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CONVEX REAL PROJECTIVE STRUCTURES ON COMPACT SURFACES

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Abstract

The space of inequivalent representations of a compact surface S with $\chi(S) < 0$ as a quotient of a convex domain in $\mathbb{RP}^2$ by a properly discontinuous group of projective transformations is a cell of dimension $-8\chi(S)$.

The purpose of this paper is to investigate convex real projective structures on compact surfaces. Let $\mathbb{RP}^2$ be the real projective plane and $\mathbf{PGL}(3, \mathbb{R})$ the group of projective transformations $\mathbb{RP}^2 \to \mathbb{RP}^2$. A *convex real projective manifold* (*convex* $\mathbb{RP}^2$*-manifold*) is a quotient $M = \Omega/\Gamma$, where $\Omega \subset \mathbb{RP}^2$ is a convex domain and $\Gamma \subset \mathbf{PGL}(3, \mathbb{R})$ is a discrete group of projective transformations acting properly on Ω. The universal covering of M may then be identified with Ω, and the fundamental group $\pi_1(M)$ with Γ. Two such quotients $M_1 = \Omega_1/\Gamma_1$ and $M_2 = \Omega_2/\Gamma_2$ are *projectively equivalent* if there is a projective transformation $h \in \mathbf{PGL}(3, \mathbb{R})$ such that $h(\Omega_1) = \Omega_2$ and $h\Gamma_1 h^{-1} = \Gamma_2$. The classification of convex $\mathbb{RP}^2$-manifolds with $\chi(M) \geq 0$ is due to Kuiper [30], [31] in early 1950's.

If S is a closed smooth surface, then a *convex* $\mathbb{RP}^2$*-structure* on S is defined to be a diffeomorphism $f: S \to M$ where M is a convex $\mathbb{RP}^2$-manifold; two such pairs (f, M) and (f', M') are regarded as equivalent if there is a projective equivalence $h: M \to M'$ such that $h \circ f$ is isotopic to f'. Let $\pi = \pi_1(S)$ by the fundamental group of S. Given a convex $\mathbb{RP}^2$-structure on S, the action of π by deck transformations on the universal covering space of S determines a homomorphism $\pi \to \mathbf{PGL}(3, \mathbb{R})$, well defined up to conjugacy in $\mathbf{PGL}(3, \mathbb{R})$. The set of

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projective equivalence classes of convex $\mathbb{RP}^2$-structures on S has a natural topology which can be identified with an open subspace of the space $\mathbf{Hom}(\pi, \mathbf{PGL}(3, \mathbb{R}))/\mathbf{PGL}(3, \mathbb{R})$ of conjugacy classes of representations $\pi \to \mathbf{PGL}(3, \mathbb{R})$. (When S has boundary, we assume that the boundary is represented by closed geodesics each having a geodesically convex collar neighborhood). We call this space *the deformation space of convex* $\mathbb{RP}^2$-*structures on* S and denote it by $\mathfrak{P}(S)$. We determine explicit coordinates on this space; our main result is the following.

Theorem 1. *Let S be a compact surface having n boundary components such that $\chi(S) < 0$. Then $\mathfrak{P}(S)$ is diffeomorphic to a cell of dimension $-8\chi(S)$ and the map which associates to a convex $\mathbb{RP}^2$-manifold M the germ of the $\mathbb{RP}^2$-structure near ∂M is a fibration of $\mathfrak{P}(S)$ over an open $2n$-cell with fiber an open cell of dimension $-8\chi(S) - 2n$.*

Corollary. *Let S be a closed orientable surface of genus $g > 1$. Then the deformation space $\mathfrak{P}(S)$ of convex $\mathbb{RP}^2$-structures on S is diffeomorphic to an open cell of dimension $16(g-1)$.*

The deformation space $\mathfrak{P}(S)$ is an analogue of the Teichmüller space $\mathfrak{T}(S)$ of S, which is classically known (Fricke and Klein [12]) to be an open cell of dimension $6(g-1)$. Using the Klein-Beltrami projective model for hyperbolic geometry, every hyperbolic structure on S defines a convex $\mathbb{RP}^2$-structure; thus $\mathfrak{T}(S)$ embeds in $\mathfrak{P}(S)$. The mapping class group of S acts properly discontinuously on $\mathfrak{P}(S)$ as well as on $\mathfrak{T}(S)$; indeed $\mathfrak{P}(S)$ admits an equivariant retraction onto $\mathfrak{T}(S)$. The space $\mathfrak{P}(S)$ promises to have very interesting geometry: there is a canonical symplectic form on $\mathfrak{P}(S)$ ([14], [19]) as well as Riemannian metrics on $\mathfrak{P}(S)$ both of which restrict to the Weil-Petersson Kähler form and the Weil-Petersson Riemannian metric on $\mathfrak{T}(S)$; perhaps these constitute a Kähler geometry on $\mathfrak{P}(S)$ extending the Weil-Petersson Kähler geometry of $\mathfrak{T}(S)$ (see [40]). Moreover projective duality defines a natural involution $\mathfrak{P}(S) \to \mathfrak{P}(S)$ whose stationary set equals $\mathfrak{T}(S)$.

Recently Suhyoung Choi, in his Princeton dissertation [7], showed that every closed $\mathbb{RP}^2$-manifold M with $\chi(M) < 0$ admits a canonical decomposition into convex subsurfaces along closed geodesics. Such decompositions (and hence the developing maps) are parametrized by a countably infinite set $\mathbb{D}(S)$ defined as follows. Consider families of pairs (γ_i, w_i) where $\{\gamma_i\}_{i \in \mathcal{J}}$ is a family of disjoint simple closed curves such that each γ_i is homotopically nontrivial and no two γ_i are homotopic. Let $w_i = w_i(x, y)$ be an element of the free semigroup on two generators x, y which has even word-length in the x, y. The set $\mathbb{D}(S)$ is defined as the set of equivalence classes of families $\{(\gamma_i w_i)\}_{i\in\mathscr{I}}$, where two such families are equivalent if the corresponding families $\{\gamma_i\}_{i\in\mathscr{I}}$ are isotopic collections of curves. As in [16], the various ways M can be decomposed into convex subsurfaces are parametrized by $\mathbb{D}(S)$, and combining Choi's theorem with the coordinates developed in this paper we obtain [8]:

Corollary. *Let S be a closed surface with $\chi(S) < 0$. Then there exists a canonical diffeomorphism*

$$\mathbb{RP}^2(S) \to \mathfrak{P}(S) \times \mathbb{D}(S).$$

In particular $\mathbb{RP}^2(S)$ is a countable union of open cells of dimension $-8\chi(S)$.

The paper is organized as follows. In §1, certain facts about the group of projective transformations are collected. The projective transformations which arise from convex $\mathbb{RP}^2$-structures on closed surfaces are all represented by matrices with three positive eigenvalues; we call such projective transformations *positive hyperbolic.* Their conjugacy classes in $\mathbf{PGL}(3, \mathbb{R})$ form an open 2-cell. In 1.4–1.8 we give three equivalent sets of coordinates for this space of conjugacy classes. In 1.9–1.11 the dynamics of a positive hyperbolic projective transformation is discussed, and in 1.12 a technical lemma on the action of $\mathbf{PGL}(3, \mathbb{R})$ by conjugation is proved.

§2 discusses general facts concerning real projective structures and their deformation spaces. The definition of a general $\mathbb{RP}^2$-structure on a manifold is given and the development theorem is stated; from this we define the deformation space $\mathbb{RP}^2(S)$ of $\mathbf{RP}^2$-structures on a closed surface S. Such deformation spaces have been studied in numerous related contexts ([13], [22], [29], [34], [35], [38]); their prototype being the Teichmüller space, regarded as the deformation space for hyperbolic structures on compact surfaces. Unlike surfaces of zero Euler characteristic ([2], [13], [20], [35]) where the deformation space is neither Hausdorff nor a manifold, we prove in 2.4:

Theorem. *Let S be a closed surface with $\chi(S) < 0$. Then the deformation space $\mathbb{RP}^2(S)$ is a Hausdorff real analytic manifold of dimension $-8\chi(S)$.*

In 2.7 these notions are extended to $\mathbb{RP}^2$-structures on surfaces with boundary. If S is a compact surface with boundary, we consider $\mathbb{RP}^2$-structures such that each boundary component possesses a convex collar neighborhood.

§3 is concerned with the property of convexity of $\mathbb{RP}^2$-structures. The usual notion of geodesic convexity is equivalent to the condition that the universal covering space is projectively equivalent to a convex domain in $\mathbb{RP}^2$. The basic results on convex $\mathbb{RP}^2$-structures on a closed surface S are due to Kuiper [31], Kac-Vinberg [23] and Benzécri [4]. It follows from this work that if M is a convex $\mathbb{RP}^2$-manifold with $\chi(M) < 0$, then the universal covering space of M is a strictly convex domain $\Omega \subset \mathbb{RP}^2$ containing no affine lines; the boundary $\partial\Omega$ is a C^1-curve which is either a conic (in which case the convex $\mathbb{RP}^2$-structure on M arises from a hyperbolic structure on M) or is nowhere $C^{1+\varepsilon}$ for some $\varepsilon > 0$. Furthermore every element of Γ is a positive hyperbolic projective transformation. Results of Koszul [28], [29] imply (Proposition 3.3) that the condition of convexity defines an open subset $\mathfrak{P}(S) \subset \mathbb{RP}^2(S)$ in the full deformation space and indeed (Proposition 3.4) $\mathfrak{P}(S)$ may be identified with an open submanifold of the space $\mathrm{Hom}(\pi, \mathbf{PGL}(3, \mathbb{R}))/\mathbf{PGL}(3, \mathbb{R})$ of equivalence classes of representations of the fundamental group π of S. The rest of this section is devoted to the proof of the following "combination theorem" (Theorem 3.7), which allows one to build convex $\mathbb{RP}^2$-structures by gluing together $\mathbb{RP}^2$-structures on surfaces with boundary along boundary components. It is here that the existence of principal collar neighborhoods of boundary components of convex $\mathbb{RP}^2$-manifolds is crucial.

Theorem. *Let M_0 be a (possibly disconnected) compact $\mathbb{RP}^2$-manifold with principal boundary, and suppose that $b_1, b_2 \subset \partial M_0$ are boundary components with collar neighborhoods $b_i \subset N(b_i) \subset M_0$ $(i = 1, 2)$. Suppose that $f\colon N(b_1) \to N(b_2)$ is a projective isomorphism. Then there exist an $\mathbb{RP}^2$-manifold $M = M_0/f$ and a simple closed geodesic $b \subset M$ such that $M|b \cong M_0$ and a tubular neighborhood $N(b) \subset M$ of b with a reflection $R\colon N(b) \to N(b)$ such that R induces f on $N(b)|b \subset M_0$. If M_0 is a convex $\mathbb{RP}^2$-manifold, then M is also a convex $\mathbb{RP}^2$-manifold.*

§4 proves Theorem 1 for the special case that S is a pair of pants (a sphere minus three discs). The argument uses the preceding theory to reduce the classification to a calculation involving 3×3 matrices. Theorem 1 is proved in general in §5, using the results of §§3 and 4 and the decomposition techniques for surfaces as in [1], [10], [17], [18], [21], [38]. In 5.6 explicit coordinates are given for $\mathfrak{P}(S)$ based on the Fenchel-Nielsen coordinates on $\mathfrak{T}(S)$.

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1. The group of projective transformations

1.1. The real projective plane $\mathbb{RP}^2$ is the space of all lines through the origin in $\mathbb{R}^3$; if $(x, y, z) \in \mathbb{R}^3 - \{0\}$ is a nonzero vector in $\mathbb{R}^3$, the corresponding point in $\mathbb{RP}^2$ will be denoted

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

in homogeneous coordinates. A linear transformation $A \in \mathbf{GL}(3, \mathbb{R})$ preserves lines through the origin; hence A induces a transformation of $\mathbb{RP}^2$, which by definition is a *projective transformation.* The group of such transformations is denoted $\mathbf{PGL}(3, \mathbb{R})$, and it is easy to see that there is an exact sequence

$$\{1\} \to \mathbb{R}^* \to \mathbf{GL}(3, \mathbb{R}) \to \mathbf{PGL}(3, \mathbb{R}) \to \{1\},$$

where $\mathbb{R}^* \lhd \mathbf{GL}(3, \mathbb{R})$ is the central subgroup consisting of scalar matrices. The analytic homomorphism $\mathbf{GL}(3, \mathbb{R}) \to \mathbf{SL}(3, \mathbb{R})$ defined by

$$A \mapsto (\det A)^{-1/3} A$$

defines an isomorphism $\mathbf{PGL}(3, \mathbb{R}) \to \mathbf{SL}(3, \mathbb{R})$ as *analytic* groups. In particular every projective automorphism g of $\mathbb{RP}^2$ lifts to an orientation-preserving linear transformation (also denoted g) of $\mathbb{R}^3$. Thus we shall henceforth use only the group $\mathbf{SL}(3, \mathbb{R})$, tacitly using the above analytic isomorphism whenever convenient.

1.2. A projective transformation $A \in \mathbf{SL}(3, \mathbb{R})$ is a *reflection* if and only if A has order two in $\mathbf{SL}(3, \mathbb{R})$. Such a transformation is represented by a diagonalizable matrix with eigenvalues ± 1. Necessarily the (-1)-eigenspace has dimension two, and the stationary set (the (1)-eigenspace) has dimension one. On the projective plane the (-1)-eigenspace determines a line $l(A)$ which is pointwise fixed, and the (1)-eigenspace determines a fixed point $p(A)$ disjoint from $l(A)$. There are coordinates near $l(A)$ in which the projective transformation A appears as a (Euclidean) reflection, and near the isolated fixed point $p(A)$ there are coordinates in which A appears as symmetry about $p(A)$.

1.3. Consider the three points

$$p_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad p = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

corresponding to the coordinate axes in $\mathbb{R}^3$. The three lines joining them

$$l_1 = \overleftrightarrow{p_2p_3}, \qquad l_2 = \overleftrightarrow{p_3p_1}, \qquad l_3 = \overleftrightarrow{p_1p_2},$$

correspond to the coordinate planes and divide $\mathbb{RP}^2$ into four triangular regions:

$$\begin{aligned}
\Delta_0 &= \{[x, y, z] \in \mathbb{RP}^2 | x > 0, \ y > 0, \ z > 0\}, \\
\Delta_1 &= \{[x, y, z] \in \mathbb{RP}^2 | x < 0, \ y > 0, \ z > 0\}, \\
\Delta_2 &= \{[x, y, z] \in \mathbb{RP}^2 | y < 0, \ x > 0, \ z > 0\}, \\
\Delta_3 &= \{[x, y, z] \in \mathbb{RP}^2 | z < 0, \ x > 0, \ y > 0\}.
\end{aligned}$$

A projective transformation $A \in \mathbf{SL}(3, \mathbb{R})$ which fixes p_1, p_2, p_3 is represented by a unique diagonal matrix in $\mathbf{SL}(3, \mathbb{R})$; the transformation A leaves invariant one triangular region Δ_i (and hence every Δ_i) if and only if it is represented by a diagonal matrix with positive eigenvalues. We shall denote the full group of diagonal matrices in $\mathbf{SL}(3, \mathbb{R})$ by $\mathscr{A}$ and the subgroup of diagonal matrices with positive eigenvalues by $\mathscr{A}_+$. Let $\mathscr{A}_0 \subset \mathbf{SL}(3, \mathbb{R})$ be the four-element group represented by diagonal matrices with eigenvalues ± 1; then $\mathscr{A}_0 - \{I\}$ consists of three reflections, each one of which fixes one of the three coordinates lines as well as the corresponding coordinate point (image of the corresponding coordinate axis in $\mathbb{RP}^2$). Clearly $\mathscr{A} = \mathscr{A}_+ \times \mathscr{A}_0$ and we see that $\mathscr{A}_0$ acts transitively and freely on the set of four invariant triangular regions, $\mathscr{A}_+$ acts transitively and freely on any one of the regions, and $\mathscr{A}$ acts transitively and freely on the union of all four invariant regions.

Invariants of positive hyperbolic projective transformations.

1.4. Consider an arbitrary element A of $\mathbf{SL}(3, \mathbb{R})$. Then A is said to be *hyperbolic* if it has three distinct real eigenvalues, and A is *positive hyperbolic* if it is conjugate in $\mathbf{SL}(3, \mathbb{R})$ to a diagonal matrix with positive eigenvalues. We denote the set of positive hyperbolic elements of $\mathbf{SL}(3, \mathbb{R})$ by $\mathbf{Hyp}_+$. We shall presently determine an invariant of conjugacy classes of positive hyperbolic elements which will be calculationally useful in §4.

