: Commutation Columns for the simulated US Male Illustrative Life-Table, Table 1.1, with APR interest-rate 6%.

Age x	l_x	D_x	N_x	M_x
0	100000	100000.00	1664794.68	94233.66
5	96997	72481.80	1227973.94	69507.96
10	96702	53997.89	904739.10	51211.65
15	96435	40238.96	663822.79	37574.87
20	95840	29883.37	484519.81	27425.65
25	95051	22146.75	351486.75	19895.48
30	94295	16417.71	252900.70	14315.13
35	93475	12161.59	179821.07	10178.55
40	92315	8975.07	125748.60	7117.85
45	90402	6567.71	85951.37	4865.17
50	87119	4729.55	56988.31	3225.75
55	82249	3336.63	36282.55	2053.73
60	75221	2280.27	21833.77	1235.87
65	65600	1486.01	12110.79	685.52
70	53484	905.34	5920.45	335.12
75	39975	505.65	2256.41	127.72

That is,

$$A^{(m)}{}^{\,1}_{x:\overline{n}|} = \frac{i}{i^{(m)}} \cdot \frac{M_x - M_{x+n}}{D_x}$$

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \alpha(m)\,\frac{N_x - N_{x+n}}{D_x} - \beta(m)\,(1 - \frac{D_{x+n}}{D_x})$$

To illustrate the realistic sizes of commutation-column numbers, we reproduce as Table 6.1 the main commutation-columns for 6% APR interest, in 5-year intervals, for the illustrative simulated life table given on page 3.

6.1.1 Variable-benefit Commutation Formulas

The only additional formulas which might be commonly needed in insurance sales are the variable-benefit term insurances with linearly increasing or decreasing benefits, and we content ourselves with showing how an additional commutation-column could serve here. First consider the infinite-duration policy with linearly increasing benefit

$$IA_x \;=\; \sum_{k=0}^{\infty} (k+1)\, v^{k+1}\, {}_kp_x \cdot q_{x+k}$$

This net single premium can be written in terms of the commutation functions already given together with

$$R_x \;=\; \sum_{k=0}^{\infty} (x+k+1)\, v^{x+k+1}\, d_{x+k}$$

Clearly, the summation defining IA_x can be written as

$$\sum_{k=0}^{\infty} (x+k+1)\, v^{k+1}\, {}_kp_x \cdot q_{x+k} \;-\; x \sum_{k=0}^{\infty} v^{k+1}\, {}_kp_x \cdot q_{x+k} \;=\; \frac{R_x}{D_x} \;-\; x\,\frac{M_x}{D_x}$$

Then, as we have discussed earlier, the finite-duration linearly-increasing-benefit insurance has the expression

$$IA^{1}_{x:\overline{n}|} \;=\; IA_x - \sum_{k=n}^{\infty} (k+1)\, v^{k+x+1}\, \frac{d_{x+k}}{D_x} \;=\; \frac{R_x - xM_x}{D_x} \;-\; \frac{R_{x+n} - xM_{x+n}}{D_x}$$

and the net single premium for the linearly-decreasing-benefit insurance, which pays benefit $n-k$ if death occurs between exact policy ages k and $k+1$ for $k = 0, \ldots, n-1$, can be obtained from the increasing-benefit insurance through the identity

$$DA^{1}_{x:\overline{n}|} \;=\; (n+1)A^{1}_{x:\overline{n}|} - IA^{1}_{x:\overline{n}|}$$

Throughout *all* of our discussions of premium calculation — not just the present consideration of formulas in terms of commutation functions — we have assumed that for ages of prospective policyholders, the same interest rate and life table would apply. In a future Chapter, we shall consider the problem of premium calculation and reserving under variable and stochastic interest-rate assumptions, but for the present we continue to fix the interest rate i. Here we consider briefly what would happen to premium calculation and the commutation formalism if the key assumption that the same life table applies to all insureds were to be replaced by an assumption involving interpolation between (the death rates defined by) two separate life tables applying to different birth cohorts. This is a particular case of a topic which we shall also take up in a future chapter, namely how extra (*'covariate'*) information about a prospective policyholder might change the survival probabilities which should be used to calculate premiums *for that policyholder*.

6.1.2 Secular Trends in Mortality

Demographers recognize that there are secular shifts over time in life-table age-specific death-rates. The reasons for this are primarily related to public health (e.g., through the eradication or successful treatment of certain disease conditions), sanitation, diet, regulation of hours and conditions of work, etc. As we have discussed previously in introducing the concept of force of mortality, the modelling of shifts in mortality patterns with respect to likely causes of death at different ages suggests that it is most natural to express shifts in mortality in terms of force-of-mortality and death rates rather than in terms of probability density or population-wide relative numbers of deaths in various age-intervals. One of the simplest models of this type, used for projections over limited periods of time by demographers (*cf.* the text *Introduction to Demography* by M. Spiegelman), is to view age-specific death-rates q_x as locally linear functions of calendar time t. Mathematically, it may be slightly more natural to make this assumption of linearity directly about the force of mortality. Suppose therefore that in calendar year t, the force of mortality $\mu_x^{(t)}$ at all ages x is assumed to have the form

$$\mu_x^{(t)} \;=\; \mu_x^{(0)} \,+\, b_x\, t \tag{6.5}$$

where $\mu_x^{(0)}$ is the force-of-mortality associated with some standard life table as of some arbitrary but fixed calendar-time origin $t = 0$. The age-dependent slope b_x will generally be extremely small. Then, placing superscripts $^{(t)}$ over all life-table entries and ratios to designate calendar time, we calculate

$${}_kp_x^{(t)} \;=\; \exp\Big(-\int_0^k \mu_{x+u}^{(t)}\, du\Big) \;=\; {}_kp_x^{(t)} \,\cdot\, \exp(-t \sum_{j=0}^{k-1} b_{x+j})$$

If we denote

$$B_x \;=\; \sum_{y=0}^{x} b_y$$

and assume that the life-table radix l_0 is not taken to vary with calendar time, then the commutation-function $D_x = D_x^{(t)}$ takes the form

$$D_x^{(t)} \;=\; v^x \, l_0 \, {}_xp_0^{(t)} \;=\; D_x^{(0)} \, e^{-tB_x} \tag{6.6}$$

Thus the commutation-columns $D_x^{(0)}$ (from the standard life-table) and B_x are enough to reproduce the time-dependent commutation column D, but now the calculation is not quite so simple, and the time-dependent commutation columns M, N become

$$M_x^{(t)} \;=\; \sum_{y=x}^{\infty} v^y \, (l_y^{(t)} - l_{y+1}^{(t)}) \;=\; \sum_{y=x}^{\infty} D_y^{(0)} \, e^{-tB_y} \left(1 - e^{-t\, b_{y+1}} \, q_y^{(0)}\right) \tag{6.7}$$

$$N_x^{(t)} \;=\; \sum_{y=x}^{\infty} D_y^{(0)} \, e^{-tB_y} \tag{6.8}$$

For simplicity, one might replace equation (6.7) by the approximation

$$M_x^{(t)} \;=\; \sum_{y=x}^{\infty} D_y^{(0)} \left(p_y^{(0)} + t\, b_{y+1} \, q_y^{(0)}\right) e^{-tB_y}$$

None of these formulas would be too difficult to calculate with, for example on a hand-calculator; moreover, since the calendar year t would be fixed for the printed tables which an insurance salesperson would carry around, the virtues of commutation functions in providing quick premium-quotes would not be lost if the life tables used were to vary systematically from one calendar year to the next.

