Contemporary Mathematics

Compact components of planar surface group representations

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Abstract. Recently Deroin, Tholozan and Toulisse found connected components of relative character varieties of surface group representations in a Hermitian Lie group G with remarkable properties. For example, although the Lie groups are never compact, these components are compact. In this way they behave more like relative character varieties for compact Lie groups. (A relative character variety comprises equivalence classes of homomorphisms of the fundamental group of a surface S, where the holonomy around each boundary component of S is constrained to a fixed conjugacy class in G.)

The first examples were found by Robert Benedetto and myself in an REU in summer 1992. Here S is the 4-holed sphere and $G = \mathsf{SL}(2,\mathbb{R})$. Although computer visualization played an important role in the discovery of these unexpected compact components, computation was invisible in the final proof, and its subsequent extensions.

Contents

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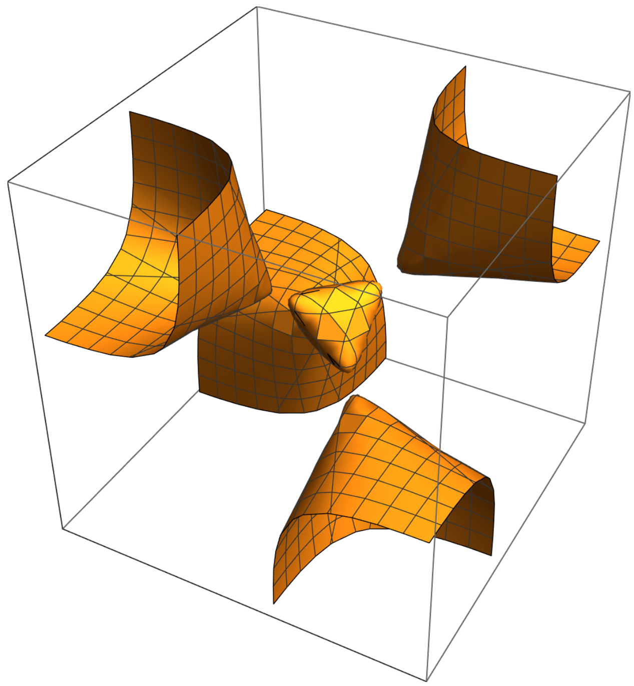

FIGURE 1. A relative character variety over $\mathbb{R}$ with four unbounded components and one compact component.

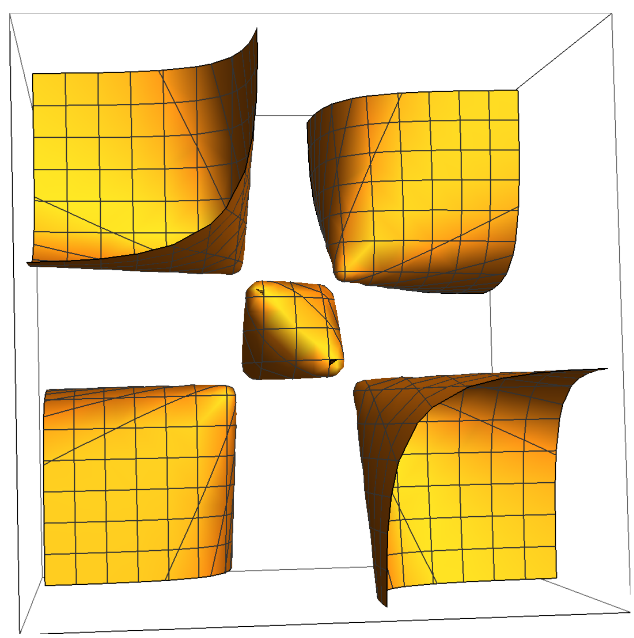

FIGURE 2. Another view of a relative character variety over $\mathbb{R}$ with four unbounded components and one compact component.

1. Relative character varieties of surfaces

Let Σ be compact oriented surface with boundary $\partial\Sigma = \partial_1 \sqcup \cdots \sqcup \partial_n$ and fundamental group $\pi = \pi_1(\Sigma)$. The *peripheral structure* consists of the conjugacy classes of subgroups $\pi_1(\partial_i) \hookrightarrow \pi$, $i = 1, \dots, n$ corresponding to the components of $\partial\Sigma$. For a compact orientable surface-with-boundary of genus g and n boundary components we shall write $\Sigma_{g,n}$.