Suppose that $A \in \mathbf{SL}(3, \mathbb{R})$. We define $\lambda(A)$ to be the real eigenvalue of $A \in \mathbf{SL}(3, \mathbb{R})$ having the smallest absolute value and $\tau(A) \in \mathbb{R}$ as the sum of the other two (possibly unreal) eigenvalues. Thus if $A \in \mathbf{Hyp}_+$ is represented by the diagonal matrix

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix} \tag{1-1}$$

with

$$\lambda\mu\nu = 1, \qquad 0 < \lambda < \mu < \nu, \tag{1-2}$$

then $\lambda(A) = \lambda$ and $\tau(A) = \mu + \nu$. Moreover $A \in \mathbf{Hyp}_+$ is determined up to $\mathbf{SL}(3, \mathbb{R})$-conjugacy by the set of eigenvalues of A which are

$$\begin{aligned} \lambda &= \lambda(A), \\ \mu &= \tfrac{1}{2}[\tau(A) - \sqrt{\tau(A)^2 - 4/\lambda(A)}], \\ \nu &= \tfrac{1}{2}[\tau(A) + \sqrt{\tau(A)^2 - 4/\lambda(A)}]. \end{aligned} \tag{1-3}$$

It follows that the pair $(\lambda(A), \tau(A))$ is a complete invariant of the $\mathbf{SL}(3, \mathbb{R})$-conjugacy class of A.

1.5. Proposition. *Consider the action of* $\mathbf{SL}(3, \mathbb{R})$ *on* $\mathbf{Hyp}_+$ *by conjugation. Then the restriction of*

$$\begin{aligned} \mathbf{SL}(3, \mathbb{R}) &\to \mathbb{R}^2 \\ A &\mapsto (\lambda(A), \tau(A)) \end{aligned}$$

to $\mathbf{Hyp}_+$ *is a* $\mathbf{SL}(3, \mathbb{R})$*-invariant fibration with image the region*

$$\mathfrak{R} = \{(\lambda, \tau) \in \mathbb{R}^2 | 0 < \lambda < 1, \ 2/\sqrt{\lambda} < \tau < \lambda + \lambda^{-2}\}$$

(depicted in Figure 1.1(a)*). Furthermore* $\mathbf{Hyp}_+ = (\lambda, \tau)^{-1}(\mathfrak{R})$ *and* $\mathbf{SL}(3, \mathbb{R})$ *acts transitively on each fiber with isotropy group the subgroup* $\mathscr{A}$ *corresponding to diagonal matrices.*

The proof will be based on the following lemma.

1.6. Lemma. *Let* $A \in \mathbf{SL}(3, \mathbb{R})$*. Suppose that* $v \in \mathbf{R}^3$ *is an eigenvector for* A *with eigenvalue* $0 < \lambda < 1$ *and* $E \subset \mathbf{R}^3$ *is a 2-dimensional invariant linear subspace not containing* v*. Let* τ *equal the trace of the restriction* $A|_E$*. Then* A *is positive hyperbolic with* $\lambda(A) = \lambda$ *if and only if*

$$0 < \lambda < 1, \qquad 2/\sqrt{\lambda} < \tau < \lambda + \lambda^{-2}.$$

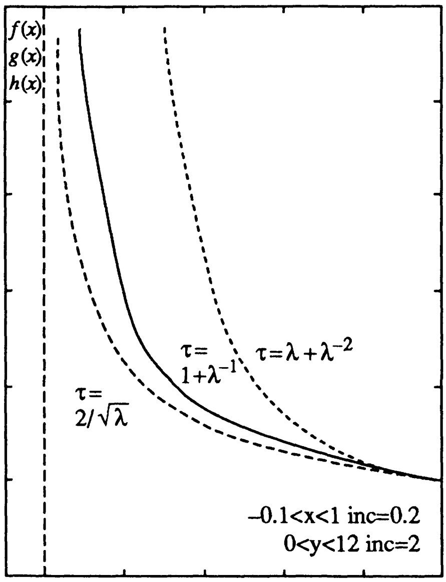

FIGURE 1.1(a)

Proof. Let $B = A|_E$; since $A \in \mathbf{SL}(3, \mathbb{R})$, we have $\det B = \lambda^{-1} > 0$. Then:

$$\begin{gathered} B \text{ has distinct positive eigenvalues} \\ \Updownarrow \\ \frac{\operatorname{tr}(B)}{\sqrt{\det B}} > 2 \\ \Updownarrow \\ \tau = \operatorname{tr}(B) > 2\sqrt{\det B} = 2/\sqrt{\lambda} \end{gathered}$$

Since $0 < \lambda < 1$, it follows that $\tau > 2/\sqrt{\lambda} > 2 > 2\lambda$. The eigenvalues of A consist of λ together with the eigenvalues of B which are given by

$$t_{\pm} = \tfrac{1}{2}(\tau \pm \sqrt{\tau^2 - 4/\lambda}).$$

These eigenvalues are real and distinct, provided $\tau > 2/\sqrt{\lambda}$. Now:

$$\begin{gathered}\lambda \text{ is the smallest eigenvalue of } A\\ \Updownarrow\\ \lambda < t_-\\ \Updownarrow\\ \sqrt{\tau^2 - 4/\lambda} < \tau - 2\lambda\\ \Updownarrow \text{ (since } \tau-2\lambda>0)\\ \tau^2 - 4/\lambda < (\tau - 2\lambda)^2\\ \Updownarrow\\ \tau < \lambda + \lambda^{-2}\end{gathered}$$

Thus A is positive hyperbolic if and only if $\tau < 2/\sqrt{\lambda}$, and in that case λ is the smallest eigenvalue of A if and only if $2/\sqrt{\lambda} < \tau < \lambda + \lambda^{-2}$. (If $0 < \lambda < 1$, then $\lambda^{1/2} + \lambda^{-3/2} > 2$ and thus $2/\sqrt{\lambda} < \lambda + \lambda^{-2}$.) This concludes the proof of Lemma 1.6.

Proof of 1.5. Clearly $(\lambda, \tau): \mathbf{Hyp}_+ \to \mathfrak{R}$ is $\mathbf{SL}(3, \mathbb{R})$-invariant. By (1-3) (λ, τ) determine the eigenvalues of A, and hence $\mathbf{SL}(3, \mathbb{R})$ acts transitively on each fiber. The isotropy over a diagonal matrix in $\mathbf{Hyp}_+$ equals its centralizer which is the full group $\mathscr{A}$ of diagonal matrices. By Lemma 1.6 the image of $\mathbf{Hyp}_+$ lies in $\mathfrak{R}$; if $(\lambda_0, \tau_0) \in \mathfrak{R}$, then (1-3) determines a diagonal matrix with $(\lambda, \tau)(A) = (\lambda_0, \tau_0)$. The converse assertion in Lemma 1.6 implies that if $A \in \mathbf{SL}(3, \mathbb{R})$ has $(\lambda, \tau)(A) \in \mathfrak{R}$, then $A \in \mathbf{Hyp}_+$. The proof of 1.5 is now complete.

1.7. The invariants (λ, τ) will be useful for the calculations later on, although more customary sets of invariants are equivalent to them. Algebraically more natural are the coefficients (x, y) of the characteristic polynomial of A,

$$\chi_A(t) = \det(tI - A) = t^3 - xt^2 + yt - 1;$$

if A is represented by the diagonal matrix (1-1), then

$$\begin{aligned}x &= \operatorname{tr}(A) = \lambda + \mu + \nu = \lambda(A) + \tau(A),\\ y &= \operatorname{tr}(A^{-1}) = \lambda^{-1} + \mu^{-1} + \nu^{-1} = \mu\nu + \lambda\nu + \lambda\mu\\ &= \lambda(A)^{-1} + \lambda(A)\tau(A).\end{aligned} \tag{1-4}$$

Now a matrix in $\mathbf{SL}(3, \mathbb{R})$ has real distinct eigenvalues if and only if its characteristic polynomial has three distinct real roots; this condition is easily expressed in terms of the *discriminant* $\delta(A)$ of χ_A:

$$\delta(A) = \mathrm{Resultant}(\chi_A, \chi_A') = \begin{vmatrix} 1 & -x & y & -1 & 0 \\ 0 & 1 & -x & y & -1 \\ 3 & -2x & y & 0 & 0 \\ 0 & 3 & -2x & y & 0 \\ 0 & 0 & 3 & -2x & y \end{vmatrix}$$

$$= -x^2y^2 + 4(x^3 + y^3) - 18xy + 27$$

and A is hyperbolic if and only if $\delta(A) < 0$; furthermore A is positive hyperbolic if and only if $\delta(A) < 0$ and $x, y > 0$.

It follows that the correspondence $(\lambda, \tau) \leftrightarrow (x, y)$ defined by (1-4) is a diffeomorphism

$$\mathfrak{R} \leftrightarrow \{(x, y) \in \mathbb{R}^2 | x, y > 0,\ x^2y^2 - 4(x^3 + y^3) + 18xy - 27 > 0\},$$

(compare Figure 1.1(b)).

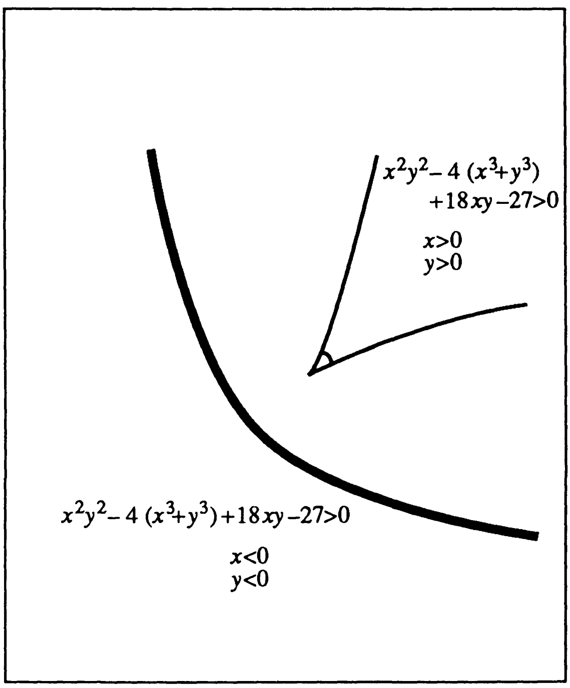

FIGURE 1.1(b)

1.8. Another pair of invariants of a positive hyperbolic element of $\mathbf{SL}(3, \mathbb{R})$ is more closely related to the geometry of convex $\mathbb{RP}^2$-manifolds.

Define

$$\ell(A) = \log(\nu/\lambda) > 0, \qquad m(A) = 3\log(\mu), \tag{1-5}$$

where A is represented by a diagonal matrix (1-1) satisfying (1-2). The conditions $0 < \lambda < \mu < \nu$ and $\lambda\mu\nu = 1$ are equivalent to the conditions

$$\ell(A) > 0, \qquad |m(A)| < \ell(A). \tag{1-6}$$

The invariants $\lambda(A)$ and $\tau(A)$ are easily expressed in terms of $\ell(A)$ and $m(A)$ by

$$\begin{aligned} \lambda(A) &= \exp\left(-\frac{\ell(A)}{2} - \frac{m(A)}{6}\right), \\ \tau(A) &= \exp\left(\frac{\ell(A)}{2} - \frac{m(A)}{6}\right) + \exp\left(\frac{m(A)}{3}\right). \end{aligned} \tag{1-7}$$

Clearly $(\ell(A), m(A))$ can be an arbitrary element of $\mathbb{R}_+ \times \mathbb{R}$ satisfying (1-6), and any $A \in \mathbf{Hyp}_+$ is determined up to conjugacy by $(\ell(A), m(A)) \in \mathbb{R}_+ \times \mathbb{R}$. The correspondence $(\lambda(A), \tau(A)) \leftrightarrow (\ell(A), m(A))$ defined by (1-7) is a diffeomorphism

$$\mathfrak{R} \leftrightarrow \{(\ell, m) \in \mathbb{R}_+ \times \mathbb{R} \,|\, |m| < \ell\}$$

giving another set of parameters for conjugacy classes in $\mathbf{Hyp}_+$.

Geometry of positive hyperbolic projective transformations.

1.9. Let $A \in \mathbf{Hyp}_+$; then the stationary set $\mathrm{Fix}(A)$ consists of three noncollinear points. By applying an inner automorphism of $\mathbf{SL}(3, \mathbb{R})$ we may assume that A is represented by a diagonal matrix (1-1). The fixed point corresponding to the eigenvector for λ is a repelling fixed point $\mathrm{Fix}_-(A)$, the fixed point corresponding to the eigenvector for ν is an attracting fixed point $\mathrm{Fix}_+(A)$, and the fixed point corresponding to the eigenvector for μ is a saddle point $\mathrm{Fix}_0(A)$. Let $\ell(A) \subset \mathbb{RP}^2$ be the line joining the attracting and repelling fixed points of A. We shall refer to $l(A)$ as the *principal line* for A. The unique reflection $R \in \mathbf{SL}(3, \mathbb{R})$ with stationary set $\mathrm{Fix}(R) = l(A) \cup \mathrm{Fix}(A)$ will be called the *principal reflection* for A; clearly R commutes with A. The two fixed points of A on $l(A)$ separate $l(A)$ into two A-invariant segments, which we call *principal segments* for A.

1.10. An *affine space* $\mathbb{A}$ in $\mathbb{RP}^2$ is by definition the complement of a (projective) line $l \subset \mathbb{RP}^2$; an *affine line* in $\mathbb{RP}^2$ is the intersection of a projective line l' distinct from l with the affine space $\mathbb{A} = \mathbb{RP}^2 - l$, i.e., the complement of a point in a projective line; we define a *half-plane* to be a component of the complement of two distinct lines in $\mathbb{RP}^2$. We say that a subset $S \subset \mathbb{RP}^2$ is *convex* if there exists an affine space $\mathbb{A} \subset \mathbb{RP}^2$ containing S such that S is convex in the usual sense, i.e., if $x, y \in S$, then the line segment $\overline{xy}$ lies in S. If $S \subset \mathbb{A}$, then its *convex hull* (with respect to $\mathbb{A}$) is the smallest convex subset of $\mathbb{A}$ containing S.

Lemma. *Let $A \in \mathbf{Hyp}_+$ and suppose that $x \in \mathbb{RP}^2$ does not lie on an A-invariant line. Then the closure of any convex set containing the $\langle A\rangle$-orbit of x contains a principal segment for A.*

Proof. Let $\mathbb{A} \subset \mathbb{RP}^2$ be an affine space and let $S \subset \mathbb{A}$ be a closed convex set containing the orbit $\langle A\rangle x$. As $n \to +\infty$ the sequence $A^n x \to \mathrm{Fix}_+(A)$, and as $n \to -\infty$ the sequence $A^n x \to \mathrm{Fix}_-(A)$. For $n > 0$, let σ_n denote the segment with endpoints $A^n(x)$ and $A^{-n}(x)$. Then clearly σ_n lies in the convex hull of $\langle A\rangle x$ (with respect to $\mathbb{A}$) and converges to a principal segment for A, which must lie in the closure of the orbit of x. Thus S contains a principal segment for A, as desired. (Compare Figure 1.2.)

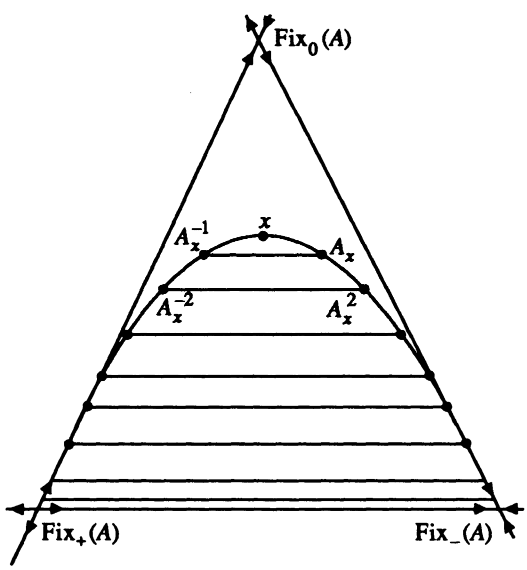

FIGURE 1.2

1.11. When A is represented by the matrix (1-1) as above, then it lies on a unique one-parameter subgroup comprised of elements

$$A^s = \begin{bmatrix} \lambda^s & 0 & 0 \\ 0 & \mu^s & 0 \\ 0 & 0 & \nu^s \end{bmatrix} \tag{1-8}$$

for $s \in \mathbb{R}$. Let $p_0 \in \mathbb{RP}^2$ be a point with homogeneous coordinates $[x_0, y_0, z_0]$ where $x_0, y_0, z_0 > 0$. Let l_∞ be a projective line not meeting the triangular region $\{[x, y, z] \in \mathbb{RP} | x > 0, \ y > 0, \ z > 0\}$. Then the convex hull of the orbit $\{A^s(p_0) | s \in \mathbb{R}\}$ in $\mathbb{RP}^2 - l_\infty$ equals the set

$$\left\{[x, y, z] \in \mathbb{RP}^2 | x, y, z > 0, \ \left(\frac{x}{x_0}\right)^{\log(\nu/\mu)} \left(\frac{z}{z_0}\right)^{\log(\mu/\lambda)} > \left(\frac{y}{y_0}\right)^{\log(\nu/\lambda)}\right\}.$$

In general we can define families of $\langle A \rangle$-invariant convex sets

$$(1\text{-}9) \quad W_\eta = \{[x, y, z] \in \mathbb{RP}^2 | x, y, z \geq 0, \ x^{\log(\nu/\mu)} z^{\log(\mu/\lambda)} \geq \eta y^{\log(\nu/\lambda)}\}$$

for each $\eta > 0$. If R is the principal reflection for A, then $W_\eta \cup R(W_\eta)$ is a closed convex neighborhood of a principal segment for A which is invariant under the one-parameter subgroup containing A.