6.2 Reserve & Cash Value of a Single Policy

In long-term insurance policies paid for by level premiums, it is clear that since risks of death rise very rapidly with age beyond middle-age, the early premium payments must to some extent exceed the early insurance costs.

Our calculation of risk premiums ensures by definition that for each insurance and/or endowment policy, the expected total present value of premiums paid in will equal the expected present value of claims to be paid out. However, it is generally *not* true *within each year* of the policy that the expected present value of amounts paid in are equal to the expected present value of amounts paid out. In the early policy years, the difference paid in versus out is generally in the insurer's favor, and the surplus must be set aside as a **reserve** against expected claims in later policy-years. It is the purpose of the present section to make all of these assertions mathematically precise, and to show how the reserve amounts are to be calculated. Note once and for all that loading plays no role in the calculation of reserves: throughout this Section, 'premiums' refer only to pure-risk premiums. The loading portion of actual premium payments is considered either as reimbursement of administrative costs or as profit of the insurer, but in any case does not represent buildup of value for the insured.

Suppose that a policyholder aged x purchased an endowment or insurance (of duration at least t) t years ago, paid for with level premiums, and has survived to the present. Define the **net (level) premium reserve** as of time t to be the excess ${}_tV$ of the expected present value of the amount to be paid out under the contract in future years over the expected present value of further *pure risk* premiums to be paid in (including the premium paid immediately, in case policy age t coincides with a time of premium-payment). Just as the notations $P^{1}_{x:\overline{n|}}$, $P_{x:\overline{n|}}^{\ \ 1}$, etc., are respectively used to denote the level annual premium amounts for a term insurance, an endowment, etc., we use the same system of sub- and superscripts with the symbol ${}_tV$ to describe the reserves on these different types of policies.

By definition, the net premium reserve of any of the possible types of contract as of policy-age $t = 0$ is $0:$ this simply expresses the balance between expected present value of amounts to be paid into and out of the insurer under the policy. On the other hand, the *terminal reserve* under the policy, which is to say the reserve ${}_nV$ just before termination, will differ from one type of policy to another. The main possibilities are the pure term insurance, with reserves denoted ${}_tV^{1}_{x:\overline{n|}}$ and terminal reserve ${}_nV^{1}_{x:\overline{n|}} = 0$, and the endowment insurance, with reserves denoted ${}_tV_{x:\overline{n|}}^{\ \ 1}$ and terminal reserve ${}_nV_{x:\overline{n|}}^{\ \ 1} = 1$. In each of these two examples, just before policy termination there are no further premiums to be received or insurance benefits to be paid, so that the terminal reserve coincides with the terminal (i.e., endowment) benefit to be paid at policy termination. Note that for simplicity, we do not consider here the insurances or endowment insurances with $m > 1$ possible payments per year. The reserves for such policies can be obtained as previously discussed from the one-payment-per-year case via interpolation formulas.

The definition of net premium reserves is by nature *prospective*, referring to future payment streams and their expected present values. From the definition, the formulas for reserves under the term and endowment insurances are respectively given by:

$$ {}_tV^1_{\overline{x:n}|} \;=\; A^1_{\overline{x+t:n-t}|} \;-\; P^1_{\overline{x:n}|} \cdot \ddot{a}_{\overline{x+t:n-t}|} \tag{6.9} $$

$$ {}_tV_{\overline{x:n}|} \;=\; A_{\overline{x+t:n-t}|} \;-\; P_{\overline{x:n}|} \cdot \ddot{a}_{\overline{x+t:n-t}|} \tag{6.10} $$

One identity from our previous discussions of net single premiums immediately simplifies the description of reserves for endowment insurances. Recall the identity

$$ \ddot{a}_{\overline{x:n}|} \;=\; \frac{1 - A_{\overline{x:n}|}}{d} \qquad \Longrightarrow \qquad A_{\overline{x:n}|} \;=\; 1 \;-\; d\,\ddot{a}_{\overline{x:n}|} $$

Dividing the second form of this identity through by the net single premium for the life annuity-due, we obtain

$$ P_{\overline{x:n}|} \;=\; \frac{1}{\ddot{a}_{\overline{x:n}|}} \;-\; d \tag{6.11} $$

after which we immediately derive from the definition the reserve-identity

$$ {}_tV_{\overline{x:n}|} \;=\; \ddot{a}_{\overline{x+t:n-t}|}\Big(P_{\overline{x+t:n-t}|} \;-\; P_{\overline{x:n}|}\Big) \;=\; 1 \;-\; \frac{\ddot{a}_{\overline{x+t:n-t}|}}{\ddot{a}_{\overline{x:n}|}} \tag{6.12} $$

6.2.1 Retrospective Formulas & Identities

The notion of *reserve* discussed so far expresses a difference between expected present value of amounts to be paid out and to be paid in to the insurer dating from t years following policy initiation, *conditionally given the survival of the insured life aged x for the additional t years to reach age $x + t$.* It stands to reason that, assuming this reserve amount to be positive, there must have been up to time t an excess of premiums collected over insurance acquired up to duration t. This latter amount is called the **cash value** of the policy when accumulated actuarially to time t, and represents a cash amount to which the policyholder is entitled (less administrative expenses) if he wishes to discontinue the insurance. (Of course, since the insured has lived to age $x+t$, in one sense the insurance has not been 'used' at all because it did not generate a claim, but an insurance up to policy age t was in any case the only part of the purchased benefit which could have brought a payment from the insurer up to policy age t.) The insurance bought had the time-0 present value $A^1_{x:\overline{t}|}$, and the premiums paid prior to time t had time-0 present value $\ddot{a}_{x:\overline{t}|} \cdot P$, where P denotes the level annualized premium for the duration-n contract actually purchased.

To understand clearly why there is a close connection between retrospective and prospective differences between expected (present value of) amounts paid in and paid out under an insurance/endowment contract, we state a general and slightly abstract proposition.