Let G reductive linear algebraic group over $\mathbf{k}$ (either $\mathbb{R}$ or $\mathbb{C}$). Then $\mathsf{Hom}(\pi, G)$ is an affine algebraic set with algebraic $\mathsf{Inn}(G)$-action. Let

$$\mathfrak{X}(\Sigma, G) := \mathsf{Hom}(\pi, G)/\!/\mathsf{Inn}(G)$$

be its categorical quotient. Restriction to $\pi_1(\partial_i)$ defines family

$$\mathfrak{X}(\Sigma, G) \longrightarrow \mathfrak{X}(\partial_1, G) \times \cdots \times \mathfrak{X}(\partial_n, G)$$

of *relative character varieties.* These have a natural Poisson structure, for which the relative character varieties are symplectic leaves. The restriction maps are Casimirs for the Poisson structure. In this paper we always work in the classical topology, not the Zariski topology. For background on character varieties, we recommend Sikora [**15**].

In the cases of interest here, this has a very explicit structure, due to the *Vogt-Fricke theorem* which describes the $\mathsf{SL}(2,\mathbb{C})$-character variety of the two-generator free group F_2. Suppose $\mathsf{F}_2 = \langle X, Y \rangle$ be a two-generator free group. Then

$$\mathsf{Hom}(\mathsf{F}_2, \mathsf{SL}(2)) \cong \mathsf{SL}(2) \times \mathsf{SL}(2)$$

and $\mathfrak{X}(\mathsf{F}_2, \mathsf{SL}(2,\mathbb{C}))$ is its *Geometric Invariant Theory* quotient under $\mathsf{Inn}\big(\mathsf{SL}(2)\big)$. See [**7**] for a modern elementary treatment.

Write $Z = (XY)^{-1}$ so that $XYZ = \mathbb{I}$. The $\mathsf{Inn}\big(\mathsf{SL}(2,\mathbb{C})\big)$-invariant mapping

$$\begin{aligned} \mathsf{Hom}(\mathsf{F}_2, \mathsf{SL}(2,\mathbb{C})) &\xrightarrow{\chi} \mathbb{C}^3 \\ \rho &\longmapsto \begin{bmatrix} x := & \mathsf{Tr}\big(\rho(X)\big) \\ y := & \mathsf{Tr}\big(\rho(Y)\big) \\ z := & \mathsf{Tr}\big(\rho(Z)\big) \end{bmatrix} \end{aligned}$$

defines an isomorphism

$$\mathfrak{X}(\mathsf{F}_2, \mathsf{SL}(2,\mathbb{C})) \xrightarrow{\ \cong\ } \mathbb{C}^3.$$

This means that every regular function $\mathsf{Hom}(\mathsf{F}_2, \mathsf{SL}(2,\mathbb{C})) \xrightarrow{f} \mathbb{C}$ which is invariant under $\mathsf{Inn}\big(\mathsf{SL}(2,\mathbb{C})\big)$ factors as $\chi \circ F$ for some polynomial $F \in \mathbb{C}[x, y, z]$. The map χ is constant on closures of $\mathsf{Inn}\big(\mathsf{SL}(2,\mathbb{C})\big)$-orbits. A point has a closed orbit if and only if it is either irreducible or is a direct sum of a pair of 1-dimensional representations in $\mathsf{SL}(2,\mathbb{C})$; in the latter case, ρ is equivalent to a representation by diagonal matrices:

$$\rho(X) = \begin{bmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{bmatrix}$$
$$\rho(Y) = \begin{bmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{bmatrix}$$
$$\rho(Z) = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}$$

where $\xi\eta\zeta = 1$, and

$$\begin{aligned} x &= \xi + \xi^{-1} \\ y &= \eta + \eta^{-1} \\ z &= \xi\eta + (\xi\eta)^{-1}. \end{aligned}$$

An example of such a representation is given in (2.3). The commutator trace

$$\begin{aligned} \mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C})) &\longrightarrow \mathbb{C} \\ \rho \qquad &\longmapsto \mathsf{Tr}[\rho(X), \rho(Y)] \end{aligned}$$

descends to the polynomial $\kappa \in \mathbb{C}[x,y,z]$ defined by:

$$\kappa(x,y,z) := x^2 + y^2 + z^2 - xyz - 2$$

Then ρ is irreducible if and only if $\kappa(x,y,z) \neq 2$. This condition is equivalent to the $\mathsf{Inn}\big(\mathsf{SL}(2,\mathbb{C})\big)$-orbit being closed and having trivial isotropy.

The level set $\kappa(x,y,z) = 2$ corresponds to reducible representations, but the character variety cannot distinguish between a representation and its *semisimplification.* Namely, if ρ is reducible, presrving a linear subspace $L \subset \mathbb{C}^2$, then ρ induces a representation ρ^{ss} on $L \oplus \mathbb{C}^2/L$, which we call its *semisimplification.* Unless ρ is completely reducible (that is, *reductive*) it is *not* equivalent as a representation to ρ^{ss} although it has the same character. The level set $\kappa^{-1}(2)$ is the *Cayley cubic,* discussed in §2 and admits a rational parametrization (2.2). As discussed below, the diagonal matrices are precisely the ones preserving the decomposition $\mathbb{C}^2 \cong L \oplus \mathbb{C}^2/L$; compare (2.3).