Consider the one-parameter subgroup of $\mathscr{A}$ comprised of elements

$$(1\text{-}10) \qquad B^s = \begin{bmatrix} e^{-s} & 0 & 0 \\ 0 & e^{2s} & 0 \\ 0 & 0 & e^{-s} \end{bmatrix}$$

for $s \in \mathbb{R}$. Then B^s commutes with A. The orbits of the one-parameter subgroup $\{B^s : s \in \mathbb{R}\}$ are line segments joining $\mathrm{Fix}_0(A)$ to the principal line of A. Furthermore B^s maps the convex set W_η to $W_{\eta'}$ where

$$n' = (\nu/\lambda)^{-3s}\eta.$$

In particular the convex sets W_η (for $\eta > 0$) are all projectively equivalent.

The invariant $\ell(A)$ defined in 1.8 may now be interpreted geometrically as follows. Consider a principal segment σ for A and choose $x \in \sigma$. Then the cross-ratio of the four points

$$\mathrm{Fix}\_(A), x, A(x), \mathrm{Fix}_+(A)$$

on the principal line l for A equals $e^{\ell(A)}$. If Ω is any $\langle A \rangle$-invariant convex domain, then [4], [5], [25]–[27], [39] for definition of the Hilbert metric on the convex domain in $\mathbb{RP}^2$). If $\partial\Omega$ is a conic, then the Hilbert metric is the hyperbolic metric, and $\ell(A)$ equals the geodesic length displacement function discussed in [15].

The action of $\mathbf{SL}(3, \mathbb{R})$ by conjugation.

1.12. For later use we prove the following results concerning the action of $\mathbf{SL}(3, \mathbb{R})$ on representations. For related material, see [22], [14].

Lemma. *Let $G = \mathbf{SL}(3, \mathbb{R})$ and $m > 1$. Let $\mathscr{U} \subset G^m$ denote the open set consisting of all $(X_1, \cdots, X_m)$ such that no line in $\mathbb{R}^3$ is simultaneously* *invariant under* $X_1, \cdots, X_m$. *Then the action of* G *on* $\mathscr{U}$ *by conjugation is proper and free.*

Proof. We first show that G acts properly on $\mathscr{U}$. Let $A, B \subset \mathscr{U}$ be compact; we must show that

$$G(A, B) = \{g \in G | gA \cap B \neq \emptyset\}$$

is compact. Thus we assume a sequence $g^{(1)}, \cdots, g^{(n)}, \cdots \in G$ satisfies $g^{(n)}A \cap B \neq \emptyset$, and we must prove that the $g^{(n)}$ remain bounded in G. Since $G(A, B)$ is necessarily closed, it suffices to replace A, B by compact sets $A' \supset A$, $B' \supset B$ and prove that $G(A', B')$ is compact, whenever convenient. Let $K = \mathbf{SO}(3) \subset \mathbf{SL}(3, \mathbb{R})$; thus we may assume that A, B are each K-invariant compact subsets of $\mathscr{U}$.

Let $\mathscr{A} \subset G$ denote the subgroup of diagonal matrices in G; since $G = K\mathscr{A}K$ and A and B are each K-invariant, it suffices to consider the case where $g^{(n)} \in \mathscr{A}$. Write

$$g^{(n)} = \begin{bmatrix} \lambda_1^{(n)} & 0 & 0 \\ 0 & \lambda_2^{(n)} & 0 \\ 0 & 0 & \lambda_3^{(n)} \end{bmatrix}.$$

Assume that the sequence $g^{(n)}$ is unbounded in $\mathscr{A} \subset G$. Then by conjugation we are led to consider two cases:

(i) $|\lambda_k^{(n)}| \to \infty$ for $k = 1, 2$ and $|\lambda_3^{(n)}|$ remains bounded;

(ii) $|\lambda_3^{(n)}| \to \infty$ and $|\lambda_k^{(n)}|$ remains bounded for $k = 1, 2$.

Since A and B are compact, there exists $M > 0$ such that if $(X_1, \cdots, X_m) \in A \cup B$, then

$$|(X_l)_{ij}| \leq M \tag{1-11}$$

for $l = 1, \cdots, m$. Let $G_3 = \{X \in \mathbf{SL}(3, \mathbb{R}) | X_{13} = X_{23} = 0\}$ be the stabilizer of the third coordinate line in $\mathbb{R}^3$. Since $A \cup B$ is compact in $\mathscr{U} \subset G^m - (G_3)^m$, we may further assume that

$$\sum_{l=1}^{m} |(X_l)_{13}| + |(X_l)_{23}| \geq M^{-1}. \tag{1-12}$$

If $X \in \mathbf{SL}(3, \mathbb{R})$, then the entries of $g^{(n)}X$ are given by

$$(g^{(n)}X)_{ij} = \frac{\lambda_i^{(n)}}{\lambda_j^{(n)}} X_{ij}.$$

In case (i) above,

$$|(g^{(n)}X_l)_{i3}| = \left|\frac{\lambda_i^{(n)}}{\lambda_3^{(n)}}(X_l)_{i3}\right| \to \infty,$$

and (1-11) implies that $(X_l)_{13} = (X_l)_{23} = 0$ for $l = 1, \cdots, m$ contradicting $X_l \notin G_3$. In case (ii) above,

$$|(g^{(n)}X_l)_{i3}| = \left|\frac{\lambda_i^{(n)}}{\lambda_3^{(n)}}(X_l)_{i3}\right| \to 0,$$

contradicting (1-12). The proof that G acts properly on $\mathscr{U}$ is complete.

We prove that G acts freely on $\mathscr{U}$. Suppose that $1 \neq g \in G$ stabilizes $X \in \mathscr{U}$. Then there exists a line $l \subset \mathbb{R}^3$ which is invariant under the full centralizer of g (for example, take the line spanned by the eigenvector for the smallest or largest real eigenvalue of g). Since X_l centralizes g, it follows that l is stabilized by X_l for $l = 1, \cdots, m$ contradicting $X \in \mathscr{U}$. Hence the proof of Lemma 1.12 is complete.

2. Real projective structures and their developing maps

2.1. In this section we describe basic general properties of real projective structures and describe several specific classes of such structures which are needed to build real projective structures on surfaces. Let $\Omega \subset \mathbb{RP}^2$ be an open set. A map $\phi: \Omega \to \mathbb{RP}^2$ is said to be *locally projective* if for each connected component $W \subset \Omega$, there is a projective transformation $g: \mathbb{RP}^2 \to \mathbb{RP}^2$ such that the restriction $\phi|_W$ equals the restriction $g|_W$. Obviously a locally projective map is a local diffeomorphism. Let M denote a connected surface, i.e., a smooth 2-dimensional manifold. An $\mathbb{RP}^2$-*atlas* on M is given by an open cover $\mathscr{U}$ of M, and a collection of coordinate charts $\{\psi_U: U \to \mathbb{RP}^2\}_{U \in \mathrm{U}}$ satisfying the following:

(i) Each ψ_U is a diffeomorphism $U \to \psi_U(U)$;

(ii) For each $U, V \in \mathscr{U}$, the change of coordinates $\psi_V \circ \psi_U^{-1}$: $\psi_U(U \cap V) \to \psi_V(U \cap V)$ is locally projective.

A *real projective structure* (or $\mathbb{RP}^2$-*structure*) on M is by definition a maximal $\mathbb{RP}^2$-atlas on M. A manifold with an $\mathbb{RP}^2$-structure will be called an $\mathbb{RP}^2$-*manifold*.

Let M, N be $\mathbb{RP}^2$-manifolds and let $f: M \to N$ be a smooth map. Then f is a *projective map* (or an $\mathbb{RP}^2$-*map*) if for each coordinate chart (U, ψ_U) on M and each coordinate chart (V, ψ_V) on N, the composition $\psi_V^{-1} \circ f \circ \psi_U$ is a locally projective map $\psi_U(f^{-1}(V) \cap U) \to$

$\psi_V(f(U) \cap V)$. An $\mathbb{RP}^2$-map between $\mathbb{RP}^2$-manifolds is necessarily a local diffeomorphism. Conversely, if $f: M \to N$ is a smooth map which is a local diffeomorphism, and N is an $\mathbb{RP}^2$-manifold, then there exists a unique $\mathbb{RP}^2$-structure on M such that f is an $\mathbb{RP}^2$-map with respect to these structures. In particular an $\mathbb{RP}^2$-structure on M induces one on every covering space of M.

The following basic theorem is well known.

2.2. Development Theorem. *Let M be an $\mathbb{RP}^2$-manifold.*

(i) *Let $\mathbf{p}: \tilde{M} \to M$ denote a universal covering space of M and let π denote the corresponding group of covering transformations. Then there exist a projective map $\mathbf{dev}: \tilde{M} \to \mathbb{RP}^2$ and a homomorphism $h: \pi \to \mathbf{SL}(3, \mathbb{R})$ such that for each $\gamma \in \pi$ the diagram*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & \mathbb{RP}^2 \\ \gamma \downarrow & & \downarrow h(\gamma) \\ \tilde{M} & \xrightarrow[\mathbf{dev}]{} & \mathbb{RP}^2 \end{array}$$

commutes.

(ii) *Suppose that $(\mathbf{dev}', h')$ is another such pair. Then there exists a projective transformation $g \in \mathbf{SL}(3, \mathbb{R})$ such that $\mathbf{dev}' = g \circ \mathbf{dev}$ and $h' = \iota_g \circ h$ where $\iota_g: \mathbf{SL}(3, \mathbb{R}) \to \mathbf{SL}(3, \mathbb{R})$ denotes the inner automorphism defined by g.*

The projective map $\mathbf{dev}: \tilde{M} \to \mathbb{RP}^2$ is called a *developing map* and the homomorphism $h: \pi \to \mathbf{SL}(3, \mathbb{R})$ is called the *holonomy homomorphism.* We shall refer to a pair $(\mathbf{dev}, h)$ as a *development pair.* Once a universal covering $\tilde{M} \to M$ has been fixed, the developing map determines the $\mathbb{RP}^2$-structure on M uniquely. The image $h(\pi)$ is called the *holonomy group* and will be denoted Γ. The *developing image* $\mathbf{dev}(\tilde{M})$ is a Γ-invariant open subset of $\mathbb{RP}^2$. In many cases (such as the $\mathbb{RP}^2$-structures discussed in this paper) the developing map is a diffeomorphism from the universal covering of M onto its image. Then the holonomy homomorphism h will be an isomorphism of $\pi_1(M)$ onto a discrete subgroup of $\mathbf{SL}(3, \mathbb{R})$ which acts properly and freely on the developing image. For examples where the developing map is not injective the reader is referred to [13], [16], [20], [36], [37].

Let M be an $\mathbb{RP}^2$-manifold. We denote by $\mathbf{Proj}(M)$ the group of all projective automorphisms of M. Fix a development pair $(\mathbf{dev}, h)$ and let $g \in \mathbf{Proj}(M)$ be an automorphism. Then there exists a lift $\tilde{g}: \tilde{M} \to \tilde{M}$ which is an automorphism of $\tilde{M}$. Any two lifts differ by a covering transformation of $\tilde{M}$, and the group of covering transformations is a normal subgroup $\pi \triangleleft \mathbf{Proj}(\tilde{M})$. The automorphism group $\mathbf{Proj}(M)$ is isomorphic to the quotient $\mathbf{Proj}(\tilde{M})/\pi$.

Let $\tilde{g} \in \mathbf{Proj}(\tilde{M})$. Then there exists a projective transformation $H(\tilde{g}) \in \mathbf{SL}(3, \mathbb{R})$ such that the diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & \mathbb{RP}^2 \\ {\scriptstyle \tilde{g}} \downarrow & & \downarrow {\scriptstyle H(\tilde{g})} \\ \tilde{M} & \xrightarrow[\mathbf{dev}]{} & \mathbb{RP}^2 \end{array}$$

commutes. Clearly $H: \mathbf{Proj}(\tilde{M}) \to \mathbf{SL}(3, \mathbb{R})$ is a homomorphism whose kernel is the (discrete) group consisting of all diffeomorphisms $f: \tilde{M} \to \tilde{M}$ such that

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & \mathbb{RP}^2 \\ {\scriptstyle f} \downarrow & & \| \\ \tilde{M} & \xrightarrow[\mathbf{dev}]{} & \mathbb{RP}^2 \end{array}$$

commutes. The image of H lies in the subgroup $\mathbf{Proj}(\mathbf{dev}(\tilde{M}))$ of $\mathbf{SL}(3, \mathbb{R})$ stabilizing the developing image $\mathbf{dev}(\tilde{M})$. If $\mathbf{dev}$ is a covering map $\tilde{M} \to \mathbf{dev}(\tilde{M})$, then $H(\mathbf{Proj}(\tilde{M}))$ actually equals the stabilizer $\mathbf{Proj}(\mathbf{dev}(\tilde{M}))$.

A *geodesic* on M is a curve $g \subset M$ such that for each component $\tilde{g}_0 \subset \mathbf{p}^{-1}(g) \subset \tilde{M}$, the developing map takes $\tilde{g}_0$ into a line in $\mathbf{RP}^2$. We shall say that a geodesic g is a *simple closed geodesic* if it is an embedded closed 1-dimensional submanifold of M.

Deformation spaces.

2.3. A developing map is uniquely determined by its restriction to any nonempty open set. Let M be a connected $\mathbf{RP}^2$-manifold and let $x \in M$. A *projective chart at* x is any projective map from a neighborhood of x into $\mathbb{RP}^2$. We then define a *projective germ at* x in the usual way as an equivalence class of projective charts at x. Thus for any $x \in M$ the projective germ at x determines a unique developing map, and the group $\mathbf{SL}(3, \mathbb{R})$ acts simply transitively on the set of projective germs at x. We can use this to construct a *deformation space* of $\mathbb{RP}^2$-structures on a fixed surface as follows.

Let S be a fixed compact smooth surface. Let $x \in S$ be a base-point, and let $\mathbf{p}: \tilde{S} \to S$ be be a corresponding universal covering space and $\pi = \pi_1(S)$ the corresponding group of covering transformations. Consider triples (M, f, ψ), where M is an $\mathbb{RP}^2$-manifold, f is a diffeomorphism and ψ is a projective germ at $f(x)$. Such a triple is equivalent to a development pair $(\mathbf{dev}, h)$, where $\mathbf{dev}: \tilde{S} \to \mathbb{RP}^2$ is a developing map and $h: \pi \to \mathbf{SL}(3, \mathbb{R})$ is the corresponding holonomy homomorphism. We shall say that two such triples (M_1, f_1, ψ_1) and (M_2, f_2, ψ_2) are *equivalent* if there is an $\mathbb{RP}^2$-isomorphism $\phi: M_1 \to M_2$ such that $\phi \circ f_1 \simeq f_2$ by an isotopy leaving x fixed and $\phi^*(\psi_2) = \psi_1$. Using the C^1 topology on developing maps, we give the set of equivalence classes of such triples a topology $\mathfrak{D}(S)$ which is Hausdorff. There is a canonical $\mathbf{SL}(3, \mathbb{R})$-action on $\mathfrak{D}(S)$ which corresponds to changing the projective germ. The map which associates to such a triple the corresponding holonomy homomorphism defines a $\mathbf{SL}(3, \mathbb{R})$-equivalent continuous map

$$\mathbf{hol}: \mathfrak{D}(S) \to \mathbf{Hom}(\pi, \mathbf{SL}(3, \mathbb{R}))$$

which is a local homeomorphism (see [6], [17], [34], and [38] for further discussion). Since $\mathbf{Hom}(\pi, \mathbf{SL}(3, \mathbb{R}))$ is a real algebraic variety, we may use this local homeomorphism to define the structure of a real analytic space on $\mathfrak{D}(S)$. This structure is clearly preserved by the action of $\mathbf{SL}(3, \mathbb{R})$. We denote the quotient $\mathfrak{D}(S)/\mathbf{SL}(3, \mathbb{R})$ by $\mathbb{RP}^2(S)$.

2.4. Theorem. *Let S be a closed surface with $\chi(S) < 0$. Then $\mathbb{RP}^2(S)$ has the structure of a (Hausdorff) real analytic manifold of dimension $-8\chi(S)$.*

By contrast if S is a torus, then $\mathbb{RP}^2(S)$ is neither Hausdorff nor a manifold. $\mathbb{RP}^2$-structures on a torus are classified in [13], see also [2], [3], [16], [30], [35], [20].

Proof of 2.4. Choose a set of generators $A_1, \cdots, A_n$ for π; then the evaluation map $\mathrm{E}: \mathbf{Hom}(\pi, \mathbf{SL}(3, \mathbb{R})) \to \mathbf{SL}(3, \mathbb{R})^n$ defined by $E(\rho) = (\rho(A_1), \cdots, \rho(A_n))$ is an $\mathbf{SL}(3, \mathbb{R})$-equivariant embedding. Let $\mathscr{U} \subset \mathbf{SL}(3, \mathbb{R})^n$ be the subset defined in 1.12 and let $\mathscr{U}(\pi) = \mathrm{E}^{-1}(\mathscr{U})$ be the set of representations $\pi \to \mathbf{SL}(3, \mathbb{R})$ whose image has no fixed point in $\mathbb{RP}^2$. It follows from [14, 1.2] that $\mathscr{U}(\pi)$ is a manifold of dimension

$$-\dim(\mathbf{SL}(3, \mathbb{R})) \cdot (\chi(S) - 1) = -8\chi(S) + 8.$$

By Lemma 1.12, $\mathbf{SL}(3, \mathbb{R})$ acts properly and freely on $\mathscr{U}(\pi)$; so the quotient $\mathscr{U}(\pi)$ is a real analytic manifold of dimension $-8\chi(S)$. Thus to prove 2.4 it will suffice to show that the space of holonomy representations of $\mathbb{RP}^2$-structures on S lies in $\mathscr{U}(\pi)$. Hence the proof of 2.4 is reduced to the following.