> Suppose that a life-contingent payment stream (of duration n at least t) can be described in two stages, as conferring a benefit of expected time-0 present value U_t on the policy-age-interval $[0,t)$ (up to but not including policy age t), and **also** conferring a benefit *if the policyholder is alive as of policy age* t with expected present value *as of time* t which is equal to F_t. **Then the total expected time-0 present value of the contractual payment stream is**
>
> $$U_t \,+\, v^t \, {}_tp_x \, F_t$$

Before applying this idea to the balancing of prospective and retrospective reserves, we obtain without further effort three useful identities by recognizing that a term life-insurance, insurance endowment, and life annuity-due (all of duration n) can each be divided into a before- and after- t component along the lines of the displayed Proposition. (Assume for the following discussion that the intermediate duration t is also an integer.) For the insurance, the benefit U_t up to t is $A^1_{x:\overline{t}|}$, and the contingent after-t benefit F_t is $A^1_{x+t:\overline{n-t}|}$. For the endowment insurance, $U_t = A^1_{x:\overline{t}|}$ and $F_t = A_{x+t:\overline{n-t}|}$.

Finally, for the life annuity-due, $U_t = \ddot{a}_{x:\overline{t}|}$ and $F_t = \ddot{a}_{\overline{x+t:n-t}|}$. Assembling these facts using the displayed Proposition, we have our three identities:

$$A^{1}_{x:\overline{n}|} \;=\; A^{1}_{x:\overline{t}|} + v^t\,{}_tp_x\, A^{1}_{x+t:\overline{n-t}|} \tag{6.13}$$

$$A_{x:\overline{n}|} \;=\; A^{1}_{x:\overline{t}|} + v^t\,{}_tp_x\, A_{x+t:\overline{n-t}|} \tag{6.14}$$

$$\ddot{a}_{x:\overline{n}|} \;=\; \ddot{a}_{x:\overline{t}|} + v^t\,{}_tp_x\, \ddot{a}_{x+t:\overline{n-t}|} \tag{6.15}$$

The factor $v^t\,{}_tp_x$, which discounts present values from time t to the present values at time 0 *contingent on survival to* t, has been called above the **actuarial present value**. It coincides with ${}_tE_x = A_{x:\overline{t}|}^{\;\;1}$, the expected present value at 0 of a payment of 1 to be made at time t if the life aged x survives to age $x+t$. This is the factor which connects the excess of insurance over premiums on $[0,t)$ with the reserves ${}_tV$ on the insurance/endowment contracts which refer prospectively to the period $[t,n]$. Indeed, substituting the identities (6.13), (6.14), and (6.15) into the identities

$$A^{1}_{x:\overline{n}|} = P^{1}_{x:\overline{n}|}\,\ddot{a}_{x:\overline{n}|} \;, \qquad A_{x:\overline{n}|} = P_{x:\overline{n}|}\,\ddot{a}_{x:\overline{n}|}$$

yields

$$v^t\,{}_tp_x\,{}_tV^{1}_{x:\overline{n}|} \;=\; -\Big[\, A^{1}_{x:\overline{t}|} - P^{1}_{x:\overline{n}|}\,\ddot{a}_{x:\overline{n}|} \,\Big] \tag{6.16}$$

$$v^t\,{}_tp_x\,{}_tV_{x:\overline{n}|} \;=\; -\Big[\, A_{x:\overline{t}|} - P_{x:\overline{t}|}\,\ddot{a}_{x:\overline{n}|} \,\Big] \tag{6.17}$$

The interpretation in words of these last two equations is that the actuarial present value of the net (level) premium reserve at time t (of either the term insurance or endowment insurance contract) is equal to the negative of the expected present value of the difference between the contract proceeds and the premiums paid on $[0,t)$.

Figure 6.1 provides next an illustrative set of calculated net level premium reserves, based upon 6% interest, for a 40-year term life insurance with face-amount $1,000$ for a life aged 40. The mortality laws used in this calculation, chosen from the same families of laws used to illustrate force-of-mortality curves in Figures 2.3 and 2.4 in Chapter 2, are the same plausible mortality laws whose survival functions are pictured in Figure 2.5. These mortality laws are realistic in the sense that they closely mirror the US male mortality in 1959 (*cf.* plotted points in Figure 2.5.) The cash-reserve curves ${}_tV_{\overline{40:40}|}$ as functions of $t = 0, \ldots, 40$ are pictured graphically in Figure 6.1. Note that these reserves can be very substantial: at policy age 30, the reserves on the $1000 term insurance are respectively $459.17, $379.79, $439.06, and $316.43.

6.2.2 Relating Insurance & Endowment Reserves

The simplest formulas for net level premium reserves in a general contract arise in the case of endowment insurance ${}_tV^1_{x:\overline{n}|}$, as we have seen in formula (6.12). In fact, for endowment and/or insurance contracts in which the period of payment of level premiums coincides with the (finite or infinite) policy duration, the reserves for term-insurance and pure-endowment can also be expressed rather simply in terms of ${}_tV_{x:\overline{n}|}$. Indeed, reasoning from first principles with the prospective formula, we find for the pure endowment

$$ {}_tV_{x:\overline{n}|}^{\;\;\;1} \;=\; v^{n-t}\;{}_{n-t}p_{x+t} \;-\; \frac{v^n\;{}_np_x}{\ddot{a}_{x:\overline{n}|}}\,\ddot{a}_{x+t:\overline{n-t}|} $$

from which, by substitution of formula (6.12), we obtain

$$ V_{x:\overline{n}|}^{\;\;\;1} \;=\; v^{n-t}\;{}_{n-t}p_{x+t} \;-\; v^n\;{}_np_x\;\left(1 \;-\; {}_tV_{x:\overline{n}|}\right) \tag{6.18} $$

Then, for net level reserves or cash value on a term insurance, we conclude

$$ V^1_{x:\overline{n}|} \;=\; (1 - \;v^n\;{}_np_x)\;{}_tV_{x:\overline{n}|} \;+\; v^n\;{}_np_x \;-\; v^{n-t}\;{}_{n-t}p_{x+t} \tag{6.19} $$

6.2.3 Reserves under Constant Force of Mortality

We have indicated above that the phenomenon of positive reserves relates in some way to the aging of human populations, as reflected in the increasing force of mortality associated with life-table survival probabilities. A simple benchmark example helps us here: we show that when life-table survival is governed at all ages x and greater by a constant force of mortality μ, the reserves for term insurances are identically 0. In other words, the expected present value of premiums paid in *within each policy year* exactly compensates, under constant force of mortality, for the expected present value of the amount to be paid out *for that same policy year*.