The two $\mathbb{R}$-forms of $\mathsf{SL}(2,\mathbb{C})$ are $\mathsf{SU}(2)$ and $\mathsf{SL}(2,\mathbb{R})$. Real characters, that is $(x,y,z) \in \mathbb{R}^3 \subset \mathbb{C}^3$, correspond to equivalence classes of representations into $\mathsf{SU}(2)$ or $\mathsf{SL}(2,\mathbb{R})$. (The common case occurs, namely for representations into $\mathsf{SO}(2) = \mathsf{SU}(2) \cap \mathsf{SL}(2,\mathbb{R})$.)

Once again this is detected by the polynomial κ. Suppose that a representation ρ has *real* trace, that is, $(x,y,z) \in \mathbb{R}^3$. If ρ is equivalent to an $\mathsf{SU}(2)$-representation, then $-2 \le x,y,z \le 2$. Furthermore, $\rho[X,Y] = [\rho(X), \rho(Y)]$ also has trace in $[-2,2]$, and in particular $\kappa(x,y,z) \le 2$. Conversely every

$$(x,y,z) \ \in \ [-2,2]^3 \cap \kappa^{-1}(-\infty, 2]$$

arises as a character of a representation of F_2 into $\mathsf{SU}(2)$.

In [**7**], this is proved by relating $\mathsf{SU}(2)$ as the universal covering of the orthogonal group $\mathsf{SO}(3)$ which is conjugate to the isometry group of a positive definite quadratic form on $\mathbb{R}^3$. The corresponding symmetric bilinear form is defined by the symmetric 3×3 matrix

$$\mathbf{B} := \begin{bmatrix} 2 & z & y \\ z & 2 & x \\ y & x & 2 \end{bmatrix}$$

which has determinant $4 - 2\kappa(x,y,z)$. The 2×2-minors of $\mathbf{B}$ are positive definite if and only if $-2 < x,y,z < 2$. Furthermore $\mathbf{B}$ itself is positive definite if and only if $-2 < x,y,z < 2$ and $\det(\mathbf{B}) > 0$. Compare [**7**] for further details.

2. The one-holed torus

The fundamental group of $\Sigma_{1,1}$ is a two-generator free group $\langle X, Y\rangle$ with redundant geometric presentation

$$\pi := \langle X, Y, K \mid K = XYX^{-1}Y^{-1}\rangle$$

wit peripheral generator K.

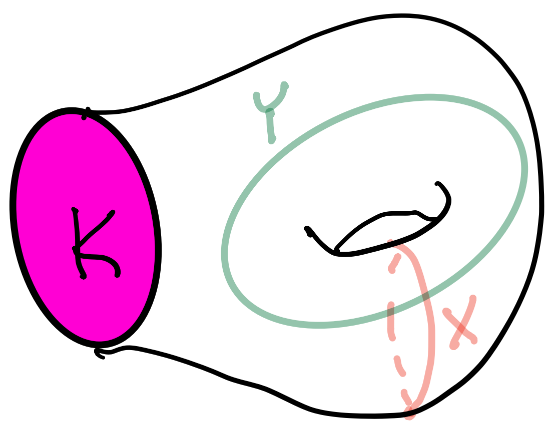

FIGURE 3. Three loops on $\Sigma_{1,1}$.

The boundary trace is defined as follows. The commutator trace function corresponds to the peripheral structure $\partial_1 = K = [X, Y] = XYX^{-1}Y^{-1}$:

$$\begin{aligned} \mathfrak{X}(\mathsf{F}_2, \mathsf{SL}(2,\mathbb{C})) \cong \mathbb{C}^3 &\xrightarrow{\kappa} \mathbb{C} \\ (x,y,z) &\longmapsto x^2 + y^2 + z^2 - xyz - 2 \\ &= \mathsf{Tr}[\rho(X), \rho(Y)] \end{aligned} \tag{2.1}$$

where $x = \mathsf{Tr}\big(\rho(X)\big), y = \mathsf{Tr}\big(\rho(Y)\big), z = \mathsf{Tr}\big(\rho(XY)\big)$.