2.5. Lemma. *Let M be a closed $\mathbb{RP}^2$-surface with holonomy group $\Gamma \subset \mathbf{SL}(3, \mathbb{R})$. If $\chi(M) < 0$, then Γ cannot fix a point in $\mathbb{RP}^2$.*

Proof. Suppose that Γ fixes a point $y \in \mathbb{RP}^2$. Let $\mathfrak{F}$ be the (singular) foliation of $\mathbb{RP}^2$ consisting of the pencil of lines through y; the only singular point of $\mathfrak{F}$ is y. Then $\mathbf{dev}^*\mathfrak{F}$ is a foliation of $\tilde{M}$ invariant under the group $\pi_1(M)$ with singularities at $\mathrm{dev}^{-1}(y)$. Thus there is a foliation $\mathfrak{F}_M$ of M such that $\mathbf{p}^*(\mathfrak{F}_M) = \mathbf{dev}^*(\mathfrak{F})$. The singularities of

$\mathfrak{F}_M$ comprise the finite set $\mathbf{p}(\mathbf{dev}^{-1}(y)) \subset M$, and since $\mathbf{p}$ and $\mathbf{dev}$ are local diffeomorphisms, the Poincaré-Hopf index of $\mathfrak{F}_M$ at each singularity equals the index of $\mathfrak{F}$ at y, which equals $+1$. Summing these indices over $\mathbf{p}(\mathbf{dev}^{-1}(y))$ we obtain $\chi(M) = |\mathbf{p}(\mathbf{dev}^{-1}(y))| \geq 0$, a contradiction.

Surfaces with boundary.

2.6. We can extend all of the above definitions to surfaces with boundary as follows. Let M be a surface with boundary. An $\mathbb{RP}^2$-*structure on* M *with geodesic boundary* is defined by a maximal atlas of coordinate charts (U, ψ_U), where (U, ψ_J) is as above when U is disjoint from ∂M, and ψ_U is a diffeomorphism restricting to a diffeomorphism from ∂U to a line in $\mathbb{RP}^2$ and to a diffeomorphism on $\operatorname{int}(U)$ when $U \cap \partial M \neq \varnothing$. Clearly the interior $\operatorname{int}(M)$ is an $\mathbb{RP}^2$-manifold.

Suppose that M is an $\mathbb{RP}^2$-manifold and $b \subset \partial M$ is a boundary component. Let $\gamma \in \pi_1(M)$ be the corresponding deck transformation of $\tilde{M}$, and $\tilde{b} \subset \tilde{M}$ the corresponding $\langle\gamma\rangle$-invariant lift of b. We say that b is *principal* if, for a fixed development pair $(\mathbf{dev}, h)$, the holonomy $h(\gamma) \in \mathbf{Hyp}_+$ and $\mathbf{dev}$ maps $\tilde{b}$ diffeomorphically onto a principal segment for $h(\gamma)$. If S is a smooth surface with boundary, we shall denote the corresponding deformation space of $\mathbb{RP}^2$-structures on S with principal boundary by $\mathbb{RP}^2(S)$.

If M is an $\mathbb{RP}^2$-manifold, and $g \subset M$ is a simple closed geodesic, then there exists an $\mathbb{RP}^2$-manifold with boundary $M|g$ (M "split along g") and an identification map $\iota: M|g \to M$. The interior of $M|g$ is projectively equivalent to the complement $M - g$, and $M|g$ has two "new" boundary components each of which is mapped diffeomorphically onto $g \subset M$.

2.7. Let $A \in \mathbf{Hyp}_+$ be a positive hyperbolic projective transformation. As usual, when we desire explicit coordinates we represent A by the diagonal matrix (1-1). We denote by $\langle A\rangle$ the cyclic group generated by A. Let $l(A)$ denote the principal line for A, and choose a principal segment $\sigma \subset l(A)$. Let Δ_i, $i = 1, 2$, denote the two invariant (open) triangular regions bounded by σ. Then for each $i = 1, 2$, the quotient $M_i(\Delta_i \cup \sigma)/\langle A\rangle$ is an $\mathbb{RP}^2$-manifold with geodesic boundary $\sigma/\langle A\rangle$ and is diffeomorphic to an annulus with a single boundary component. Furthermore the quotient $M = (\Delta_1 \cup \sigma \cup \Delta_2)/\langle A\rangle$ is an open annulus with an $\mathbb{RP}^2$-structure. We shall call M_i a *principal half-annulus with holonomy* $\langle A\rangle$, and M a *principal annulus with holonomy* $\langle A\rangle$. The projective automorphism group $\mathbf{Proj}(M)$ of M acts simply transitively on the complement of the closed geodesic $\sigma_M = \sigma/\langle A\rangle$ and is generated by the images in $\mathbf{Proj}(M)$ of a principal reflection R and the one-parameter groups defined by (1–8) and (1–10). The image in $\mathbf{Proj}(M)$ of the one-parameter group $\{A^s | s \in \mathbb{R}\}$ is a circle group C of rotations on M and acts simply transitively on the closed geodesic σ_M. If $x_0 \in \sigma_M$, then its stabilizer in $\mathbf{Proj}(M)$ is the image of the group $\{R^j B^s | \; j \in \mathbb{Z}/2, s \in \mathbb{R}\}$ which is isomorphic to the multiplicative group of real numbers.

The annulus M is foliated by the C-orbits which are all circles. Moreover the images of the convex sets $W_\eta \cup RW_\eta$ defined in (1–9) are open subsets M_η which define tubular neighborhoods of the geodesic σ_M, called *principal annular neighborhoods.* These tubular neighborhoods are all projectively equivalent submanifolds since the image of B^s maps M_η to $M_{\eta'}$ where $\eta' = (\nu/\lambda)^{3s/2}\eta$. We shall refer to an $\mathbb{RP}^2$-manifold of the form $W_\eta/\langle A\rangle$ as a *principal collar.*

The principal annulus M covers a unique $\mathbb{RP}^2$-structure on a Möbius band defined as follows. The projective transformation B defined by

$$\begin{bmatrix} -\sqrt{\lambda} & 0 & 0 \\ 0 & \sqrt{\mu} & 0 \\ 0 & 0 & -\sqrt{\nu} \end{bmatrix}$$

defines an orientation-reversing involution on M leaving invariant σ_M and interchanges the two half-annuli comprising M. The quotient is then an $\mathbb{RP}^2$-manifold diffeomorphic to a Möbius band which we call a *principal cross-cap with holonomy* $\langle B\rangle$. The quotient $(W_\eta \cup RW_\eta)/\langle B\rangle$ we call a *principal cross-cap neighborhood.* These $\mathbb{RP}^2$-manifolds will be used to build convex $\mathbb{RP}^2$-structures on nonorientable surfaces.

2.8. Lemma. *Let M be an $\mathbb{RP}^2$-manifold with boundary and suppose $b \subset \partial M$ is a compact principal boundary component. Then there exists a principal collar neighborhood $N(g) \subset M$.*

Proof. The essential point here is that b is compact. Let $N'(b) \subset M$ be a collar neighborhood of b in M. We shall find a principal collar $N(b)$ contained in N'. Fix a development pair so that the holonomy $\gamma \in \mathbf{Hyp}_+$ of b is represented by the diagonal matrix (1–1) and that $\mathbf{dev}(\mathrm{int}(\tilde{N}'))$ lies in the triangular region

$$\Delta = \{[x, y, z] \in \mathbb{RP}^2 | x, y, z > 0\}.$$

Let $\phi: \Delta \to \mathbb{R}$ be the γ-invariant function defined by

$$\phi([x, y, z]) = x^{\log(\nu/\mu)} y^{-\log(\nu/\lambda)} z^{\log(\mu/\lambda)}.$$

Let $c = \partial N'(b) - b \subset \partial N'(b)$ and choose a lift $\tilde{c} \subset \widetilde{N'(b)}$. Let $\tilde{c}_0$ be a fundamental interval for the cyclic group $\langle\gamma\rangle$ acting on $\tilde{c}$. Then since $\tilde{c}_0$

is compact and $\mathbf{dev}(\tilde{c}) \subset \Delta$, there exists $\phi_0 > 0$ such that $\phi \circ \mathbf{dev}(x) > \phi_0$ for $x \in \tilde{c}_0$. As ϕ is γ-invariant, it follows that $\phi \circ \mathbf{dev}(x) > \phi_0$ for $x \in \tilde{c}$. Then

$$W = \{[x, y, z] \in \mathbb{RP}^2 | x, y, z > 0, \ f(x, y, z) \le f_0\} \cup \mathbf{dev}(\tilde{b})$$

is a principal $\langle\gamma\rangle$-invariant collar neighborhood of $\mathbf{dev}(\tilde{b})$ inside $\mathbf{dev}(\widetilde{N'(b)})$ which projects to a principal collar neighborhood $N(b)$ of b inside $N'(0)$.

2.9. Suppose that M is an $\mathbb{RP}^2$-manifold and C is a principal boundary component. It follows from 2.7 that any collar neighborhood constructed in the above way is projectively equivalent to any other one; the projective equivalence class depends solely on the $\mathbf{SL}(3, \mathbb{R})$-conjugacy class of the holonomy transformation $h(\gamma) \in \mathbf{Hyp}_+$ of b. Thus the germ of an $\mathbb{RP}^2$-structure near a principal boundary component is determined by the conjugacy invariants (λ, τ) or (ℓ, m) of $H(\gamma) \in \mathbf{Hyp}_+$ discussed in 1.4–1.8. For example, $\ell(h(\gamma))$ is the Hilbert length of the closed geodesic b, measured in any principal tubular neighborhood of b. If C is a closed 1-dimensional manifold, we denote the space of germs of $\mathbb{RP}^2$-structures on principal collar (or tubular) neighborhoods of C by $\mathfrak{P}(C)$. Clearly $\mathfrak{P}(C)$ is a product of open 2-cells, one for each component of C; the coordinates (ℓ, m) define a diffeomorphism

$$\mathfrak{P}(C) \to \prod_{\pi_0(C)} \{(\ell, m) \in \mathbb{R}_+ \times \mathbb{R} | \, |m| < \ell\}.$$

Thus associating to an $\mathbb{RP}^2$-manifold M the germ of its $\mathbb{RP}^2$-structure near a collection B of principal boundary components $b_1, \cdots, b_k$ defines a map

$$\Theta_B : \mathbb{RP}^2(S) \to \mathfrak{P}(\partial S).$$

In a similar way, if $C \subset M$ is a principal two-sided simple closed geodesic, then the preceding discussion applies to the $\mathbb{RP}^2$-manifold $M|C$. There exist principal annular neighborhoods of $C \subset M$, the germ of which is recorded by the invariant $(\ell, m)(C)(M) \in \mathbb{R}_+ \times \mathbb{R} + \mathfrak{P}(C)$. If $C \subset M$ is a principal one-sided simple closed geodesic, then a similar invariant $(\ell, m)(C)(M)$ expresses the germ of a principal cross-cap neighborhood of C in M. In §5 these invariants will be used as coordinates for convex $\mathbb{RP}^2$-structures.

3. Convex $\mathbb{RP}^2$-structures

In this section we shall describe basic properties of convex $\mathbb{RP}^2$-structures on closed surfaces. This important class of projective structures can be characterized in several ways. Recall (1.10) that a domain $\Omega \subset \mathbb{RP}^2$ is *convex* if there exists a projective line $l \subset \mathbb{RP}^2$ such that $\Omega \cap l = \emptyset$ and Ω is a convex subset of the affine plane $\mathbb{RP}^2 - l$. Equivalently $\Omega \subset \mathbb{RP}^2$ is convex if for any two points $x, y \in \Omega$ there is a unique geodesic joining x and y. Another equivalent condition is that there exists an open convex cone $\Omega' \subset \mathbb{R}^3 - \{0\}$ such that $\Omega = \mathbf{P}(\Omega')$ where $\mathbf{P}: \mathbb{R}^3 - \{0\} - \mathbb{RP}^2$ denotes projection. With this definition, $\mathbb{RP}^2$ itself is *not* convex, while the affine plane $\mathbb{R}^2$ is convex.

3.1. Proposition. *Let M be an $\mathbb{RP}^2$-manifold. Then the following are equivalent*:

(1) *Every path in M is homotopic (rel endpoints) to a unique geodesic path*;

(2) *A developing map $\mathbf{dev}: \tilde{M} \to \mathbb{RP}^2$ is a diffeomorphism onto a convex domain in $\mathbb{RP}^2$.*

(3) *M is projectively isomorphic to a quotient Ω/Γ, where $\Omega \subset \mathbb{RP}^2$ is a convex domain and $\Gamma \subset \mathbf{Proj}(\Omega) \subset \mathbf{SL}(3, \mathbb{R})$ is a discrete group acting properly and freely on Ω.*

Proof. (1) $\Rightarrow$ (2). Let $\mathbf{p}: \tilde{M} \to M$ be a universal covering space and $(\mathbf{dev}, h)$ be a development pair. We first show that $\mathbf{dev}$ is injective. Suppose that $\tilde{x}, \tilde{y} \in \tilde{M}$ satisfy $\mathbf{dev}(\tilde{x}) = \mathbf{dev}(\tilde{y})$; let $\tilde{r}$ be a path in $\tilde{M}$ joining $\tilde{x}$ to $\tilde{y}$ and let $r = \mathbf{p} \circ \tilde{r}$. By (1), r is homotopic to a geodesic path r_0 in M joining $x = \mathbf{p}(\tilde{x})$ to $y = \mathbf{p}(\tilde{y})$. Lift r_0 to a geodesic path $\tilde{r}_0$ in $\tilde{M}$ joining $\tilde{x}$ to $\tilde{y}$; then $\mathbf{dev}|_{\tilde{r}_0}$ maps $\tilde{r}_0$ diffeomorphically to the geodesic joining $\mathbf{dev}(\tilde{x})$ and $\mathbf{dev}(\tilde{y})$, which must be a point (a constant geodesic) since $\mathbf{dev}(\tilde{x}) = \mathbf{dev}(\tilde{y})$ and r_0 is unique. It follows that $\tilde{r}_0$ must be a point and hence $\tilde{x} = \tilde{y}$.

Thus $\mathbf{dev}$ maps $\tilde{M}$ bijectively onto a domain $\Omega \subset \mathbb{RP}^2$, which we presently show is convex. Given $u, v \in \Omega$, there exist unique inverse images $\mathbf{dev}^{-1}(u)$, $\mathbf{dev}^{-1}(v) \in \tilde{M}$ which can be joined by a path; by (1) we may assume this path is a geodesic $\tilde{r}$. Then $\mathbf{dev} \circ \tilde{r}$ is a line segment joining u and v, whence Ω is convex.

(2) $\Rightarrow$ (3) Let $\Omega = \mathbf{dev}(\tilde{M})$ and $\Gamma = h(\pi_1(M))$. Then $\mathbf{dev}: \tilde{M} \to \Omega$ is a projective isomorphism which induces an isomorphism $M \to \Omega/\Gamma$.

(3) $\Rightarrow$ (1) Let r be a path in M and lift to a path $\tilde{r}$ in $\tilde{M}$. The path $\mathbf{dev} \circ \tilde{r}$ is a path in the convex domain $\Omega = \mathbf{dev}(\tilde{M})$, which is homotopic (rel endpoints) to a unique line segment $r_0 \subset \Omega$. Composing this homotopy with the inverse of the diffeomorphism $\mathbf{dev}: \tilde{M} \to \Omega$ one obtains a homotopy from r to the geodesic $\mathbf{dev}^{-1}(r_0)$. q.e.d.

Property (1) is the usual condition of *geodesic convexity.* An $\mathbb{RP}^2$-

manifold satisfying any of the above equivalent conditions is said to be *convex*. If $M = \Omega/\Gamma$ is a convex $\mathbb{RP}^2$-manifold, then its universal covering we identify with Ω and its fundamental group with Γ.

The following fundamental facts are due to Kuiper [31], Kac-Vinberg [23] and Benzécri [4] (see Kobayashi [25]–[27] and Vey [39], [40] as well as Goldman [20] for related matter):

3.2. Theorem. *Let $M = \Omega/\Gamma$ be a closed surface with a convex $\mathbb{RP}^2$-structure. Suppose that $\chi(M) < 0$. Then the following hold.*

(1) *$\Omega \subset \mathbb{RP}^2$ is a strictly convex domain with C^1 boundary and therefore contains no affine line.*

(2) *Either $\partial\Omega$ is a conic in $\mathbb{RP}^2$ or is not $C^{1+\varepsilon}$ for some $0 < \varepsilon < 1$.*

(3) *If $\gamma \in \Gamma$ is nontrivial, then $\gamma \in \mathbf{Hyp}_+$. Furthermore every homotopically nontrivial closed curve on M is freely homotopic to a unique closed geodesic which must be principal.*

(4) *The attracting and repelling fixed points of elements of Γ form a dense subset of $\partial\Omega$. Furthermore given any pair $(x, y) \in \partial\Omega \times \partial\Omega$, there exists a sequence $\gamma_n \in \Gamma$ such that $\mathrm{Fix}_+(\gamma_n) \to x$ and $\mathrm{Fix}_-(\gamma_n) \to y$.*

(Examples of such domains are drawn in Figure 3.1.) Let S be a closed surface; define $\mathfrak{P}(S) \subset \mathbb{RP}^2(S)$ to be the subset of $\mathbb{RP}^2(S)$ corresponding to convex $\mathbb{RP}^2$-manifolds.