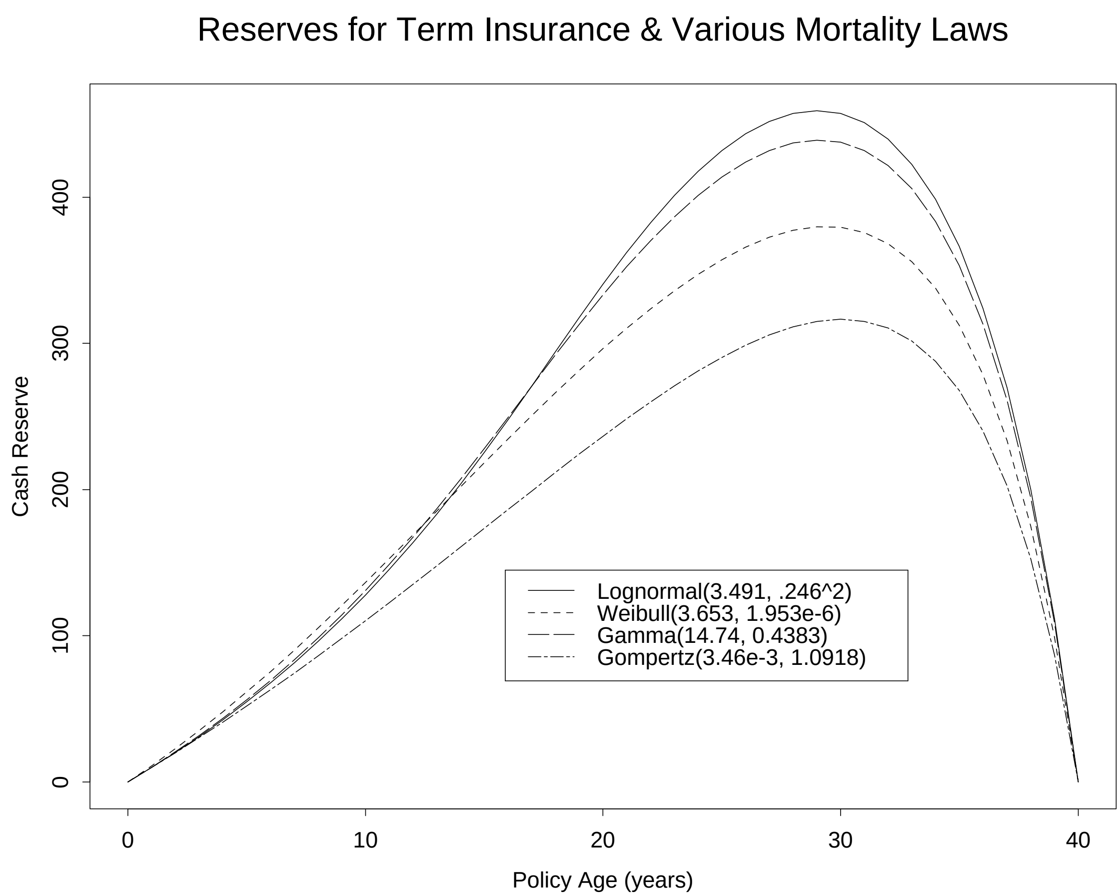

Figure 6.1: Net level premium reserves as a function of policy age for a 40-year term insurance to a life aged 40 with level benefit 1000, at 6%, under the same mortality laws pictured in Figures 2.5, with median age 72 at death. For definitions of the mortality laws see the Examples of analytical force-of-mortality functions in Chapter 2.

The calculations here are very simple. Reasoning from first principles, under constant force of mortality

$$A^{1}_{x:\overline{n}|} \;=\; \sum_{k=0}^{n-1} v^{k+1}\, e^{-\mu\, k}\, (1 - e^{-\mu}) \;=\; (e^{\mu} - 1)\, (ve^{-\mu})\, \frac{1 \,-\, (ve^{-\mu})^n}{1 \,-\, ve^{-\mu}}$$

while

$$\ddot{a}_{x:\overline{n}|} \;=\; \sum_{k=0}^{n-1} v^k\, e^{-\mu\, k} \;=\; \frac{1 \,-\, (ve^{-\mu})^n}{1 \,-\, ve^{-\mu}}$$

It follows from these last two equations that

$$P^{1}_{x:\overline{n}|} \;=\; v\, (1 \,-\, e^{-\mu})$$

which does not depend upon either age x (within the age-range where constant force of mortality holds) or n. The immediate result is that for $0 < t < n$

$${}_tV^{1}_{x:\overline{n}|} \;=\; \ddot{a}_{x+t:\overline{n-t}|}\left(P^{1}_{x+t:\overline{n-t}|} - P^{1}_{x:\overline{n}|}\right) \;=\; 0$$

By contrast, since the terminal reserve of an endowment insurance must be 1 even under constant force of mortality, the intermediate net level premium reserves for endowment insurance must be positive and growing. Indeed, we have the formula deriving from equation (6.12)

$${}_tV_{x:\overline{n}|} \;=\; 1 \,-\, \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} \;=\; 1 \,-\, \frac{1 - (ve^{-\mu})^{n-t}}{1 - (ve^{-\mu})^n}$$

6.2.4 Reserves under Increasing Force of Mortality

Intuitively, we should expect that a force-of-mortality function which is everywhere increasing for ages greater than or equal to x will result in intermediate reserves for term insurance which are positive at times between $0, n$ and in reserves for endowment insurance which increase from 0 at $t = 0$ to 1 at $t = n$. In this subsection, we prove those assertions, using the simple observation that when $\mu(x + y)$ is an increasing function of positive y,

$${}_kp_{x+y} \;=\; \exp(-\int_0^k \mu_(x + y + z)\, dz) \qquad \searrow \quad \text{in } y \tag{6.20}$$

First suppose that $0 \le s < t \le n$, and calculate using the displayed fact that ${}_kp_{x+y}$ decreases in y,

$$\ddot{a}_{\overline{x+t:n-t|}} = \sum_{k=0}^{n-t-1} v^{k+1} \, {}_kp_{x+t} \le \sum_{k=0}^{n-t-1} v^{k+1} \, {}_kp_{x+s} < \ddot{a}_{\overline{x+s:n-s|}}$$

Therefore $\ddot{a}_{\overline{x+t:n-t|}}$ is a decreasing function of t, and by formula (6.12), ${}_tV_{\overline{x:n|}}$ is an increasing function of t, ranging from 0 at $t = 0$ to 1 at $t = 1$.