Level sets of κ are relative character varieties:

- For $k < -2$, level set $\mathcal{S}_k$ has four components, each component parametrizing convex hyperbolic structures with totally geodesic boundary whose length l relates to k by:

  $$k = -2\cosh(l/2)$$

  These convex hyperbolic structures extend uniquely to complete hyperbolic structures with ideal boundary parallel to a unique closed geodesic of length l. Each component is homeomorphic to a disc and the various components are parametrized by spin structures on Σ.
- For $k = -2$, the level set $\mathcal{S}_{-2}$ is the *Markoff surface,* with five components. One component is $\{\mathbf{o}\}$ where $\mathbf{o}$ is the *origin* $(0,0,0)$, the unique $\mathsf{SU}(2)$-character with $\kappa = -2$. It is an isolated point in the real level set $\mathcal{S}_{-2}$,

although it is a node in the complexification $\kappa^{-1}(-2)$. The origin is the character of the representation given by the *Pauli spin matrices:*

$$\rho(X) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ \rho(Y) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \ \rho(Z) = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}.$$

The other four components correspond complete finite area hyperbolic structures on $\Sigma_{1,1}$ with spin structures. The *Markoff triples* form the orbit of $(3,3,3)$ and correspond to hyperbolic structures on $\Sigma_{1,1}$ with triple symmetry. $\mathbf{o}$ is a singular point in $\kappa^{-1}(-2)$ (an ordinary double point) and is isolated in

$$\mathcal{S}_{-2} = \kappa^{-1}(-2) \cap \mathbb{R}^3.$$

- For $-2 < k < 2$, the level set has five components, one of which is a compact component corresponding to $\mathsf{SU}(2)$-representations (see Figure 5). The four noncompact level sets for $-2 < k < 2$ correspond to hyperbolic structures on torus with an isolated singularity of cone angle θ, where $k = -2\cos(\theta/2)$.
- The level set $\kappa^{-1}(+2)$ corresponds to *reducible characters* and forms the *Cayley cubic*

$$x^2 + y^2 + z^2 - xyz = 4.$$

(Compare Figure 6.) This is the one case when a level set of κ admits a rational parametrization:

$$\begin{aligned} \mathbb{C}^* \times \mathbb{C}^* &\longrightarrow \kappa^{-1} \subset \mathbb{C}^3 \\ (\xi, \eta) &\longmapsto \begin{bmatrix} \xi + \xi^{-1} \\ \eta + \eta^{-1} \\ \xi\eta + (\xi\eta)^{-1} \end{bmatrix} \end{aligned} \tag{2.2}$$

which is a quotient map by the involution

$$(\xi, \eta) \longmapsto (\xi^{-1}, \eta^{-1}).$$

The corresponding reducible representation is:

$$\begin{aligned} X &\xmapsto{\rho} \begin{bmatrix} \xi & * \\ 0 & \xi^{-1} \end{bmatrix} \\ Y &\xmapsto{\rho} \begin{bmatrix} \eta & * \\ 0 & \eta^{-1} \end{bmatrix} \end{aligned} \tag{2.3}$$

- For $k \geq 2$, the level set is homeomorphic to $\Sigma_{0,4}$. In particular it is connected and *noncompact.*

Other famous cubic surfaces (or rather their complex projectivizations) occur in this family: the *Fermat cubic* defined by $A^3+B^3+C^3+D^3 = 0$ arises for $k = -10/3$ and the *Clebsch diagonal cubic* defined by $A^3+B^3+C^3+D^3+E^3 = A+B+C+D+E = 0$ arises for $k = 18$.

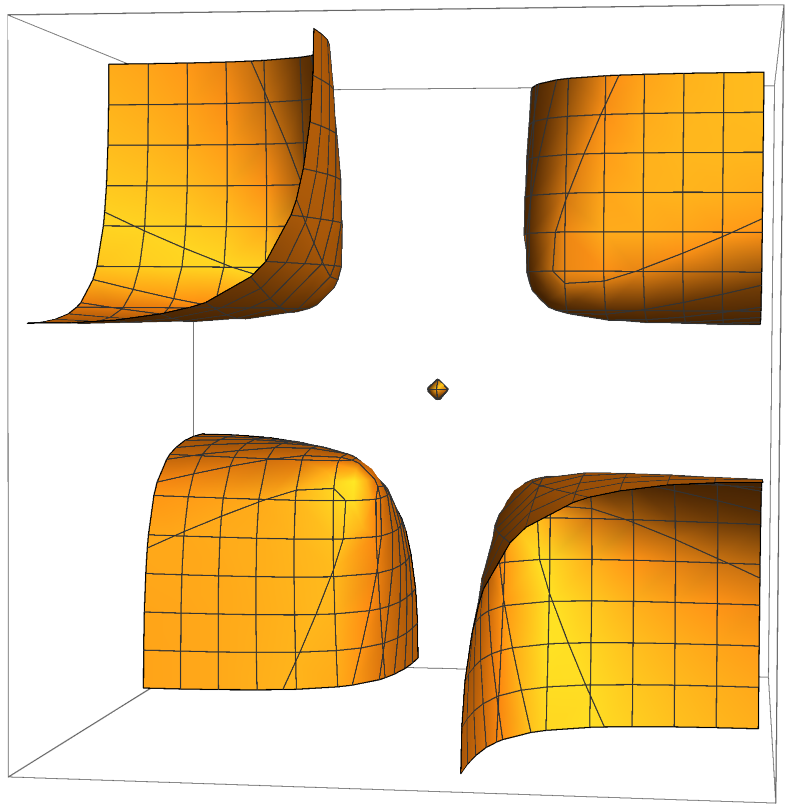

FIGURE 4. The Markoff cubic surface, with the origin and four components corresponding to the Fricke-Teichmüller space of $\Sigma_{1,1}$