3.3 Proposition. *$\mathfrak{P}(S)$ is open in $\mathbb{RP}^2(S)$.*

Proof. By [4] (see also [36], [13], [19]) to every $\mathbb{RP}^2$-manifold M there is a naturally associated flat affine manifold $\mathbf{A}(M)$ diffeomorphic to $M \times S^1$. Suppose that S is a closed surface. Then by 3.2(1) an $\mathbb{RP}^2$-manifold M representing a point in $\mathbb{RP}^2(S)$ is convex if and only if a developing map for $\mathbf{A}(M)$ is a diffeomorphism onto a convex cone in $\mathbb{R}^3$ containing no complete straight line. By Koszul [28], [29] (see also Kobayashi [27, 6.22]), the class of such affine structures is open in the deformation space of affine structures on $M \times S^1$. It follows that the space of convex $\mathbb{RP}^2$-structures on S is open in $\mathbb{RP}^2(S)$.

3.4. Proposition. *The restriction of*

$$\mathbf{hol}\colon \mathbb{RP}^2(S) \to \mathbf{Hom}(\pi, \mathbf{SL}(3, \mathbb{R}))/\mathbf{SL}(3, \mathbb{R})$$

to $\mathfrak{P}(S)$ is an embedding of $\mathfrak{P}(S)$ onto a Hausdorff real analytic manifold of dimension $-8\chi(S)$.

Proof. By 3.3, $\mathfrak{P}(S)$ is open in $\mathbb{RP}^2(S)$. By 2.4, $\mathbb{RP}^2(S)$ is a Hausdorff real analytic manifold of dimension $-8\chi(S)$, and **hol** is a local diffeomorphism

$$\mathbb{RP}^2(S) \to \mathbf{Hom}(\pi, \mathbf{SL}(3, \mathbb{R}))/\mathbf{SL}(3, \mathbb{R}).$$

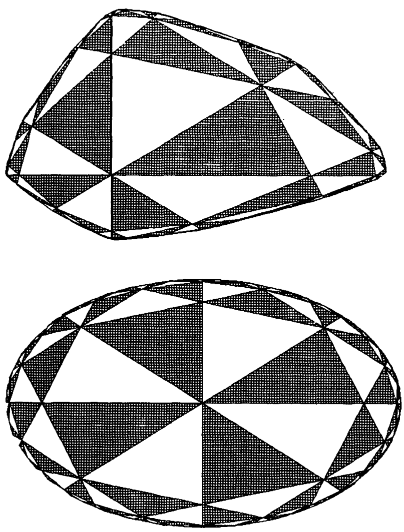

FIGURE 3.1(a)

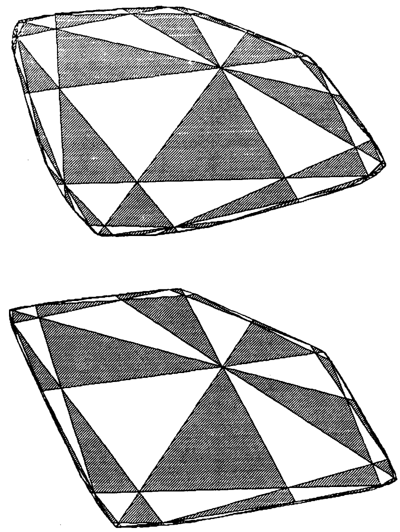

FIGURE 3.1(b)

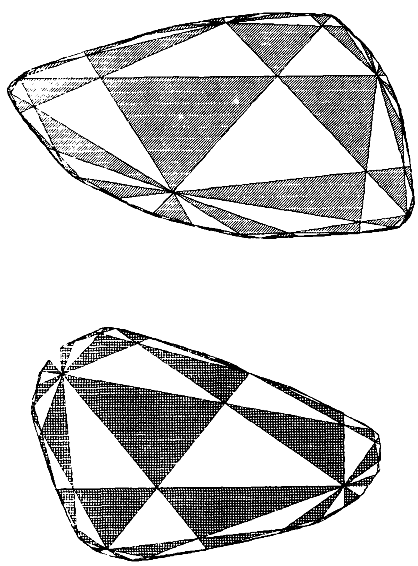

FIGURE 3.1(c)

Thus all that remains to prove 3.4 is that the restriction of **hol** to $\mathfrak{P}(S)$ is injective.

Suppose that M_1 and M_2 are convex $\mathbb{RP}^2$-manifolds such that

$$\mathbf{hol}(M_1) = \mathbf{hol}(M_2) \in \mathbf{Hom}(\pi, \mathbf{SL}(3, \mathbb{R}))/\mathbf{SL}(3, \mathbb{R}).$$

We may assume that $M_i = \Omega_i/\Gamma$ where $\Gamma \subset \mathbf{SL}(3, \mathbb{R})$ is the holonomy group of either structure. Let $\gamma \in \Gamma$. Since $\partial\Omega_1$ is a γ-invariant C^1 convex curve, the two tangents to $\partial\Omega_1$ at the attracting and repelling fixed points of γ intersect in a fixed point of γ which lies outside Ω_1 and thus equals the saddle fixed-point $\mathrm{Fix}_0(\gamma)$. It follows from 3.2(4) that the set $\{\mathrm{Fix}_0(\gamma): \gamma \in \Gamma\}$ is dense in the complement $\mathbb{RP}^2 - \Omega_1$. Therefore if Γ acts properly on an open subset $W \subset \mathbb{RP}^2$, then $W \subset \Omega_1$. (Compare the discussion in Kulkarni [33, 7.1].) Thus $\Omega_2 \subset \Omega_1$, and replacing Ω_1 by Ω_2 gives that $\Omega_1 \subset \Omega_2$, so that $\Omega_1 = \Omega_2$. Hence

$$M_1 \cong \Omega_1/\Gamma = \Omega_2/\Gamma \cong M_2$$

as desired. This completes the proof of 3.4.

Gluing convex $\mathbb{RP}^2$-manifolds.

3.5. Suppose M is a closed convex $\mathbb{RP}^2$-surface Ω/Γ with $\chi(M) < 0$. If $c \subset M$ is a homotopically nontrivial closed curve, there exists a unique closed geodesic $g_c \subset M$ homotopic to c. If $c \subset M$ is simple, then g_c must also be simple. Moreover if $c_1, \cdots, c_n \subset M$ is a disjoint family of simple closed geodesics, then the corresponding simple closed geodesics $g_{c_1}, \cdots, g_{c_n}$ are also disjoint. These facts follow from the usual arguments for hyperbolic surfaces (see [1, exposé 3], [38, 5.3.3]): If $c \subset M$ is a homotopically nontrivial closed curve, then the corresponding deck transformation $h(c) \in \Gamma$ has two fixed points on $\partial\Omega$ and the image in $M = \Omega/\Gamma$ of the line segment in Ω which they span is g_c. The condition that c be simple is that all the components of the inverse image $\mathbf{p}^{-1}(c) \subset \tilde{M}$ be disjoint, implying that for each pair of components $\tilde{c}_1, \tilde{c}_2 \subset \mathbf{p}^{-1}(c)$, the endpoints of $\tilde{c}_1$ in $\partial\Omega$ do not separate the endpoints of $\tilde{c}_2$. But for any closed curve $c \subset M$, the endpoints of a component $\tilde{c} \subset \mathbf{p}^{-1}(c)$ are the endpoints of the corresponding geodesic $\tilde{g}_c$—the condition that the endpoints of $\tilde{c}_i$ in $\partial\Omega$ do not separate each other is equivalent to the condition that g_c be simple. Thus g_c is simple if c is. Similarly, if c_1, c_2 are disjoint simple closed curves in M, then for any pair of lifts $\tilde{c}_1 \subset \mathbf{p}^{-1}(c_1), \tilde{c}_2 \subset \mathbf{p}^{-1}(c_1)$ the endpoints of $\tilde{c}_1$ do not separate the endpoints of $\tilde{c}_2$, i.e., the corresponding geodesics g_{c_1} and g_{c_2} are disjoint.

Thus if c_1 and c_2 are disjoint simple closed curves, then the corresponding simple closed geodesics g_{c_1} and g_{c_2} are disjoint.

3.6. The main result of this section concerns identifying convex $\mathbb{RP}^2$-manifolds along boundary components to obtain convex $\mathbb{RP}^2$-manifolds. We carefully describe the process of gluing surfaces, first for smooth manifolds, and then for $\mathbb{RP}^2$-manifolds.

Let M_0 denote a (not necessarily connected) smooth manifold with a pair of boundary components $b_i \subset \partial M_0$, $i = 1, 2$. We wish to glue M_0 along these boundary components by identifications of b_1 with b_2. That is, we seek a manifold M with a submanifold $b \subset M$ such that the (split) manifold with boundary $M|b$ is diffeomorphic to M_0 and the boundary components of $M|b$ corresponding to b are b_1 and b_2. Let $\iota: M_0 \to M$ be the identification map. Choose a tubular neighborhood $N(b) \subset M$ and an orientation-reversing involution (a "reflection") $\rho: N(b) \to N(b)$ with $\mathrm{Fix}(\rho) = b$; the inverse image $\iota^{-1}(N(b))$ is a disjoint union of collar neighborhoods $N(b_1)$ of b_1 and $N(b_2)$ of b_2 in M_0. Moreover ρ induces a diffeomorphism $f: N(b_1) \to N(b_2)$.

Conversely, suppose M_0 is a manifold with boundary, and $b_1, b_2 \subset \partial M_0$ are boundary components with collar neighborhoods $b_i \subset N(b_i) \subset M_0$ $(i = 1, 2)$. Suppose that $f: N(b_1) \to N(b_2)$ is a diffeomorphism. Then there exists a unique smooth manifold M (denoted M_0/f) with an identification map $\iota: M_0 \to M$ which induces a diffeomorphism $M|b \to M_0$ (where $b = \iota(b_i)$) such that a reflection in a tubular neighborhood of b induces f.

Now let M_0 be an $\mathbb{RP}^2$-manifold with boundary, and suppose that $b_1, b_2 \subset \partial M_0$ are boundary components with collar neighborhoods $b_i \subset N(b_i) \subset M_0$ $(i = 1, 2)$. Suppose that $f: N(b_1) \to N(b_2)$ is a projective isomorphism. Then there exists a unique $\mathbb{RP}^2$-structure on the identification space such that the identification map $\iota: M_0 \to M_0/f$ induces a projective isomorphism $M|b \to M_0$ where a reflection in a tubular neighborhood of b induces f.

In terms of a developing map, this construction may be described as follows. Choose a universal covering $\mathbf{p}: \tilde{M}_0 \to M_0$, a development pair $(\mathbf{dev}_0, h_0)$ for M_0, and lifts $\tilde{b}_i \subset \mathbf{p}^{-1}(b_i)$ for $i = 1, 2$. Let $\gamma_i \in \pi$ be the corresponding elements of the fundamental group. Let $\tilde{b}_i \subset \widetilde{N(b_i)} \subset \tilde{M}$ be the corresponding lifts of the collar neighborhoods; corresponding to the projective isomorphism $f: N(b_1) \to N(b_2)$ is a projective isomorphism $\tilde{f}: \widetilde{N(b_1)} \to \widetilde{N(b_2)}$. Thus there exists a projective transformation $g \in \mathbf{SL}(3, \mathbb{R})$ such that

$$\begin{array}{ccc} \widetilde{N(b_1)} & \xrightarrow{\mathbf{dev}_0} & \mathbb{RP}^2 \\ \tilde{f}\downarrow & & \downarrow g \\ \widetilde{N(b_2)} & \xrightarrow[\mathbf{dev}_0]{} & \mathbb{RP}^2 \end{array}$$

commutes and $g^{-1}h_0(\gamma_1)g = h_0(\gamma_2)$.

Consider the product $\tilde{M}_0 \times \langle\gamma_1\rangle\backslash\pi_1(M_0)/\langle\gamma_2\rangle$ as a collection of "copies" of $\tilde{M}_0$; for each double coset $[\gamma] \in \langle\gamma_1\rangle\backslash\pi_1(M_0)/\langle\gamma_2\rangle$ let $\tilde{M}_0^{[\gamma]} = \tilde{M}_0 \times [\gamma]$ be the corresponding copy of $\tilde{M}_0$, and $j^{[\gamma]}: \tilde{M}_0 \to \tilde{M}_0^{[\gamma]}$ the corresponding diffeomorphism. We build $\tilde{M}$ from copies of $\tilde{M}_0$ as follows. Attach $\tilde{M}_0^{[\gamma]}$ to $\tilde{M}_0 = \tilde{M}_0^{[1]}$ by identifying boundary components $\gamma\tilde{b}_1 \subset \partial\tilde{M}_0$ and $\tilde{b}_2^{[\gamma]} \subset \partial\tilde{M}_0^{[\gamma]}$; the resulting identification space will be a combinatorial neighborhood of $\tilde{M}_0$ inside $\tilde{M}$. Then $\tilde{M}$ is obtained by repeating this process for each new copy of $\tilde{M}_0$ which has been added. The developing map for $\tilde{M}_0$ is extended to an adjacent $\tilde{M}_0^{[\gamma]}$ as the composition

$$R^{[\gamma]} \circ g \circ \mathbf{dev}_0 \circ j^{[\gamma]},$$

where $R^{[\gamma]}$ is the principal reflection for the holonomy transformation $h(\gamma_2) \in \mathbf{Hyp}_+$.

3.7. Theorem. *Let M_0 be a compact convex $\mathbb{RP}^2$-manifold with principal boundary, and suppose that $b_1, b_2 \subset \partial M_0$ are boundary components with collar neighborhoods $b_i \subset N(b_i) \subset M_0$ $(i = 1, 2)$. Suppose that $f: N(b_1) \to N(b_2)$ is a projective isomorphism. Then the $\mathbb{RP}^2$-manifold M_0/f obtained by identification by $f: N(b_1) \to N(b_2)$ is a convex $\mathbb{RP}^2$-manifold.*

The proof of this theorem is based on several lemmas dealing with the structure of collar neighborhoods of principal boundary components in convex $\mathbb{RP}^2$-manifolds.

3.8. Lemma. *Let M_0 be a compact convex $\mathbb{RP}^2$-manifold with a set B of principal geodesic boundary components $b \subset \partial M_0$. Then there exists a convex $\mathbb{RP}^2$-manifold M with a set B' of simple closed geodesics $b' \subset M$ such that $M|B'$ is isomorphic to a disjoint union of M_0 and principal half-annuli, one for each $b \in B$. If m_0 is not a principal half-annulus, then the developing image of M is properly contained in a half-plane.*

We call M the *enlargement* of M_0 along B and denote it $\mathscr{E}(M_0, B)$. If M_0 is a convex $\mathbb{RP}^2$-manifold with principal boundary, the enlargement of M_0 along its full boundary will be called the *enlargement* of M_0 and denoted $\mathscr{E}(M_0)$.

Proof. For each $b \in B$, let $N(b)$ be a principal collar neighborhood of b constructed as above. Choose a generator γ_b for the image β of $\pi_1(N(b))$ in $\pi = \pi_1(M_0)$. Let M_b be the principal half-annulus with convex $\mathbb{RP}^2$-structure such that there exists a projective monomorphism $\imath_b : N(b) \hookrightarrow M_b$. Explicitly, choose a universal covering $\mathbf{p} : \tilde{M}_0 \to M_0$, a development pair $(\mathbf{dev}_0, h)$ for M_0, and a component $\widetilde{N(b)}$ of $\mathbf{p}^{-1}(N(b)) \subset \tilde{M}$. Consider the holonomy transformation $h(\gamma_b)$ around b. Let Δ be a $h(\beta)$-invariant triangular region containing $\mathbf{dev}(\widetilde{N(b)})$, and $R \in \mathbf{SL}(3, \mathbb{R})$ the principal reflection for $h(\gamma_b)$. Then

$$\Omega = \mathbf{dev}_0(\widetilde{N(b)}) \cup R(\Delta)$$

is a β-invariant convex region, and the quotient $M_b = \Omega/\beta$ is the desired annulus. Let M_B denote the disjoint union $\bigcup_{b \in B} M_b$, $N(B)$ the disjoint union $\bigcup_{b \in B} N(b) \subset M_0$, and let $\imath : N(B) \to M_B$ be the map induced by all $\imath_b$, $b \in B$. Let M be the $\mathbb{RP}^2$-manifold obtained by identifying the disjoint union of $M_0 \cup M_B$ via $\imath$.