It remains to show that for force of mortality which is increasing in age, the net level premium reserves ${}_tV^1_{\overline{x:n|}}$ for term insurances are positive for $t = 1, \ldots, n-1$. By equation (6.9) above, this assertion is equivalent to the claim that

$$A^1_{\overline{x+t:n-t|}} / \ddot{a}_{\overline{x+t:n-t|}} \;>\; A^1_{\overline{x:n|}} / \ddot{a}_{\overline{x:n|}}$$

To see why this is true, it is instructive to remark that each of the displayed ratios is in fact a multiple v times a weighted average of death-rates: for $0 \le s < n$,

$$\frac{A^1_{\overline{x+s:n-s|}}}{\ddot{a}_{\overline{x+s:n-s|}}} = v \left\{ \frac{\sum_{k=0}^{n-s-1} v^k \, {}_kp_{x+s} \, q_{x+s+k}}{\sum_{k=0}^{n-s-1} v^k \, {}_kp_{x+s}} \right\}$$

Now fix the age x arbitrarily, together with $t \in \{1, \ldots, n-1\}$, and define

$$\overline{q} = \frac{\sum_{j=t}^{n-1} v^j \, {}_jp_x \, q_{x+j}}{\sum_{j=t}^{n-1} v^j \, {}_jp_x}$$

Since μ_{x+y} is assumed increasing for all y, we have from formula (6.20) that $q_{x+j} = 1 - p_{x+j}$ is an increasing function of j, so that for $k < t \le j$, $q_{x+k} < q_{x+j}$ and

$$\text{for all} \quad k \in \{0, \ldots, t-1\}, \qquad q_{x+k} < \overline{q} \tag{6.21}$$

Moreover, dividing numerator and denominator of the ratio defining $\overline{q}$ by $v^t \, {}_tp_x$ gives the alternative expression

$$v \, \overline{q} = \frac{\sum_{a=0}^{n-t-1} v^a \, {}_ap_{x+t} \, q_{x+t+a}}{\sum_{a=0}^{n-t-1} v^a \, {}_ap_{x+t}} = P^1_{\overline{x+t:n-t|}}$$

Finally, using equation (6.21) and the definition of $\overline{q}$ once more, we calculate

$$\frac{A^1_{\overline{x:n|}}}{\ddot{a}_{\overline{x:n|}}} \;=\; v\Big(\frac{\sum_{k=0}^{t-1}\, v^k\, {}_kp_x\, q_{x+k} \;+\; \sum_{j=t}^{n-t-1}\, v^j\, {}_jp_x\, q_{x+j}}{\sum_{k=0}^{t-1}\, v^k\, {}_kp_x \;+\; \sum_{j=t}^{n-t-1}\, v^j\, {}_jp_x}\Big)$$

$$=\; v\Big(\frac{\sum_{k=0}^{t-1}\, v^k\, {}_kp_x\, q_{x+k} \;+\; \overline{q}\,\sum_{j=t}^{n-t-1}\, v^j\, {}_jp_x}{\sum_{k=0}^{t-1}\, v^k\, {}_kp_x \;+\; \sum_{j=t}^{n-t-1}\, v^j\, {}_jp_x}\Big) \;<\; v\,\overline{q} \;=\; P^1_{\overline{x+t:n-t|}}$$

as was to be shown. The conclusion is that under the assumption of increasing force of mortality for all ages x and larger, ${}_tV^1_{\overline{x:n|}} \;>\; 0$ for $t = 1, \ldots, n-1$.

6.2.5 Recursive Calculation of Reserves

The calculation of reserves can be made very simple and mechanical in numerical examples with small illustrative life tables, due to the identities (6.13) and (6.14) together with the easily proved recursive identities (for integers t)

$$A^1_{\overline{x:t+1|}} \;=\; A^1_{\overline{x:t|}} \;+\; v^{t+1}\, {}_tp_x\, q_{x+t}$$

$$A_{\overline{x:t+1|}} \;=\; A_{\overline{x:t|}} \;-\; v^t\, {}_tp_x \;+\; v^{t+1}\, {}_tp_x \;=\; A_{\overline{x:t|}} \;-\; d\, v^t\, {}_tp_x$$

$$\ddot{a}_{\overline{x:t+1|}} \;=\; \ddot{a}_{\overline{x:t|}} \;+\; v^t\, {}_tp_x$$

Combining the first and third of these recursions with (6.13), we find

$${}_tV^1_{\overline{x:n|}} \;=\; \frac{-1}{v^t\, {}_tp_x}\left(A^1_{\overline{x:t|}} \;-\; P^1_{\overline{x:n|}}\, \ddot{a}_{\overline{x:t|}}\right)$$

$$=\; \frac{-1}{v^t\, {}_tp_x}\left(A^1_{\overline{x:t+1|}} \;-\; v^{t+1}\, {}_tp_x\, q_{x+t} \;-\; P^1_{\overline{x:n|}}\left[\ddot{a}_{\overline{x:t+1|}} \;-\; v^t\, {}_tp_x\right]\right)$$

$$=\; v\, p_{x+t}\;\; {}_{t+1}V^1_{\overline{x:n|}} \;+\; v\, q_{x+t} \;-\; P^1_{\overline{x:n|}}$$

The result for term-insurance reserves is the single recursion

$${}_tV^1_{\overline{x:n|}} \;=\; v\, p_{x+t}\;\; {}_{t+1}V^1_{\overline{x:n|}} \;+\; v\, q_{x+t} \;-\; P^1_{\overline{x:n|}} \tag{6.22}$$

which we can interpret in words as follows. The reserves at integer policy-age t are equal to the sum of the one-year-ahead actuarial present value of the reserves at time $t+1$ and the excess of the present value of this year's expected insurance payout $(v\, q_{x+t})$ over this year's received premium $(P^1_{x:\overline{n}|})$.

A completely similar algebraic proof, combining the one-year recursions above for endowment insurance and life annuity-due with identity (6.14), yields a recursive formula for endowment-insurance reserves (when $t < n$) :

$$ {}_tV_{\overline{x:n}|} \;=\; v\, p_{x+t}\;\; {}_{t+1}V_{\overline{x:n}|} \;+\; v\, q_{x+t} \;-\; P^1_{\overline{x:n}|} \tag{6.23} $$

The verbal intepretation is as before: the future reserve is discounted by the one-year actuarial present value and added to the expected present value of the one-year term insurance minus the one-year cash (risk) premium.

6.2.6 Paid-Up Insurance

An insured may want to be aware of the cash value (equal to the reserve) of an insurance or endowment either in order to discontinue the contract and receive the cash or to continue the contract in its current form and borrow with the reserve as collateral. However, it may also happen for various reasons that an insured may want to continue insurance coverage but is no longer able or willing to par further premiums. In that case, for an administrative fee the insurer can convert the premium reserve to a single premium for a new insurance (either with the same term, or whole-life) with lesser benefit amount. This is really not a new topic, but a combination of the previous formulas for reserves and net single premiums. In this sub-section, we give the simplified formula for the case where the cash reserve is used as a single premium to purchase a new whole-life policy. Two illustrative worked examples on this topic are given in Section 6.6 below.