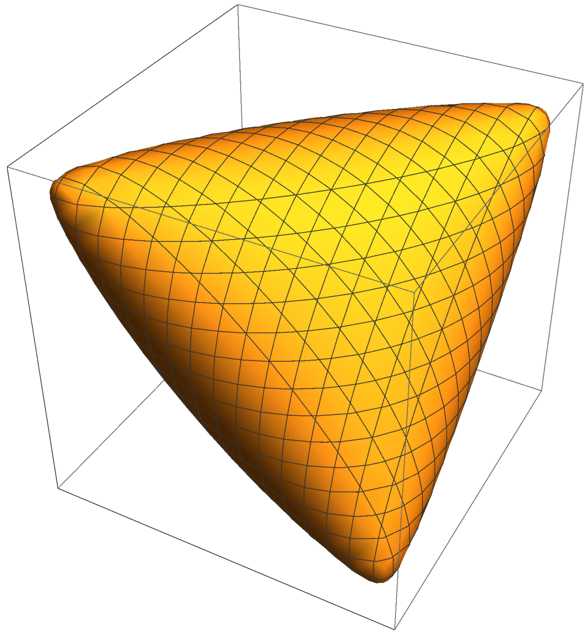

FIGURE 5. A compact component, corresponding to $\mathsf{SU}(2)$-representations of F_2

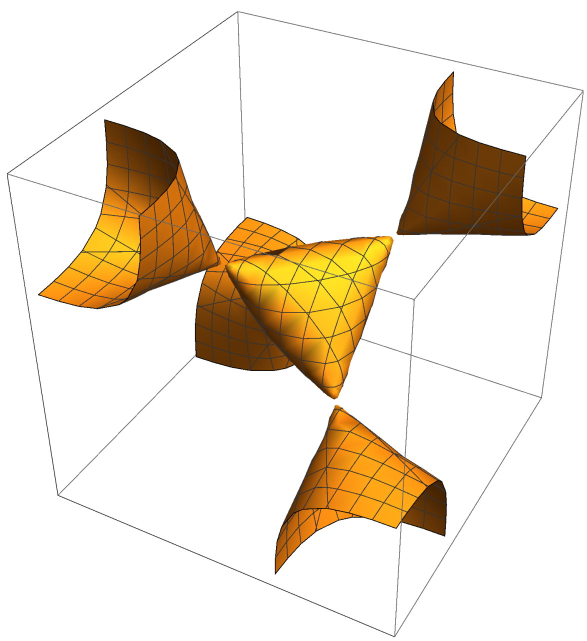

FIGURE 6. The Cayley cubic surface, corresponding to reducible representations. Its four nodes correspond to the (central) $\{\pm\mathbf{1}\}$-representations and form the vertices of a curvilinear tetrahedron.

3. The four-holed sphere

The fundamental group of $\Sigma_{0,4}$ is a three-generator free group $\langle A, B, C\rangle$ given by redundant geometric presentation

$$\pi := \langle A, B, C, D \mid ABCD = \mathbb{I}\rangle$$

with peripheral generators A, B, C, D. For a $\mathsf{SL}(2, \mathbb{C})$-representation ρ, we denote the boundary traces by $a, b, c, d \in \mathbb{C}$.

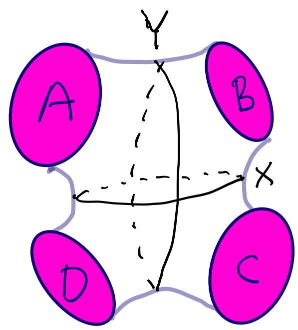

FIGURE 7. Some loops on $\Sigma_{0,4}$.

The traces [1] of the interior curves are:

$$x := -\mathsf{Tr}\big(\rho(AB)\big), \qquad y := -\mathsf{Tr}\big(\rho(BC)\big), \qquad z := -\mathsf{Tr}\big(\rho(CA)\big).$$

These seven functions are related by the defining equation

$$\begin{aligned} x^2 + y^2 + z^2 - xyz + &\\ (ab+cd)x + (bc+ad)y + (ca+bd)z &\\ &= 4 - (a^2+b^2+c^2+d^2+abcd) \end{aligned} \tag{3.1}$$

which we write as

$$x^2 + y^2 + z^2 - xyz + px + qy + rz = s$$

where the linear and constant terms are defined as:

$$p = ab + cd, \tag{3.2}$$
$$q = bc + ad, \tag{3.3}$$
$$r = ca + bd, \tag{3.4}$$
$$s = 4 - (a^2+b^2+c^2+d^2+abcd), \tag{3.5}$$