We must show that M is a convex $\mathbb{RP}^2$-manifold. By induction it suffices to consider the case that B consists of one boundary component b. To show that M is convex, we must show that a developing map for M is a homeomorphism onto a convex set. Let $\mathbf{dev} : \tilde{M} \to \mathbb{RP}^2$ be a developing map for M extending $\mathbf{dev}_0$.

We begin by showing that $\mathbf{dev}$ is injective. Since the natural inclusion $M_0 \to M$ is a homotopy equivalence, there is an embedding of universal covering spaces $\tilde{M}_0 \to \tilde{M}$. Indeed, $\tilde{M}$ is obtained from $\tilde{M}_0$ by attaching to each component of $\mathbf{p}^{-1}(b)$ a copy of the universal covering $\tilde{M}_b$ of the annulus M_b. Choose a component $\tilde{b}$ of $\mathbf{p}^{-1}(b)$. The inverse image $\mathbf{p}^{-1}(b)$ is the disjoint union of $\gamma\tilde{b}$, where γ ranges over a system π/β of coset representatives of β in π.

Since M_0 is convex, its developing map $\mathbf{dev}_0$ is a diffeomorphism of $\tilde{M}_0$ onto a convex domain $\Omega_0 \subset \mathbb{RP}^2$ invariant under the holonomy group $\Gamma \subset \mathbf{SL}(3, \mathbb{R})$. Let σ be the open interval $\mathbf{dev}(\tilde{b})$; then the inverse image $\mathbf{p}^{-1}(b)$ develops to the disjoint union of open intervals

$$\bigcup_{\gamma \in \pi/\beta} h(\gamma)\sigma \subset \partial\Omega_0.$$

Let Δ be as above; then Ω_0 is an $h(\beta)$-invariant convex domain whose boundary contains σ as a maximal line segment, and, it follows that $\Omega_0 \subset \Delta$. The component $\tilde{M}_b$ of $\mathbf{p}^{-1}(M_b)$ which is attached to $\tilde{M}_0$ along $\tilde{b}$

develops diffeomorphically onto $R\Delta$ where R is the principal reflection for $h(\beta)$. Since Δ and $R\Delta$ are disjoint, it follows that the restriction of $\mathbf{dev}\colon \tilde{M} \to \mathbb{RP}^2$ to $\tilde{M}_0 \cup \tilde{M}_b$ is injective. Suppose that $\gamma \in \pi - \beta$; then $h(\gamma)\sigma$ is a line segment inside Δ. Let T denote the triangular region in Δ which is bounded by σ and whose boundary contains the endpoints of $h(\gamma)\sigma$. Since the intersection $T \cap \Omega_0$ is bounded by $h(\gamma)\sigma$ and $\partial\Omega_0$ is an $h(\gamma\gamma_b\gamma^{-1})$-invariant convex curve, the image $h(\gamma)R\Delta \subset T$ (Figure 3.2); the inclusion is proper unless $\Omega_0 \subset T$, in which case Ω_0 is a quadrilateral. In particular $h(\gamma)R\Delta \subset T \subset \Delta$ is disjoint from $R\Delta$. Thus the restriction of $\mathbf{dev}$ to $\tilde{M}_b \cup \tilde{M}_0 \cup \gamma\tilde{M}_b$ is injective. Now it follows easily that $\mathbf{dev}\colon \tilde{M} \to \mathbb{RP}^2$ is injective: if $x \in \gamma_1\tilde{M}_b$ and $y \in \gamma_2\tilde{M}_b$, and $\mathbf{dev}(x) = \mathbf{dev}(y)$, then $\gamma_1^{-1}x \in \tilde{M}_b$ and $\gamma_2\gamma_1^{-1}y \in \gamma_2 M_b$ have the same developing image, whence $x = y$ by the preceding fact.

Let $\Omega = \mathbf{dev}(\tilde{M})$. We have just seen that $\Omega = \Omega_0 \cup \bigcup_{\gamma\in\pi\beta} h(\gamma)R\Delta$. The boundary $\partial\Omega$ is obtained by replacing each segment $h(\gamma)\sigma$ by the union of the remaining sides of $h(\gamma)R\Delta$. Clearly Ω is locally convex at each point of $\partial\Omega$; it follows that Ω is a convex domain.

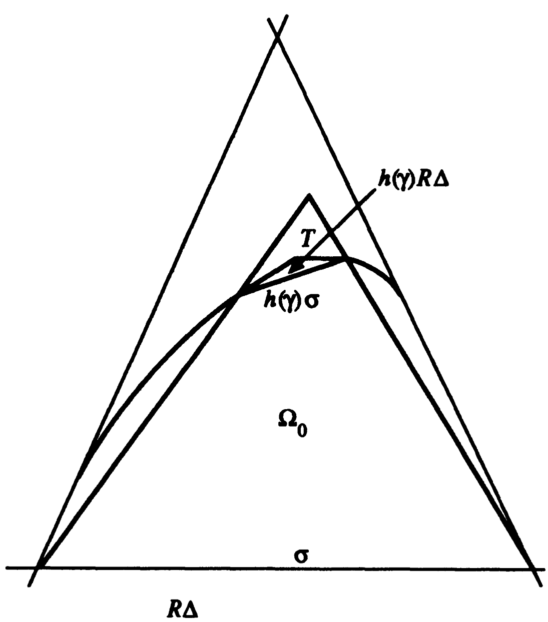

FIGURE 3.2

Finally we show that Ω is properly contained in a half-plane. We have already seen that $\Omega \subset \Delta \cup \sigma \cup R\Delta$. Suppose that Ω equals the half-plane $\Delta \cup \sigma \cup R\Delta$. Unless $\pi_1(b) = \pi_1(M_0)$, there will exist a line segment $h(\gamma)\sigma \subset \partial\Omega_0 \cap \Delta$. Then either Ω is properly contained in $\Delta \cup \sigma \cup R\Delta$ or (using the above notation) $h(\gamma)R\Delta = T$, in which case Ω_0 must be a quadrilateral. But this implies that

$$\Omega = T \cup R\Delta \neq \Delta \cup \sigma \cup R\Delta.$$

Therefore $\beta \hookrightarrow \pi$ is an isomorphism whence M_0 is an annulus. If Ω equals $\Delta \cup \sigma \cup R\Delta$, then Ω_0 equals $\Delta \cup \sigma$ and M_0 is a principal annulus. The proof of Lemma 3.8 is now complete.

Proof of Theorem 3.7. It suffices to consider the case that M is connected, whence M_0 has either one or two connected components. We discuss these cases separately, starting with the case that M_0 has two components M_1 and M_2. In that case $M = M_1 \cup_f M_2$, the fundamental group $\pi = \pi_1(M)$ is the amalgamated free product $\pi_1(M_1) \coprod_{\pi_1(b)} \pi_1(M_2)$, and the map $i_*: \pi_1(M_1) \to \pi_1(M)$ induced by the inclusion $i: M_1 \hookrightarrow M$ is injective. Let $\hat{M}$ denote the covering space of M with fundamental group $i_*\pi_1(M_1) \subset \pi$, and let $\widehat{M_2}$ denote the covering space of M_2 having a fundamental group the cyclic group $\beta = i_*\pi_1(b) \subset \pi_1(M_2)$; there is a compact boundary component $\hat{b}_2 \subset \partial\widehat{M_2}$ corresponding to b. Let $N(b) \subset M_1$ be a principal collar of b in M_1. There is a projective monomorphism $f: N(b) \to M_2$, which lifts to a projective monomorphism $\hat{f}: N(\hat{b}_2) \hookrightarrow \widehat{M_2}$. Then $\hat{M}$ is projectively equivalent to the union $M_1 \cup_{\hat{f}} \widehat{M_2}$. Since M_2 is convex with principal boundary, it follows that the image of its developing map $\mathbf{dev}_2: \tilde{M}_2 \to \mathbb{RP}^2$ lies in a triangular region Δ bounded by the developing image σ of a lift of b. The composition

$$\tilde{M}_2 \xrightarrow{\mathbf{dev}_2} \mathbf{dev}_2(\tilde{M}_2) \hookrightarrow \Delta \cup \sigma$$

induces a projective monomorphism $\widehat{M_2} \to (\Delta \cup \sigma)/h(\beta)$. Thus there is a projective monomorphism $\hat{M} = M_1 \cup_{\hat{f}} \widehat{M_2} \to \mathscr{E}(M_1)$. Since $\hat{M}$ embeds into the enlargement $\mathscr{E}(M_1)$ and the developing map for $\mathscr{E}(M_1)$ is injective, the developing map for $\hat{M}$ (and hence for M) is injective.

We claim that the developing image $\mathbf{dev}(\tilde{M})$ is convex. To this end we show that any path in $\hat{M}$ is homotopic to a geodesic path. Let r be a path in $\hat{M}$; since $\hat{M} \hookrightarrow \mathscr{E}(M_1)$ and $\mathscr{E}(M_1)$ is convex, the path r is homotopic (rel endpoints) to a geodesic path r_0 to $\mathscr{E}(M_1)$. Since $\hat{M} \hookrightarrow \mathscr{E}(M_1)$ is a homotopy equivalence, it will suffice to show that $r_0 \subset \hat{M}$.

If both endpoints of r lie in $M_1 \subset \hat{M}$, then r is homotopic to a path lying entirely in M_1, and to a geodesic path in M_1 since M_1 is convex. By uniqueness, $r_0 \subset M_1 \subset \hat{M}$. Let r_1 be a component of $r_0 \cap M_b$; it will suffice to show that $r_1 \subset \widehat{M}_2$. If $r_1 \subset \operatorname{int}(r_0)$, then both endpoints of r_1 lie on $\hat{b}$, so by uniqueness $r_1 \subset \hat{b} \subset M_1$. Since $r_0 \supset r_1$ is a geodesic, $r_0 \subset \hat{b} \subset M_1 \subset \hat{M}$. Thus r_1 must contain an endpoint of r_0. If r_1 contains both endpoints of r_0, then $r_1 = r_0 \subset M_b$ has both endpoints lying in $\widehat{M}_2$. Since M_2 is convex, it follows that $r_0 = r_1 \subset \widehat{M}_2 \subset \widehat{M}$. It remains to consider the case that one endpoint of r_1 lies in M_b. Since $\widehat{M}_2 \hookrightarrow M_b$ is a homotopy equivalence, r_1 is homotopic (rel endpoints) to a geodesic $r_2 \subset \widehat{M}_2$. By the uniqueness of homotopic geodesic paths, we must have $r_1 = r_2 \subset \widehat{M}_2$. It follows that $\hat{M}$ is convex.

In the case that M_0 is connected, $\pi_1(M)$ is obtained by an HNN-construction on $\pi_1(M_0)$ where the inclusions $\pi_1(b_i) \to \pi_1(M_0)$ are identified. In particular the homomorphism $i_*: \pi_1(M_0) \to \pi_1(M)$ induced by inclusion $i: M_0 \to M$ is injective. Let $\hat{M}$ denote the covering space of M having fundamental group $i_*\pi_1(M_0)$, and $\widehat{M}_j$ $(j = 1, 2)$ the covering space of M_0 having fundamental group $i_*\pi_1(b_j) \subset \pi_1(M_0)$. Then $\hat{M}$ can be identified with the union $\widehat{M}_1 \cup_{b_1} M_0 \cup_{b_2} \widehat{M}_2$ which embeds in the enlargement of M_0. The rest of the proof proceeds as before. This concludes the proof of Theorem 3.6.

Attaching cross-caps.

3.9. If M is a convex $\mathbb{RP}^2$-manifold, and $C \subset M$ is an orientation-reversing simple closed curve, then there exists a tubular neighborhood of C which is a principal cross-cap neighborhood as defined in 2.7. Let $f: \hat{M} \to M$ denote the oriented double covering of M. Since $\hat{M}$ is convex, $\hat{c} = f^{-1}(c)$ has a principal annular neighborhood $N(\hat{c})$ invariant under the deck transformation of $\hat{M}$ and the image $f(N(\hat{c})) \subset M$ is a principal cross-cap neighborhood $N(c)$ of c.

Conversely suppose that M_0 is an $\mathbb{RP}^2$-manifold, and $b \subset \partial M_0$ is a principal boundary component. In the enlargement $\mathscr{E}(M_0, b)$ let $N(b)$ be a principal annular neighborhood of b; then there exists a unique orientation-reversing free involution J of $N(b)$ such that $N(b)/J$ is a principal cross-cap neighborhood. Let $\zeta: N(b) \to N(b)/J$ be the quotient map. The resulting identification space $M = \mathscr{E}(M_0, b) \cup_\zeta N(b)/J$ is an $\mathbb{RP}^2$-manifold with an orientation-reversing simple closed geodesic β such that $M|\beta \cong M_0$; indeed M is diffeomorphic to M_0 with a cross-cap attached along b.

Doubling convex $\mathbb{RP}^2$-manifolds.

3.10. Theorem 3.6 can be used to construct doubles of convex $\mathbb{RP}^2$-manifolds with boundary. This construction is useful for deducing results about convex $\mathbb{RP}^2$-structures on manifolds with boundary from corresponding results on closed manifolds. Let M be a closed convex $\mathbb{RP}^2$-manifold with principal geodesic boundary. Consider the product $M \times \{1, 2\}$. For each component $C \subset \partial M$ let $N(C)$ be a principal collar. The double $2M$ of M is by definition the $\mathbb{RP}^2$-manifold obtained by identifying $M \times \{1\}$ with $M \times \{2\}$ by means of projective isomorphisms $N(C) \times \{1\} \to N(C) \times \{2\}$ as above. Thus the double of M has a natural $\mathbb{RP}^2$-structure. An immediate consequence of 3.7 is the following.

Corollary. *Let M be a compact convex $\mathbb{RP}^2$-manifold with principal geodesic boundary. Then the natural $\mathbb{RP}^2$-structure on the double $2M$ of M is convex. In particular every compact convex $\mathbb{RP}^2$-manifold with principal geodesic boundary embeds in a closed convex $\mathbb{RP}^2$-manifold.*

We may apply this doubling procedure to deformation spaces as follows. Recall that if S is a surface with boundary, then $\mathbb{RP}^2(S)$ is the deformation space of $\mathbb{RP}^2$-structures on S with principal boundary, and $\mathfrak{P}(S)$ is the deformation space of convex $\mathbb{RP}^2$-structures on S with principal boundary.

3.11. Corollary. *Let S be a compact surface with boundary such that $\chi(S) < 0$. Then $\mathbb{RP}^2(S)$ is a Hausdorff real analytic manifold of dimension $-8\chi(S)$, and $\mathfrak{P}(S)$ is an open subset of $\mathbb{RP}^2(S)$.*

Proof. Let $\Gamma \subset \mathbf{SL}(3, \mathbb{R})$ be the holonomy group of an $\mathbb{RP}^2$-manifold M representing a point in $\mathbb{RP}^2(S)$. Let Γ' be the subgroup of $\mathbf{SL}(3, \mathbb{R})$ generated by Γ and the principal reflections R_i for the holonomy of the boundary components b_i of M; the holonomy group for the double $2M$ lies in Γ'.

We claim that Γ fixes no point $p \in \mathbb{RP}^2$. Suppose not; then since $p \in \mathbb{RP}^2$ is stationary under the holonomy of a boundary component b_i, it is also stationary under R_i. Thus Γ' fixes p, and hence the holonomy group of $2M$ fixes p, contradicting 2.5. It follows that the holonomy homomorphism of M lies in the open subset $\mathscr{U}(\pi) \subset \mathbf{Hom}(\pi, \mathbf{SL}(3, \mathbb{R}))$ comprising representations with no stationary point in $\mathbb{RP}^2$, as in 2.4. Since π is a free group of rank $1 - \chi(S)$, the dimension of $\mathbf{Hom}(\pi, \mathbf{SL}(3, \mathbb{R})) \approx \mathbf{SL}(3, \mathbb{R})^{1-\chi(S)}$ equals $8(1 - \chi(S))$. Applying 1.12 as in the proof of 2.4 we see that $\mathbb{RP}^2(S)$ is a Hausdorff real analytic manifold of dimension $-8\chi(S)$, and by 3.4 the deformation space $\mathfrak{P}(S)$ is open in $\mathbb{RP}^2(S)$. This concludes the proof of 3.11.

4. Convex $\mathbb{RP}^2$-structures on a pair-of-pants

4.1. In this section we classify $\mathbb{RP}^2$-structures on a *pair-of-pants*, that is, a compact oriented surface of genus zero with three boundary components. Throughout this section S will denote such a surface, and A, B, C its three boundary components. The main result of this section is the following. Recall (§1.5) that the set of conjugacy classes of positive hyperbolic projective transformations can be parametrized by an open 2-disk $\mathfrak{R}$; a closed curve γ in a convex $\mathbb{RP}^2$-manifold has well defined invariant $((\lambda, \gamma)_\gamma = (\lambda(\gamma), \tau(\gamma)) \in \mathfrak{R}$. The main result of this section is the following.

Theorem. *The deformation space $\mathfrak{P}(S)$ of convex $\mathbb{RP}^2$-structures on S is an open 8-dimension cell. Furthermore the map*

$$\Theta_{\partial S}: \mathfrak{P}(S) \to \mathfrak{R}^3$$

obtained by associating to a convex structure the boundary invariants

$$((\lambda, \tau)_A, (\lambda, \tau)_B, (\lambda, \tau)_C)$$

is a fibration over an open 6-cell with fiber a 2-dimensional open cell.