The general formula for reserves, now specialized for whole-life insurances, is

$$ {}_tV_x \;=\; \mathrm{A}_{x+t} \;-\; \frac{\mathrm{A}_x}{\ddot{\mathrm{a}}_x} \cdot \ddot{\mathrm{a}}_{x+t} \;=\; 1 \;-\; \frac{\ddot{\mathrm{a}}_{x+t}}{\ddot{\mathrm{a}}_x} $$

This formula, which applies to a unit face amount, would be multiplied through by the level benefit amount B. Note that loading is disregarded in this calculation. The idea is that any loading which may have applied has been collected as part of the level premiums; but in practice, the insurer might apply some further (possibly lesser) loading to cover future administrative costs. Now if the cash or reserve value ${}_tV_x$ is to serve as net single premium for a new insurance, the new face-amount F is determined as of the t policy-anniversary by the balance equation

$$B \cdot {}_t\mathrm{V}_x \;=\; F \cdot \mathrm{A}_{x+t}$$

which implies that the equivalent **face amount of paid-up insurance** as of policy-age t is

$$F \;=\; \frac{B\, {}_t\mathrm{V}_x}{\mathrm{A}_{x+t}} \;=\; B\left(1 \,-\, \frac{\ddot{\mathrm{a}}_{x+t}}{\ddot{\mathrm{a}}_x}\right) \Big/ \,(1 \,-\, d\,\ddot{\mathrm{a}}_{x+t}) \qquad (6.24)$$

6.3 Select Mortality Tables & Insurance

Insurers are often asked to provide life insurance converage to groups and/or individuals who belong to special populations with mortality significantly worse than that of the general population. Yet such **select populations** may not be large enough, or have a sufficiently long data-history, within the insurer's portfolio for survival probabilities to be well-estimated in-house. In such cases, insurers may provide coverage under special premiums and terms. The most usual example is that such select risks may be issued insurance with restricted or no benefits for a specified period of time, e.g. 5 years. The stated rationale is that after such a period of deferral, the select group's mortality will be sufficiently like the mortality of the general population in order that the insurer will be adequately protected if the premium in increased by some standard multiple. In this Section, a slightly artifical calculation along these lines illustrates the principle and gives some numerical examples.

Assume for simplicity that the general population has constant force of mortality μ, and that the select group has larger force of mortality μ^*. If the interest rate is i, and $v = 1/(1+i)$, then the level yearly risk premium for a k-year deferred whole-life insurance (of unit amount, payable at the end of the year of death) to a randomly selected life (of any specified age x) from the general population is easily calculated to be

$$\text{Level Risk Premium} \;=\; v^k\, {}_kp_x\, A_{x+k}/\ddot{\mathrm{a}}_x \;\;=\;\; (1 - e^{-\mu})\, v^{k+1}\, e^{-\mu k} \qquad (6.25)$$

If this premium is multiplied by a factor $\kappa > 1$ and applied as the risk premium for a k-year deferred whole-life policy to a member of the select

Table 6.2: Excess payout as given by formula (6.26) under a k-year deferred whole life insurance with benefit \$1000, issued to a select population with constant force of mortality μ^* for a level yearly premium which is a multiple κ times the pure risk premium for the standard population which has force-of-mortality μ. Interest rate is $i = 0.06$ APR throughout.

k	μ	μ^*	κ	Excess Payout
0	0.02	0.03	1	109
0	0.02	0.03	2	-112
5	0.02	0.03	1	63
5	0.02	0.03	1.42	0
0	0.05	0.10	1	299
0	0.05	0.10	3	-330
5	0.05	0.10	1	95
5	0.05	0.10	1.52	0
3	0.05	0.10	1	95
3	0.05	0.10	1.68	0

population, then the expected excess (per unit of benefit) of the amount paid out under this select policy over the risk premiums collected, is

$$\text{Excess Payout} \;=\; v^k \, {}_kp^*_x \, A^*_{x+k} \;-\; \kappa \, (1 - e^{-\mu}) \, v^{k+1} \, e^{-\mu k} \, \ddot{a}^*_x$$
$$= \; \frac{v^{k+1}}{1 - ve^{-mu^*}} \, \Big\{ (1 - e^{-\mu}) e^{-\mu k} \;-\; (1 - e^{-\mu^*}) e^{-\mu^* k} \Big\} \qquad (6.26)$$

where the probability, insurance, and annuity notations with superscripts * are calculated using the select mortality distribution with force of mortality μ^*. Because of the constancy of forces of mortality both in the general and the select populations, the premiums and excess payouts do not depend on the age of the insured. Table 6.3 shows the values of some of these excess payouts, for $i = 0.06$, under several combinations of k, μ, μ^*, and κ. Note that in these examples, select mortality with force of mortality multiplied by 1.5 or 2 is offset, with sufficient protection to the insurer, by an increase of 40–60% in premium on whole-life policies deferring benefits by 3 or 5 years.

Additional material will be added to this Section later. A calculation *along the same lines as the foregoing Table, but using the Gompertz*$(3.46e-3, 1.0918)$ *mortality law previously found to approximate well the realistic Life Table data in Table 1.1, will be included for completeness.*

6.4 *Exercise Set 6*

For the first problem, use the Simulated Illustrative Life Table with commutator columns given as Table 6.1 on page 150, using 6% APR as the going rate of interest. (Also assume, wherever necessary, that the distribution of deaths within whole years of age is uniform.)

(1). (a) Find the level premium for a 20-year term insurance of \$5000 for an individual aged 45, which pays at the end of the half-year of death, where the payments are to be made semi-annually.

(b) Find the level annual premium for a whole-life insurance for an individual aged 35, which pays \$30,000 at the end of year of death if death occurs before exact age 55 and pays \$60,000 at the instant (i.e., day) of death at any later age.

(2). You are given the following information, indirectly relating to the fixed rate of interest i and life-table survival probabilities ${}_kp_x$.

(i) For a one-payment-per-year level-premium 30-year endowment insurance of 1 on a life aged x, the amount of reduced paid-up endowment insurance at the end of 10 years is 0.5.

(ii) For a one-payment-per-year level-premium 20-year endowment insurance of 1 on a life aged $x+10$, the amount of reduced paid-up insurance at the end of 5 years is 0.3.

Assuming that cash values are equal to net level premium reserves and reduced paid-up insurances are calculated using the equivalence principle, so that the cash value is equated to the net single premium of an endowment insurance with reduced face value, calculate the amount of reduced paid-up insuranceat the end of 15 years for the 30-year endowment insuranceof 1 on a life aged x. See the following Worked Examples for some relevant formulas.

(3). Give a formula for A_{45} in terms of the following quantities alone:

$$ {}_{25}p_{20}\ , \quad \ddot{a}_{\overline{20:25}|}\ , \quad \mathrm{P}_{\overline{20:25}|}\ , \quad {}_{25}P_{20}\ , \quad v^{25} $$

where

$$ \mathrm{P}_{\overline{x:n}|} \;=\; \mathrm{A}_{\overline{x:n}|} \Big/ \ddot{a}_{\overline{x:n}|} \qquad \text{and} \qquad {}_tP_x \;=\; A_x \Big/ \ddot{a}_{\overline{x:t}|} $$

(4). A life aged 32 purchases a life annuity of 3000 per year. From tables, we find commutation function values

$$ N_{32} \;=\; 2210 \quad , \qquad N_{34} \;=\; 1988 \quad , \qquad D_{33} = 105 $$

Find the net single premium for the annuity purchased if the first yearly payment is to be received (a) immediately, (b) in 1 year, and (c) in 2 years.