Then (3.1) defines a quartric hypersurface $\mathcal{V} \subset \mathbb{C}^7$, which we regard as the total space of a family of cubic surfaces with coordinates $(x, y, z) \in \mathbb{C}^3$. This family lives over $\mathbb{C}^4$ with parameters the boundary traces $(a, b, c, d) \in \mathbb{C}^4$. To describe this

[1]This convention departs from some previous works ([**1, 3, 4, 8, 7**]). The minus sign is introduced to make the defining equation (3.1) more compatible with the defining equation (2.1); in particular the higher order terms in both are $x^2 + y^2 + z^2 - xyz$, not $x^2 + y^2 + z^2 + xyz$.

family of surfaces explicitly, denote the coordinate projection $\mathbb{C}^7 \to \mathbb{C}^3$ by Π_{XYZ}. Then the *relative character varieties*

$$\mathcal{V}_{a,b,c,d} := \Pi_{XYZ}\Big(\mathcal{V} \cap \big(\{(a,b,c,d)\} \times \mathbb{C}^3\big)\Big)$$

form a family of cubic surfaces in $\mathbb{C}^3$ with coordinates (x, y, z) parametrized by $(a, b, c, d) \in \mathbb{C}^4$. Cantat and Loray [**3, 4**] prove that the mapping

$$\begin{aligned} \mathbb{C}^4 &\longrightarrow \mathbb{C}^3 \\ (a,b,c,d) &\longmapsto (p,q,r,s) \end{aligned}$$

defined by (3.2),(3.3),(3.4),(3.5) has degree 24, and in particular is surjective. (Compare also Goldman-Toledo [**10**] for a proof of surjectivity.)

When $(a, b, c, d) \in \mathbb{R}^4$ (so $p, q, r \in \mathbb{R}$ as well), the real solutions of (3.1) are either $\mathsf{SL}(2,\mathbb{R})$-characters or $\mathsf{SU}(2)$-characters (or both). In analogy with the case of the one-holed torus, $-2 \le a, b, c, d \le 2$ and

$$\kappa(a,b,x), \kappa(c,d,x), \kappa(b,c,y), \kappa(a,d,y), \kappa(c,a,z), \kappa(b,d,z) < 2$$

are *necessary* conditions for a relative character $(a, b, c, d; x, y, z)$ to correspond to an $\mathsf{SU}(2)$-representation. We conjecture that these inequalities are also *sufficient.*.

Denote the set of real solutions by

$$\mathcal{S}_{a,b,c,d} := \mathcal{V}_{a,b,c,d} \cap \mathbb{R}^3.$$

In 1992, Benedetto and Goldman [**1**] proved that, for certain $(a, b, c, d) \in [-2, 2]^4$, the real algebraic set $\mathcal{S}_{(a,b,c,d)}$ has a connected component of $\mathsf{SL}(2,\mathbb{R})$-characters which is *compact.* This markedly contrasts the case of relative character varieties of $\Sigma_{1,1}$, when the compact components of $\kappa^{-1}(k) \cap \mathbb{R}^3$ correspond to exactly to $\mathsf{SU}(2)$-representations and not $\mathsf{SL}(2,\mathbb{R})$-representations. (These components exist only if $-2 \le k < 2$.)

In §9.3 of [**3**],Cantat and Loray show that even if $\mathcal{V}_{a,b,c,d}$ admits a compact component corresponding to $\mathsf{SL}(2,\mathbb{R})$-representations, then by changing (a, b, c, d) but keeping (p, q, r, s) fixed, the compact component corresponds to $\mathsf{SU}(2)$-representations.

The surfaces $\Sigma_{0,4}$ and $\Sigma_{1,1}$ closely relate. When the linear coefficients $p, q, r = 0$, then the defining equation for relative character varieties of $\Sigma_{1,1}$ agrees with that of $\Sigma_{0,4}$ where $k = s - 2$. (Compare [**10**], Theorem 6 and Lemma 7.)

Lemma 3.1. *Suppose $p, q, r = 0$. Then one of two (not exclusive) possibilities occur:*

- *At least three of a, b, c, d vanish;*
- *Three of a, b, c, d are equal and the fourth equals their negative.*

Both possibilties occur when $a = b = c = d = 0$.

When $a = b = c = d = 0$, we call the character *bi-dihedral* [2]. The corresponding representation sends A, B, C, D to involutions in geodesics in H^3 which admit a common orthogonal geodesic. Such a reprensentation is a double extension of a reducitble representation into $\mathbb{C}^\times < \mathsf{SL}(2,\mathbb{C})$. Here is an example of a bi-dihedral

[2]Apologies for the linguistic impurity.

representation:

$$A \longmapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$B \longmapsto \begin{bmatrix} 0 & \xi \\ -\xi^{-1} & 0 \end{bmatrix}$$
$$C \longmapsto \begin{bmatrix} 0 & \xi\eta \\ -\xi^{-1}\eta^{-1} & 0 \end{bmatrix}$$
$$D \longmapsto \begin{bmatrix} 0 & \eta \\ -\eta^{-1} & 0 \end{bmatrix}$$

It contains the reducible representation (2.3), when the matices are diagonal, with index two.