(Equivalently we could use the more geometric invariants (ℓ, m); they are easily related to (λ, τ) by (1–6).)

S. Choi has pointed out that there exist convex $\mathbb{RP}^2$-structures on pairs-of-pants such that the holonomy of a boundary component has real repeated eigenvalues (although not diagonalizable); such structures cannot be embedded, however, in closed $\mathbb{RP}^2$-manifolds.

The proof centers on a computation. We shall show that a convex $\mathbb{RP}^2$-structure on S determines a geometric configuration (equivalent to its developing map) for which the holonomy representation will be given by a triple of 3×3 matrices satisfying certain equations and inequalities. Given the boundary invariants (a point in $\mathfrak{R}^3$) we shall solve the resulting equations to explicitly parametrize the possible solutions by points in a 2-cell. We shall denote the space of solutions by $\mathscr{Q}$. What is not immediately obvious is that a point of $\mathscr{Q}$ (which represents a $\mathbb{RP}^2$-manifold M with the correct behavior at the boundary) determines a convex structure. This shall be demonstrated indirectly as follows. The double $2S$ of S admits a natural $\mathbb{RP}^2$-structure, which by Corollary 3.10 is convex if M is convex. By applying Proposition 3.3 to $2S$ the convex structures on $2S$ form an open subset $\mathfrak{P}(S) \subset \mathscr{Q}$. Furthermore this subset is nonempty since it contains points corresponding to hyperbolic structures on S. Since $\mathscr{Q}$ is connected, it suffices to show that $\mathfrak{P}(S)$ is closed in $\mathscr{Q}$. To this end we consider a limit in $\mathscr{Q}$ of a sequence of structures in $\mathfrak{P}(S)$. Since an injective developing map determines a closed subset in the deformation space of geometric structures, the developing map of the limiting structure must be injective. By the explicit description of the developing map it will be shown that the developing image is indeed convex.

4.2. We establish the following conventions concerning the fundamental group and the universal covering space of S. Choose a basepoint $s_0 \in S$, and let $\mathbf{p}: \tilde{S} \to S$ be the corresponding universal covering space with covering group $\pi = \pi_1(S)$. Then π is generated by three loops (denoted $A, B, C \in \pi$) obtained by joining s_0 to the three boundary components $A, B, C \subset \partial S$, and there is a corresponding presentation

$$\pi = \langle A, B, C | CBA = I \rangle.$$

We shall decompose S into two open 2-simplices T_+, T_- along three simple arcs a, b, c. The curve a will spiral in the positive direction towards boundary components B and C; similarly b (resp. c) will spiral positively towards C and A (resp. A and B). (Compare Figure 4.1.) Choose a component T_0 of $\mathbf{p}^{-1}(T_+) \subset \tilde{S}$. Inside $\tilde{S}$ there are lifts T_a (resp. T_b, T_c) of T_- which are adjacent to $\tilde{T}_-$ along $\tilde{a}$ (resp. $\tilde{b}, \tilde{c}$). The deck transformation A maps T_b to T_c, the deck transformation B maps T_c to T_a and the deck transformation C maps T_a to T_b. (Compare Figure 4.2.)

Let $F = \overline{T}_0 \cup \overline{T}_c \subset \tilde{S}$, where $\overline{T}_0$ and $\overline{T}_c$ denote closed 2-simplices with their vertices removed; then F is a fundamental domain for π acting on $\tilde{S}$. Thus F is (combinatorially) a quadrilateral minus its vertices and we label its four edges: there are edges $e_a, e_b, e_{\overline{a}}, e_{\overline{b}}$, where e_a, e_b are the edges corresponding to a, b in $T_0 \subset F$, and $e_{\overline{a}}, e_{\overline{b}}$ are the edges corresponding to a, b in $T_c \subset F$. A model for the universal covering is given by the identification space of the disjoint union

$$\coprod_{\gamma \in \pi} \gamma F,$$

where the edges of γF are identified by the following instructions (compare Figure 4.3):

$$\begin{aligned}
\gamma e_a &\leftrightarrow \gamma B e_{\overline{a}} \subset \gamma B F, \\
\gamma e_b &\leftrightarrow \gamma A^{-1} e_{\overline{b}} \subset \gamma A^{-1} F, \\
\gamma e_{\overline{a}} &\leftrightarrow \gamma B^{-1} e_a \subset \gamma B^{-1} F, \\
\gamma e_{\overline{b}} &\leftrightarrow \gamma A e_b \subset \gamma A F.
\end{aligned}$$

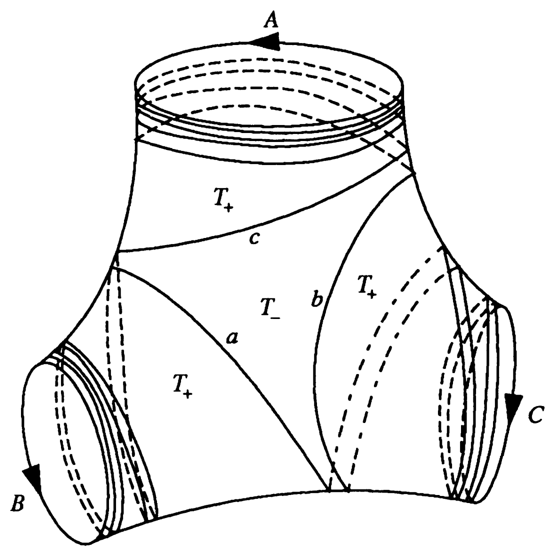

FIGURE 4.1

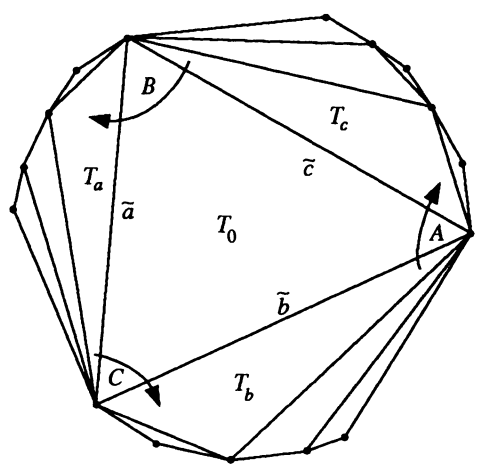

FIGURE 4.2

A model developing map is obtained by mapping T_0 diffeomorphically onto Δ_0 (where the edges a, b, c are mapped to the corresponding edges of Δ_0) and T_c diffeomorphically onto Δ_c. This map $F \to \mathbb{RP}^2$ extends uniquely to a developing map $\tilde{S} \to \mathbb{RP}^2$ which is equivariant with respect to $h{:} \to \mathbf{SL}(3, \mathbb{R})$.

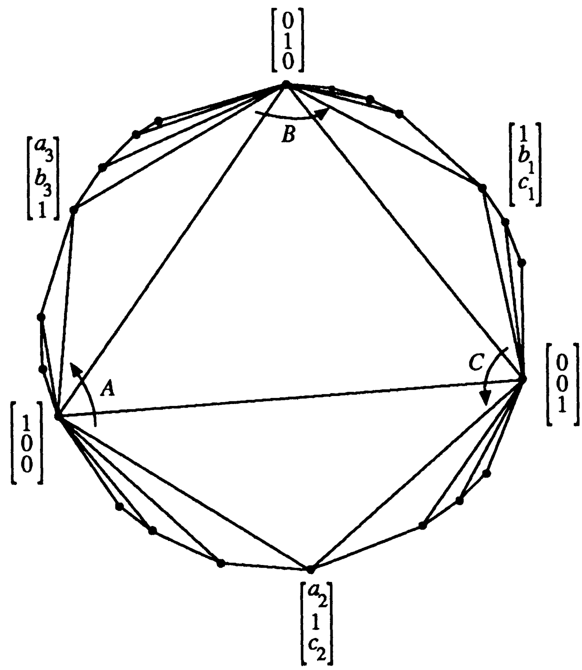

FIGURE 4.3

Suppose that M is given a convex $\mathbb{RP}^2$-structure with principal boundary and let $(\mathbf{dev}, h)$ be a development pair. Then $\mathbf{dev}: \tilde{M} \to \mathbb{RP}^2$ is a projective isomorphism from $\tilde{M}$ to a convex domain $\Omega \subset \mathbb{RP}^2$. Let $\tilde{a}, \tilde{b}, \tilde{c}$ be the arcs bounding T_0; then the endpoints of $\mathbf{dev}(\tilde{a})$ (resp. $\mathbf{dev}(\tilde{b})$ and $\mathbf{dev}(\tilde{c})$) are the repelling fixed points of $h(B), h(C)$ (resp. $h(C), h(A)$ and $h(A), h(B)$). Let $\hat{a}$ (resp. $\hat{b}, \hat{c}$) be geodesic segments joining the repelling fixed points of $h(B)$ and $h(C)$ (resp. $h(C), h(A)$ and $h(A), h(B)$). Then the collection of geodesics $\mathbf{p}(\hat{a}), \mathbf{p}(\hat{b}), \mathbf{p}(\hat{c})$ on M is isotopic to the original collection of curves a, b, c. We shall henceforth replace the original curves a, b, c by these geodesic curves. In that case $\mathbf{dev}(T_0)$, $\mathbf{dev}(T_a)$, $\mathbf{dev}(T_b)$, $\mathbf{dev}(T_c)$ are triangular regions in Ω and their union is a convex hexagon in Ω. We henceforth abbreviate the holonomy transformations $h(A), h(B), h(C)$ by A, B, C respectively.

Thus we associate to a convex $\mathbb{RP}^2$-structure M representing a point in $\mathfrak{P}(S)$ the following: four triangular regions $\Delta_0, \Delta_a, \Delta_b, \Delta_c \subset \mathbb{RP}^2$ and three projective transformations $A, B, C \in \mathbf{SL}(3, \mathbb{R})$ which satisfy the following conditions:

(i) $\overline{\Delta}_a, \overline{\Delta}_b, \overline{\Delta}_c$ each intersect $\overline{\Delta}_0$ along each of the three edges of $\overline{\Delta}_0$;
(ii) the union $\overline{\Delta}_0 \cup \overline{\Delta}_a \cup \overline{\Delta}_b \cup \overline{\Delta}_c$ is a convex hexagon;
(iii) $CBA = I$ and

$$A(\Delta_b) = \Delta_c, \quad B(\Delta_c) = \Delta_a, \quad C(\Delta_a) = \Delta_b;$$

(iv) $A, B, C \in \mathbf{Hyp}_+$ and the vertices of Δ_0 are the repelling fixed points

$$\mathrm{Fix}_-(A), \mathrm{Fix}_-(B), \mathrm{Fix}_-(C)$$

of A, B, C respectively and satisfy

$$\overline{\Delta}_a \cap \overline{\Delta}_b = \mathrm{Fix}_-(C), \quad \overline{\Delta}_b \cap \overline{\Delta}_c = \mathrm{Fix}_-(A), \quad \overline{\Delta}_c \cap \overline{\Delta}_a = \mathrm{Fix}_-(B).$$

We denote the set of all $(\Delta_0, \Delta_a, \Delta_b, \Delta_c, A, B, C)$ satisfying (i)–(iv) by $\mathscr{C}'$; the projective group $\mathbf{SL}(3, \mathbb{R})$ acts properly and freely on $\mathscr{C}'$ and we denote the quotient by $\mathscr{C}$. Since a convex $\mathbb{RP}^2$-structure determines a projective class of such configurations, there is an embedding $\mathfrak{P}(S) \to \mathscr{C}$. Furthermore each point in $\mathscr{C}'$ determines an $\mathbb{RP}^2$-structure on S with principal boundary. The proof of Theorem 4.1 will be broken up into two steps.

4.3. Proposition. *$\mathscr{C}$ is an open cell of dimension 8 and the map*

$$\mathscr{C} \to \mathfrak{R}^3$$
$$(\Delta_0, \Delta_a, \Delta_b, \Delta_c, A, B, C) \mapsto ((\lambda, \tau)_A, (\lambda, \tau)_B, (\lambda, \tau)_C)$$

is a fibration with fiber an open 2-cell over the 6-cell $\mathfrak{R}^3$. Furthermore there is an embedding $\mathfrak{T}(S) \subset \mathfrak{P}(S) \subset \mathscr{C}$ where $\mathfrak{T}(S)$ is the deformation space (an open 3-cell) of convex hyperbolic structures on S.

4.4. Proposition. *Each $\mathbb{RP}^2$-structure corresponding to a point in $\mathscr{C}$ is convex, i.e., $\mathscr{C} = \mathfrak{P}(S)$.*

Proof of 4.4 *assuming* 4.3. Let M be an $\mathbb{RP}^2$-manifold whose structure corresponds to a point of $\mathscr{C}$; such an $\mathbb{RP}^2$-manifold has principal boundary, so its double $2M$ is an $\mathbb{RP}^2$-manifold diffeomorphic on a closed surface Σ of genus two. Thus there is an embedding E_2 of the space $\mathscr{C}$ in the deformation space $\mathbb{RP}^2(\Sigma)$. The image $E_2(\mathscr{C})$ consists of $\mathbb{RP}^2$-structures on Σ which admit the symmetry of a double of a pair-of-pants—i.e., structures which admit an orientation-reversing involution whose stationary set consists of three disjoint simple closed curves. Now $\mathfrak{T}(S) \subset \mathfrak{P}(S) \subset \mathscr{C}$, so that $\mathfrak{P}(S)$ is nonempty. Furthermore by Proposition 3.3, the subset $E_2(\mathfrak{P}(S))$ is open in $E_2(\mathscr{C}) \subset \mathbb{RP}^2(\Sigma)$. It follows that $\mathfrak{P}(S)$ is a nonempty open subset of $\mathscr{C}$.

We claim that $\mathfrak{P}(S)$ is in addition closed in $\mathscr{Q}$. To this end suppose that $Q_n \in \mathfrak{P}(S)$ is a sequence converging to a point $Q_\infty \in \mathscr{Q}$. The corresponding sequence $E_2(Q_n)$ of convex structures on Σ converge to an $\mathbb{RP}^2$-structure $E_2(Q_\infty)$ on Σ, which we must show is convex. Concerning convergence of geometric structures we have the following elementary fact (compare [6, 1.5.3]):

4.5. Lemma. *Let X be a manifold upon which a Lie group G acts strongly effectively (i.e., two transformations of G which agree on a non-empty open set are identical), and let M_n be a sequence of (X, G)-structures on a manifold S converging to a (X, G)-structure M_∞. If each M_n has injective developing map, so does M_∞.*

Proof of 4.5. Convergence of the structures implies that there are developing maps $\mathbf{dev}_n: \tilde{M}_n \to X$ which converge to $\mathbf{dev}_\infty: \tilde{M}_\infty \to X$ in the C^1-topology. Suppose that $\mathbf{dev}_\infty$ is not injective. Then there exist disjoint open sets $U_1, U_2 \subset \tilde{M}_\infty$ such that $\mathbf{dev}_\infty(U_1) = \mathbf{dev}_\infty(U_2)$. For n sufficiently large, $\mathbf{dev}_n$ will be C^1-close to $\mathbf{dev}_\infty$ and $\mathbf{dev}_n(U_1) \cap \mathbf{dev}_\infty(U_2) \neq \emptyset$, contradicting the injectivity of $\mathbf{dev}_n$. q.e.d.

Thus the $\mathbb{RP}^2$-structure $E_2(Q_\infty)$ has injective developing map. It follows that Q_∞ has injective developing map as well. To see that it is actually convex, it suffices to show that its developing image is convex.

The universal covering $\tilde{M}$ can be represented by an increasing union $\bigcup_{k=1}^\infty P_k$ where each P_k is a connected union of closed 2-simplices with their vertices removed. Indeed there exists a space $\overline{M}$ with a $\pi_1(M)$-action having the following properties:

(1) $\overline{M}$ is a union of closed 2-simplices permuted by $\pi_1(M)$;

(2) $\overline{M}$ contains $\tilde{M}$ as a dense open $\pi_1(M)$-invariant subset;

(3) the complement $\overline{M} - M$ is a disjoint union of closed 1-simplices, the interiors of which are permuted freely by $\pi_1(M)$ with exactly three orbits corresponding to the three faces of a 2-simplex;

(4) the developing map $\mathbf{dev}: \tilde{M} \to \mathbb{RP}^2$ extends to an embedding $\overline{M} \to \mathbb{RP}^2$ with closed image.

Explicitly $\overline{M}$ can be constructed from a convex structure on M as follows. Choose a developing pair $(\mathbf{dev}, h)$ and let g denote a Riemannian metric on $\mathbb{RP}^2$. Then $(\tilde{M}, \mathbf{dev}^* g)$ is a metric space for which $\mathbf{dev}: \tilde{M} \to \mathbb{RP}^2$ is an isometric embedding onto the interior of a compact convex set Ω. Let $\overline{M}$ denote the metric completion of $(\tilde{M}, \mathbf{dev}^* g)$. Then $\mathbf{dev}$ extends to an isometric embedding $\overline{M} \to \mathbb{RP}^2$. The simplices comprising $\overline{M}$ are the inverse images under $\mathbf{dev}$ of simplices in Ω having vertices the fixed points of elements of $\pi_1(M)$. Convex polygons P_k are constructed by taking successive star-neighborhoods: P_{k+1} is the union of all 2-simplices in $\overline{M}$ having a face in ∂P_k. Clearly each P_k is convex.