(5). Henry, who has just reached his 70^{th} birthday, is retiring immediately with a monthly pension income of 2500 for life, beginning in 1 month. Using the uniform-failure assumption between birthdays and the commutation function values $M_{70} = 26.2$ and $D_{70} = 71$ associated with the interest rate $i = 0.05$, find the expected present value of Henry's retirement annuity.

(6). Find the cash value of a whole-life insurance for $\$100,000$ on a life aged 45 with yearly premium-payments (which began at issuance of the policy) after 25 years, assuming interest at 5% and constant force-of-mortality $\mu_{40+t} \;=\; 0.07$ for all $t > 0$.

(7). Suppose that 25 years ago, a life then aged 40 bought a whole-life insurance policy with quarterly premium payments and benefit payable at the end of the quarter of death, with loading-factor 4%. Assume that the interest rate used to figure present values and premiums on the policy was 6% and that the life-table survival probabilities used were ${}_tp_{40} = (60 - t)/60$. If the insured is now alive at age 65, then find the face amount of paid-up insurance which he is entitled to — with no further premiums paid and no further loading applied — on a whole-life policy with benefits payable at the end of quarter-year of death.

(7). Verify formulas (6.25) and (6.26).

6.5 Illustration of Commutation Columns

Consider the following articial life-table fragment, which we imagine to be available together with data also for all older ages,on a population of potential insureds:

x	l_x	d_x
45	75000	750
46	74250	760
47	73490	770
48	72720	780
49	71940	790
50	71150	

Let us imagine that the going rate of interest is 5% APR, and that we are interested in calculating various life insurance and annuity risk-premiums for level-benefit contracts and payments only on policy anniversaries ($m = 1$), on lives aged 45 to 50. One way of understanding what commutation columns do is to remark that *all* whole-life net single premiums of this type are calculable directly from the table-fragment given along with the single additional number $A_{50} = 0.450426$. The point is that all of the commutation columns $D_x,\ N_x,\ M_x$ for ages 45 to 49 can now be filled in. First, we use the identity (6.4) to obtain

$$D_{50} = 1.05^{-50}\, 71150 = 6204.54, \qquad M_{50} = D_{50}\, 0.450426 = 2794.69$$

$$N_{50} = D_{50}\, \ddot{a}_{50} = \frac{D_{50}}{d}\,(1 - A_{50}) = \frac{1.05}{0.05}\,(D_{50} - M_{50}) = 71606.99$$

Next we fill in the rest of the columns for ages 45 to 49 by the definition of D_x as $1.05^x\, l_x$ and the simple recursive calculations

$$N_x = N_{x+1} + D_x \quad , \qquad M_x = M_{x+1} + v^{x+1}\, d_x$$

Filling these values in for $x = 49,\ 48,\ \ldots,\ 45$ gives the completed fragment

x	l_x	d_x	D_x	N_x	M_x
45	75000	750	8347.24	106679.11	3267.81
46	74250	760	7870.25	98808.85	3165.07
47	73490	770	7418.76	91390.10	3066.85
48	72720	780	6991.45	84398.64	2972.47
49	71940	790	6587.11	77811.53	2891.80
50	71150		6204.54	71606.99	2794.69

From this table fragment we can deduce, for example, that a whole-life annuity-due of \$2000 per year to a life aged 47 has expected present value $2000\, N_{47}/D_{47} = 24637.57$, or that a five-year term insurance of $100,000$ to a life aged 45 has net single premium $100000 \cdot (M_{45} - M_{50})/D_{50} = 5661.66$, or that a whole-life insurance of $200,000$ with level premiums payable for 5 years to a life aged 45 has level pure-risk premiums of $200,000 \cdot M_{45}/(N_{45} - N_{50}) = 18631.78$.

6.6 Examples on Paid-up Insurance

Example 1. Suppose that a life aged 50 has purchased a whole life insurance for \$100,000 *with level annual premiums and has paid premiums for* 15 *years. Now, being aged 65, the insured wants to stop paying premiums and convert the cash value of the insurance into (the net single premium for) a fully paid-up whole-life insurance. Using the APR interest rate of* 6% *and commutator columns given in Table 6.1 and disregarding premium loading, calculate the face amount of the new, paid-up insurance policy.*

Solution. Applying formula (6.24) to the Example, with $x = 50$, $t = 15$, $B = 100,000$, and using Table 6.1,gives

$$F \;=\; 100,000 \left(D_{65} - \frac{N_{65}\, D_{50}}{N_{50}} \right) \Big/ \left(D_{65} - d\, N_{65} \right)$$

$$= \; 100,000 \, \frac{1486.01 - \frac{4729.55}{56988.31}\, 12110.79}{1486.01 - \frac{0.06}{1.06}\, 12110.79} \;=\; 60077.48$$

If the new insurance premium were to be figured with a loading such as $L' = 0.02$, then the final amount figured using pure-risk premiums would be *divided* by 1.02, because the cash value would then be regarded as a single risk premium which *when inflated by the factor* $1+L'$ purchases the contract of expected present value $F \cdot \mathrm{A}_{x+t}$.

The same ideas can be applied to the re-figuring of the amount of other insurance contracts, such as an endowment, based upon an incomplete stream of premium payments.

Example 2. Suppose that a 20*-year pure endowment for* 50,000 *on a newborn governed by the life-table and commutator columns in Table 6.1, with seminannual premiums, but that after* 15 *years the purchaser can afford no more premium payments. Disregarding loading, and assuming uniform distribution of death-times within single years of age, what is the benefit amount which the child is entitled to receive at age* 20 *if then alive ?*

Solution. Now the prospective formula for cash value or reserve based on initial benefit amount B is

$$B \left({}_5p_{15}\, v^5 \; - \; {}_{20}p_0\, v^{20} \, \frac{\ddot{a}^{(2)}_{\overline{15:5|}}}{\ddot{a}^{(2)}_{\overline{0:20|}}} \right)$$

which will be used to balance the endowment $F\, A_{\overset{\;\;1}{15:5|}}$. Therefore, substituting the approximate formula (5.6), we obtain

$$F \;=\; B \cdot \left({}_5p_{15}\, v^5 \; - \; {}_{20}p_0\, v^{20} \, \frac{\alpha(2)\, \ddot{a}^{(2)}_{\overline{15:5|}} - \beta(2)(1 - \, {}_5p_{15}\, v^5)}{\alpha(2)\, \ddot{a}^{(2)}_{\overline{0:20|}} - \beta(2)(1 - \, {}_{20}p_0\, v^{20})} \right) \Big/ \left({}_5p_{15}\, v^5 \right)$$

In the particular example, where $i = 0.06$, $\alpha(2) = 1.000212$, and $\beta(2) = 0.257391$, we find

$$F \;=\; 50000 \cdot \left(1 \; - \; \frac{1.000212\, (N_{15} - N_{20}) \; - \; 0.257391\, (D_{15} - D_{20})}{1.000212\, (N_0 - N_{20}) \; - \; 0.257391\, (D_0 - D_{20})} \right)$$

and using Table 6.1 it follows that $F \;=\; 42400.91$.