Proof of Lemma 3.1. Suppose that $d = 0$ but $a, b \neq 0$. Then (3.2)implies that

$$0 \neq ab = ab + cd = p = 0,$$

a contradiction. Thus one of a, b must vanish. If, for example, $a = 0$ and $b, c \neq 0$, then (3.3) implies

$$0 \neq bc = bc + da = q = 0,$$

a contradiction. Thus if one of a, b, c, d vanish, then at least three of them vanish, as claimed.

When $a = b = c = d = 0$, the character is bi-dihedral, as discussed above. Therefore we assume that all $a, b, c, d \neq 0$, and show that three of them are equal and the fourth is their negative.

Definitions (3.3) and (3.4) and $q = r = 0$ imply

$$\frac{a}{b} = -\frac{c}{d}, \qquad \frac{c}{d} = -\frac{b}{a}, \tag{3.6}$$

and $(a/b)^2 = 1$. Thus $a = \pm b$. Suppose first that $a = b$. Then (3.6) implies $d = -c$. Now apply (3.2) with $p = 0$ to similarly conclude that $c = \pm b$. Suppose first that $c = b$. Then $c = b = a = -d$ as desired. Similarly $c = -b$ implies $c = -b = -d = -a$ as desired. The case that $a = -b$ is completely analogous. □

The first case, when the peripheral traces are $(0, 0, 0, d)$, has been treated by §6.3 of Goldman [**6**], and §2.4 of Cantat-Loray [**3**]. In that case the $\Sigma_{0,4}$-character corresponds to a representation ρ where $\rho(A), \rho(B), \rho(C)$ are lifts to $\mathsf{SL}(2, \mathbb{C})$ of involutions in $\mathsf{PSL}(2, \mathbb{C})$ and

$$\rho(D) = \rho(C)\rho(B)\rho(A)$$

is (essentially) their product. The representation thus descends from $\pi_1(\Sigma_{0,4}) \cong \mathsf{F}_3$ to the fundamental group of an orbifold-with-boundary $\mathfrak{O}$ whose underlying surface is a disc and has three branch points of order two. The (hyper-) elliptic involution on $\Sigma_{1,1}$ defines an orbifold-covering space $\Sigma_{1,1} \to \mathfrak{O}$ and the corresponding $\Sigma_{1,1}$-character corresponds to the restriction of the representaion to the induced monomorphism

$$\begin{aligned} \pi_1(\Sigma_{1,1}) &\hookrightarrow \pi_1(\mathfrak{O}) \\ X &\longmapsto A'B' \\ Y &\longmapsto B'C' \end{aligned}$$

where

$$\pi_1(\mathfrak{O}) = \langle A', B', C', D' \mid (A')^2 = (B')^2 = (C')^2 = A'B'C'D' = e\rangle$$

in the notation of [**6**], §6.3, to which we refer for details. A similar interpretation for the relative characters for $(a, a, a, -a)$ for $a \neq 0$ would be interesting.

4. Recent developments

Recently Deroin-Tholozan [**5**] found compact components of $\mathsf{PSL}(2,\mathbb{R})$-characters in $\mathfrak{X}(\Sigma_{0,n})$ for all $n \geq 3$. They called these representations *supra-maximal* since their relative Euler class "exceeded" the presumed maximum value in the Milnor-Wood inequality. Namely, for a surface with nonempty boundary, to define the Toledo invariant (which agrees with the Euler class when $G = \mathsf{PSL}(2,\mathbb{R})$), one requires some boundary conditions. As in [**2**], one must correct the definition of the Toledo invariant when the holonomy around a boundary component c is elliptic. The correction term is the rotation number. Denoting the subset of $\mathsf{PSL}(2,\mathbb{R})$ consisting of elliptic elements by Ell, and the subset of hyperbolic elements by Hyp. The *rotation angle mapping* $\mathsf{Ell} \xrightarrow{\theta} (0, 2\pi)$ is $\mathsf{Inn}\big(\mathsf{PSL}(2,\mathbb{R})\big)$-invariant and assigns to

$$\begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

the parameter θ.[3] Although it is continuous on Ell, it does *not* extend continuously to $\mathsf{PSL}(2,\mathbb{R})$. Deroin and Tholozan extend θ to an invariant upper-semicontinuous function which vanishes on Hyp and takes the identity element $\mathbf{1} \in \mathsf{PSL}(2,\mathbb{R})$ to 2π. ($\theta = 0$ on the positive parabolic elements and $\theta = 2\pi$ on the negative parabolic elements.[4]) Then the corrected relative Euler number $e(\rho)$ of a $\mathsf{PSL}(2,\mathbb{R})$-representation ρ is obtained by lifting the interior generators of $\pi_1(\Sigma)$ to the universal covering $\widetilde{\mathsf{SL}(2,\mathbb{R})}$ and correcting by contributions of θ for each boundary component. Specifically, in terms of the standard presentation

$$\pi_1(\Sigma) = \langle A_1, B_1, \dots, A_g, B_g, C_1, \dots C_n \mid [A_1, B_1] \dots [A_g, B_g] C_1 \dots C_n = \mathbb{I}\rangle$$

and a representation ρ, the expression

$$[\widetilde{\rho(A_1)}, \widetilde{\rho(B_1)}] \dots [\widetilde{\rho(A_g)}, \widetilde{\rho(B_g)}] \quad \widetilde{\rho(C_1)} \dots \widetilde{\rho(C_n)}$$

lies in

$$\pi_1\big(\mathsf{PSL}(2,\mathbb{R})\big) = \mathsf{Ker}\big(\widetilde{\mathsf{SL}(2,\mathbb{R})} \longrightarrow \mathsf{PSL}(2,\mathbb{R})\big) \cong \mathbb{Z},$$

where $\widetilde{X}$ denotes the lift of $X \in \mathsf{PSL}(2,\mathbb{R})$ which is compatible with the above choice of θ. This is the *relative Euler number* $e(\rho)$.

Using the theory of the Toledo invariant and maximal representations developed by Burger-Iozzi-Wienhard [**2**], Deroin and Tholozan find compact components of relative $\mathsf{SL}(2,\mathbb{R})$-characters exist only for *planar surfaces,* that is, when Σ has genus 0. They show that for planar surfaces, their invariant may exceed the Milnor-Wood bound $|\chi(\Sigma)| = n - 2$, but is no greater than n. The case of n only occurs

[3] A. Maret has pointed out that this parameter θ is not completely well-defined until one requires, for example, that $\theta < \pi$ for elliptic elements close to $\mathbf{1}$ in the positive segment of an ellptic one-parameter subgroup.

[4] A positive parabolic element is one $\mathsf{PSL}(2,\mathbb{R})$-conjugate to $\left[\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right]$ for $t > 0$ and negative parabolic for $t > 0$. This distinction disappears for $\mathsf{PGL}(2,\mathbb{R})$-conjugacy.

for the trivial representation, and n for the *Deroin-Tholozan* supra-maximal representations. Furthermore they show that if $e(\rho) > n-2$, then the surface is planar ($g = 0$) and compact components arise.

For every choice of boundary traces, each compact component of $\mathfrak{X}(\Sigma_{0,n}, \mathsf{SL}(2,\mathbb{R}))$ is symplectomorphic to $\mathbb{C}\mathsf{P}^{n-3}$ with its standard Fubini-Study symplectic structure, at least up to scale. Using Delzant's theory of moment polytopes, they compute the symplectic volume in terms of the boundary parameters. Arnaud Maret [**13**] found action-angle coordinates for the Hamiltonian twist flows on Deroin-Tholozan components.

Unlike components of Fuchsian characters, $\rho(x)$ is elliptic for every $x \in \pi$ corresponding to a *simple closed curve.* This is easy to see because otherwise the Hamiltonian twist flow would be unbounded, contradicting compactness. They also showed that $\mathsf{Mod}(\Sigma)$-orbit of $[\rho]$ is bounded. Maret [**12**] showed that the the $\mathsf{Mod}(\Sigma)$-action is ergodic with respect to the symplectic measure.

Following suggestions of Olivier Biquard, Gabriele Mondello [**14**] interpreted these results in terms of parabolic Higgs bundles. This is closely related to the fact (see [**5**]: For *every* complex structure on $\Sigma_{0,n}$, $\exists$ ρ-equivariant holomorphic map $\widetilde{\Sigma_{0,n}} \longrightarrow G/K$. This is analogous to constant map when G is compact. Furthermore it contrasts the situation in *higher Teichmüller theory* that for many classes of surface group representations (for example Hitchin representations into low rank simple real forms), that there is a unique conformal structure giving an equivariant holomorphic metric (Labourie [**11**]). This gives a holomorphic identification of the symplectic leaves as above.

Finally we mention that the $\mathsf{SL}(2,\mathbb{R})$-theory extends to higher rank Lie groups. Tholozan-Toulisse [**16**]have found compact components of representations in higher rank Hermitian Lie groups:

$$\mathsf{PU}(p,q), \quad \mathsf{Sp}(2m,\mathbb{R}), \quad \mathsf{SO}*(2m)$$

and many of the results proved in [**5**] generalize to these groups.

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