6.7 Useful formulas from Chapter 6

Commutation Columns $D_y = v^y \, l_y \quad , \qquad M_x = \sum_{y=x}^{\infty} v^{y+1} \, d_y$

p. 148

$$ {}_nE_x = \frac{D_{x+n}}{D_x} \quad , \qquad A_x = \frac{M_x}{D_x} \quad , \qquad A^{1}_{x:\overline{n}|} = \frac{M_x - M_{x+n}}{D_x} $$

p. 148

$$ N_x = \sum_{y=x}^{\infty} v^y \, l_y = \sum_{y=x}^{\infty} D_y \quad , \qquad \ddot{a}_x = \frac{N_x}{D_x} $$

p. 149

$$ \ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+x} \, \frac{l_{x+k}}{D_x} = \frac{N_x - N_{x+n}}{D_x} $$

p. 148

$$ M_x = D_x - d \, N_x $$

p. 149

$$ A^{(m)1}_{x:\overline{n}|} = \frac{i}{i^{(m)}} \cdot \frac{M_x - M_{x+n}}{D_x} $$

p. 150

$$ \ddot{a}^{(m)}_{x:\overline{n}|} = \alpha(m) \, \frac{N_x - N_{x+n}}{D_x} - \beta(m) \, (1 - \frac{D_{x+n}}{D_x}) $$

p. 150

$$ {}_tV^{1}_{x:\overline{n}|} = A^{1}_{x+t:\overline{n-t}|} - P^{1}_{x:\overline{n}|} \cdot \ddot{a}_{x+t:\overline{n-t}|} $$

p. 155

$$ {}_tV_{\overline{x:n}|} \; = \; A_{\overline{x+t:n-t}|} \; - \; P_{\overline{x:n}|} \cdot \ddot{a}_{\overline{x+t:n-t}|} $$

p. 155

$$ P_{\overline{x:n}|} \; = \; \frac{1}{\ddot{a}_{\overline{x:n}|}} \; - \; d $$

p. 155

$$ {}_tV_{\overline{x:n}|} \; = \; \ddot{a}_{\overline{x+t:n-t}|} \Big(P_{\overline{x+t:n-t}|} \; - \; P_{\overline{x:n}|} \Big) \; = \; 1 \; - \; \frac{\ddot{a}_{\overline{x+t:n-t}|}}{\ddot{a}_{\overline{x:n}|}} $$

p. 155

$$ A^1_{\overline{x:n}|} \; = \; A^1_{\overline{x:t}|} \; + \; v^t \, {}_tp_x \; A^1_{\overline{x+t:n-t}|} $$

p. 157

$$ A_{\overline{x:n}|} \; = \; A^1_{\overline{x:t}|} \; + \; v^t \, {}_tp_x \; A_{\overline{x+t:n-t}|} $$

p. 157

$$ \ddot{a}_{\overline{x:n}|} \; = \; \ddot{a}_{\overline{x:t}|} \; + \; v^t \, {}_tp_x \; \ddot{a}_{\overline{x+t:n-t}|} $$

p. 157

$$ v^t \, {}_tp_x \; {}_tV^1_{\overline{x:n}|} \; = \; -\Big[\, A^1_{\overline{x:t}|} \; - \; P^1_{\overline{x:n}|} \; \ddot{a}_{\overline{x:n}|} \, \Big] $$

p. 157

$$ v^t \, {}_tp_x \; {}_tV_{\overline{x:n}|} \; = \; -\Big[\, A_{\overline{x:t}|} \; - \; P_{\overline{x:t}|} \; \ddot{a}_{\overline{x:n}|} \, \Big] $$

p. 157

$$ \mathrm{V}^{\;1}_{\overline{x:n}|} \; = \; v^{n-t} \; {}_{n-t}p_{x+t} \; - \; v^n \; {}_np_x \; \big(1 \; - \; {}_t\mathrm{V}_{\overline{x:n}|} \big) $$

p. 158

$$\mathrm{V}^1_{x:\overline{n}|} = (1 - v^n\,{}_np_x)\,{}_t\mathrm{V}_{x:\overline{n}|} + v^n\,{}_np_x - v^{n-t}\,{}_{n-t}p_{x+t}$$

p. 158

$${}_tV^1_{x:\overline{n}|} = v\,p_{x+t}\;{}_{t+1}V^1_{x:\overline{n}|} + v\,q_{x+t} - P^1_{x:\overline{n}|}$$

p. 162

$$\text{Paid-up insurance Amt} \quad = B\left(1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}\right) \Big/ (1 - d\,\ddot{a}_{x+t})$$

p. 164

Bibliography

[1] Bowers, N., Gerber, H., Hickman, J., Jones, D. and Nesbitt, C. **Actuarial Mathematics** Society of Actuaries, Itasca, Ill. 1986

[2] Cox, D. R. and Oakes, D. **Analysis of Survival Data**, Chapman and Hall, London 1984

[3] Feller, W. **An Introduction to Probability Theory and its Applications**, vol. I, 2nd ed. Wiley, New York, 1957

[4] Feller, W. **An Introduction to Probability Theory and its Applications**, vol. II, 2nd ed. Wiley, New York, 1971

[5] Gerber, H. **Life Insurance Mathematics**, 3rd ed. Springer-Verlag, New York, 1997

[6] Hogg, R. V. and Tanis, E. **Probability and Statistical Inference**, *5th ed.* Prentice-Hall Simon & Schuster, New York, 1997

[7] Jordan, C. W. **Life Contingencies**, 2nd ed. Society of Actuaries, Chicago, 1967

[8] Kalbfleisch, J. and Prentice, R. **The Statistical Analysis of Failure Time Data**, Wiley, New York 1980

[9] Kellison, S. **The Theory of Interest**. Irwin, Homewood, Ill. 1970

[10] Larsen, R. and Marx, M. **An Introduction to Probability and its Applications**. Prentice-Hall, Englewood Cliffs, NJ 1985

[11] Larson, H. **Introduction to Probability Theory and Statistical Inference**, 3rd ed. Wiley, New York, 1982

[12] Lee, E. T. **Statistical Models for Survival Data Analysis**, Lifetime Learning, Belmont Calif. 1980

[13] **Splus**, version 3.4. MathSoft Inc., 1988, 1996

[14] Spiegelman, M. *Introduction to Demography*, Revised ed. Univ. of Chicago, Chicago, 1968

[15] Venables, W. and Ripley, B. **Modern Applied Statistics with Splus**, 2nd ed. Springer-Verlag, New York, 1998

Solutions and Hints to Selected Problems