Applying this construction to the sequence of convex structures Q_n, each P_k develops to a convex polygon. Since the development of P_k in the limiting structure Q_∞ is the C^1-limit of the development of P_k in the structures Q_n, it follows that the restriction $\mathbf{dev}_\infty(P_k)$ is a convex polygon. Thus the developing image of Q_∞ is the increasing union $\bigcup_{k=1}^{\infty} \mathbf{dev}_\infty(P_k)$ of convex sets and is therefore convex. This concludes the proof of 4.4 assuming 4.3.

4.6. Proof of 4.3. We turn now to the main computation. We choose coordinates in $\mathbb{RP}^2$ so that the vertices of Δ_0 have homogeneous coordinates

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where the repelling fixed point of A is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

the repelling fixed point of B is

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and the repelling fixed point of C is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Furthermore Δ_0 will be the triangular region defined by

$$\Delta_0 = \{[x, y, z] \in \mathbb{RP}^2 | x, y, z > 0\}.$$

We respectively assign to the remaining vertices of Δ_a, Δ_b, Δ_c the homogeneous coordinates

$$\begin{bmatrix} -1 \\ b_1 \\ c_1 \end{bmatrix}, \quad \begin{bmatrix} a_2 \\ -1 \\ c_2 \end{bmatrix}, \quad \begin{bmatrix} a_3 \\ b_3 \\ -1 \end{bmatrix}$$

so that the other triangular regions are given by

$$
\begin{aligned}
&\Delta_a = \{[x, y, z] \in \mathbb{RP}^2 | x < 0,\ 0 < y < -b_1 x,\ 0 < z < -c_1 x\},\\
(4\text{-}1)\quad &\Delta_b = \{[x, y, z] \in \mathbb{RP}^2 | 0 < x < -a_2 y,\ y < 0,\ 0 < z < -c_2 z\},\\
&\Delta_c = \{[x, y, z] \in \mathbb{RP}^2 | 0 < x < -a_3 z,\ 0 < y < -b_3 z,\ z < 0\},
\end{aligned}
$$

respectively (see Figure 4.4). At each of the vertices of Δ_0, the cross-ratios of the four lines containing edges of the incident triangles define invariants

$$(4\text{-}2)\qquad \rho_1 = b_3 c_2, \quad \rho_2 = a_3 c_1, \quad \rho_3 = a_2 b_1$$

(see Figure 4.5) and the hexagon $\Delta_0 \cup \Delta_a \cup \Delta_b \cup \Delta_c$ is convex if and only if

$$(4\text{-}3)\qquad b_1 > 0, \quad c_1 > 0, \qquad a_2 > 0, \quad c_2 > 0, \qquad a_3 > 0, \quad b_3 > 0,$$

and the cross-ratio invariants satisfy

$$(4\text{-}4)\qquad \rho_1 > 1, \quad \rho_2 > 1, \quad \rho_3 > 1.$$

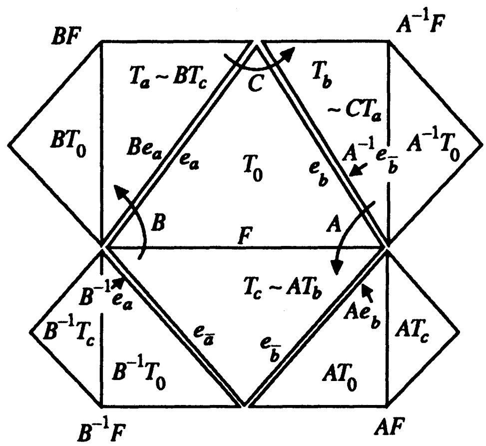

FIGURE 4.4

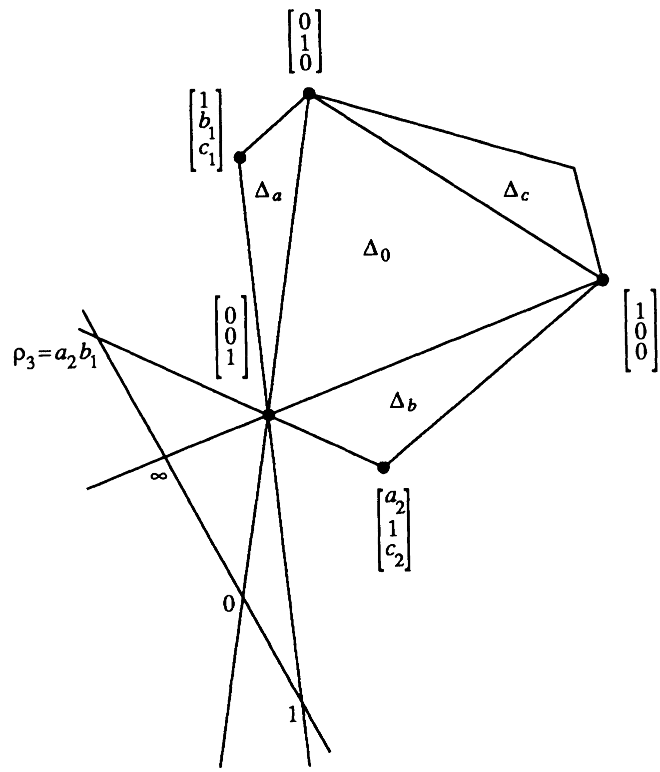

FIGURE 4.5

In this way configurations $(\Delta_0, \Delta_a, \Delta_b, \Delta_c)$ are parametrized by their coordinates

$$(b_1, c_1, a_2, c_2, a_3, b_3) \in \mathbb{R}^6$$

satisfying (4-2), (4-3) and (4-4).

Ultimately interested in the space of projective equivalence classes $\mathscr{C} = \mathscr{C}'/\mathbf{SL}(3, \mathbb{R})$, we replace the actual coordinates of the fixed points by alternate parameters invariant under a larger subgroup of $\mathbf{SL}(3, \mathbb{R})$. The only choice of coordinates thus far made has involved identifying Δ_0 with the simplex in $\mathbb{RP}^2$ with positive homogeneous coordinates. Since the stabilizer of Δ_0 consists of the group $\mathscr{A}_+$ consisting of positive diagonal matrices, the group $\mathscr{A}_+$ acts upon the set of configurations. The action of such a diagonal matrix

$$\begin{bmatrix}\lambda & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \nu\end{bmatrix} \tag{4-5}$$

on the corresponding coordinates is given by

$$\begin{bmatrix} b_1 \\ c_1 \\ a_2 \\ c_2 \\ a_3 \\ b_3 \end{bmatrix} \mapsto \begin{bmatrix} (\mu/\lambda)b_1 \\ (\nu/\lambda)c_1 \\ (\lambda/\mu)a_2 \\ (\nu/\mu)c_2 \\ (\lambda/\nu)a_3 \\ (\mu/\nu)b_3 \end{bmatrix}.$$

It is easy to see that the cross-ratio invariants ρ_1, ρ_2, ρ_3 defined in (4-2) are invariant under this action and together with

$$\text{(4-6)} \qquad \sigma_1 = a_2 b_3 c_1, \qquad \sigma_2 = a_3 b_1 c_2$$

generate all such invariants. They satisfy the relation

$$\rho_1 \rho_2 \rho_3 = \sigma_1 \sigma_2$$

as well as inequalities $\rho_i > 1$, $\sigma_i > 0$. Given $\rho_1, \rho_2, \rho_3 > 1$ and an arbitrary value of $\sigma_1 > 0$, then the choice

$$\sigma_2 = \frac{\rho_1 \rho_2 \rho_3}{\sigma_1} > 0$$

is dictated.

4.7. Of particular interest is the special case that A, B, C preserve a conic—then the $\mathbb{RP}^2$-structure on S is a hyperbolic structure. If A, B, C preserve a conic $\mathscr{C} \subset \mathbb{RP}^2$, then necessarily the vertices of Δ_0 lie on $\mathscr{C}$. Such a conic is the locus

$$\mathscr{C} = \{\mathbf{P}(v) \in \mathbb{RP}^2 | v^\dagger \mathbf{B} v = 0\},$$

where $v \in \mathbb{R}^3 - \{0\}$, $\mathbf{P}: \mathbb{R}^3 - \{0\} \to \mathbb{RP}^2$ denotes projection and

$$\mathbf{B} = \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}$$

is a symmetric matrix with entries $a, b, c \neq 0$. The conic $\mathscr{C}$ circumscribes the triangle Δ_0 if and only if the real numbers a, b, c all have the same sign. The conic $\mathscr{C}$ is determined by the projective class of this matrix in the projective space associated to the vector space of nonzero 3×3 matrices. The diagonal matrix (4–5) transforms such conics in terms of these coordinates by:

$$a \mapsto \nu\lambda a, \quad b \mapsto \mu\nu b, \quad c \mapsto \lambda\mu c,$$

and we shall fix a choice of conic $\mathscr{C}_0$ by taking $a = b = c = 1$:

$$\mathscr{C}_0 = \{[x, y, z] \in \mathbb{RP}^2 | yz + zx + xy = 0\}.$$

Given $\rho_1, \rho_2, \rho_3 > 1$ and $\sigma_1 > 0$, we obtain explicit choices for the coordinates by assigning coordinates to one of the vertices not on Δ_0. For example suppose that the vertex

$$\begin{bmatrix} a_3 \\ b_3 \\ -1 \end{bmatrix}$$

of Δ_c has homogeneous coordinates

$$\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \in \mathscr{C}_0.$$

Then (4–2) and (4–5) imply that the original coordinates are given by

(4-7)
$$a_2 = t, \quad a_3 = 2, \qquad b_1 = \frac{\rho_3}{t}, \quad b_3 = 2, \qquad c_1 = \frac{\rho_2}{2}, \quad c_2 = \frac{\rho_1}{2},$$

where $t = \sigma_1/\rho_2 > 0$ is an arbitrary positive real number.

4.8. We have expressed the geometric conditions (i) and (ii) analytically. Now we compute the possible projective transformations $A, B, C \in \mathbf{SL}(3, \mathbb{R})$ satisfying conditions (iii)–(iv). The most general projective transformation taking Δ_b to Δ_c is given by the matrix

(4-8)
$$\begin{aligned} A &= \alpha_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot [1 \quad a_2 \quad 0] + \beta_1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot [0 \quad -1 \quad 0] \\ &\quad + \gamma_1 \cdot \begin{bmatrix} a_3 \\ b_3 \\ -1 \end{bmatrix} \cdot [0 \quad c_2 \quad 1] \\ &= \begin{bmatrix} \alpha_1 & \alpha_1 a_2 + \gamma_1 a_3 c_1 & \gamma_1 a_3 \\ 0 & -\beta_1 + \gamma_1 b_3 c_2 & \gamma_1 b_3 \\ 0 & -\gamma_1 c_2 & -\gamma_1 \end{bmatrix}, \end{aligned}$$

where

$$\alpha_1 > 0, \quad \beta_1 > 0, \quad \gamma_1 > 0,$$

having determinant $\alpha_1\beta_1\gamma_1 = \det A = 1$ and $\tau(A) = -\beta_1 + \gamma_1(\rho_1 - 1)$. Similarly a projective transformation taking Δ_c to Δ_a corresponds to a matrix

$$
\begin{aligned}
B &= \alpha_2 \cdot \begin{bmatrix} -1 \\ b_1 \\ c_1 \end{bmatrix} \cdot [1 \quad 0 \quad a_3] + \beta_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot [0 \quad 1 \quad b_3] \\
&+ \gamma_2 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot [0 \quad 0 \quad -1] \\
&= \begin{bmatrix} -\alpha_2 & 0 & -\alpha_2 a_3 \\ \alpha_2 b_1 & \beta_2 & \beta_2 b_3 + \alpha_2 a_3 b_1 \\ \alpha_2 c_1 & 0 & -\gamma_2 + \alpha_2 a_3 c_1 \end{bmatrix},
\end{aligned} \tag{4-9}
$$

where

$$\alpha_2 > 0, \quad \beta_2 > 0, \quad \gamma_2 > 0,$$

$\alpha_2\beta_2\gamma_2 = \det B = 1$ and $\tau(B) = -\gamma_2 + \alpha_2(\rho_2 - 1)$. Finally a projective transformation taking Δ_c to Δ_a corresponds to a matrix

$$
\begin{aligned}
C &= \alpha_3 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot [-1 \quad 0 \quad 0] + \beta_3 \cdot \begin{bmatrix} a_2 \\ -1 \\ c_2 \end{bmatrix} \cdot [b_1 \quad 1 \quad 0] \\
&+ \gamma_3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot [c_1 \quad 0 \quad 1] \\
&= \begin{bmatrix} -\alpha_3 + \beta_3 a_2 b_1 & \beta_3 a_2 & 0 \\ -\beta_3 b_1 & -\beta_3 & 0 \\ \gamma_3 c_1 + \beta_3 b_1 c_2 & \beta_3 c_2 & \gamma_3 \end{bmatrix},
\end{aligned} \tag{4-10}
$$

where

$$a_3 > 0, \quad \beta_3 > 0, \quad \gamma_3 > 0,$$

$\alpha_3\beta_3\gamma_3 = \det C = 1$ and $\tau(C) = -\alpha_3 + \beta_3(\rho_3 - 1)$. Since the matrix A maps vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a_2 \\ -1 \\ c_2 \end{bmatrix} \mapsto \beta_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \gamma_1 \begin{bmatrix} a_3 \\ b_3 \\ -1 \end{bmatrix},$$

the matrix B maps vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \alpha_2 \begin{bmatrix} -1 \\ b_1 \\ c_1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \beta_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a_3 \\ b_3 \\ -1 \end{bmatrix} \mapsto \gamma_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and the matrix C maps vectors

$$\begin{bmatrix} -1 \\ b_1 \\ c_1 \end{bmatrix} \mapsto \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \beta_3 \begin{bmatrix} a_2 \\ -1 \\ c_2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \gamma_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

it follows that CBA maps

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \mapsto \alpha_1\alpha_2\alpha_3 \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}a_2\\-1\\c_2\end{bmatrix} \mapsto \beta_1\beta_2\beta_3 \begin{bmatrix}a_2\\-1\\c_2\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\end{bmatrix} \mapsto \gamma_1\gamma_2\gamma_3 \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Thus $CBA = I$ if and only if

$$\alpha_1\alpha_2\alpha_3 = \beta_1\beta_2\beta_3 = \gamma_1\gamma_2\gamma_3 = 1. \tag{4-11}$$

From $\det A = \det B = \det C = 1$ we obtain the further conditions

$$\alpha_1\beta_1\gamma_1 = \alpha_2\beta_2\gamma_2 = \alpha_3\beta_3\gamma_3 = 1. \tag{4-12}$$

The invariants λ, τ of A, B, C are then given by:

$$\begin{aligned} \lambda(A) &= \lambda_1 = \alpha_1, \quad & \tau(A) &= \tau_1 = -\beta_1 + \gamma_1(\rho_1 - 1), \\ \lambda(B) &= \lambda_2 = \beta_2, \quad & \tau(B) &= \tau_2 = -\gamma_2 + \alpha_2(\rho_2 - 1), \\ \lambda(C) &= \lambda_3 = \gamma_3, \quad & \tau(C) &= \tau_3 = -\alpha_3 + \beta_3(\rho_3 - 1). \end{aligned} \tag{4-13}$$

4.9. Now we may parametrize the space of convex $\mathbb{RP}^2$-structures on M. Choose boundary invariants

$$\begin{aligned} &(\lambda_1, \tau_1), (\lambda_2, \tau_2), (\lambda_3, \tau_3) \in \mathfrak{R} \\ &\quad = \{(\lambda, \tau) \in \mathbf{R}^2 | 0 < \lambda < 1, 2/\sqrt{\lambda} < \tau < \lambda + \lambda^{-2}\}. \end{aligned}$$

Then we seek

$$\begin{aligned} \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3 &> 0, \\ \rho_1, \rho_2, \rho_3 &> 1, \\ \sigma_1, \sigma_2 &> 0 \end{aligned} \tag{4-14}$$

with $\alpha_1 = \lambda_1$, $\beta_2 = \lambda_2$, $\gamma_3 = \lambda_3$ satisfying the equations

$$\begin{aligned} -\beta_1 + \gamma_1(\rho_1 - 1) &= \tau_1, \\ -\gamma_2 + \alpha_2(\rho_2 - 1) &= \tau_2, \\ -\alpha_3 + \beta_3(\rho_3 - 1) &= \tau_3, \end{aligned} \tag{4-15}$$

$$\alpha_1\beta_1\gamma_1 = \alpha_2\beta_2\gamma_2 = \alpha_3\beta_3\gamma_3 = \alpha_1\alpha_2\alpha_3 = \beta_1\beta_2\beta_3 = \gamma_1\gamma_2\gamma_3 = 1, \tag{4-16}$$

$$\sigma_1\sigma_2 = \rho_1\rho_2\rho_3. \tag{4-17}$$

To fix the origin in our coordinates on $\mathscr{C}$, we observe that a hyperbolic structure on S with geodesic boundary is determined up to isometry by the lengths of the boundary curves, and thus that the corresponding $\mathbb{RP}^2$-structure is determined by the germ of the structure near ∂S. Since a projective transformation preserving a conic has 1 as an eigenvalue, its invariants satisfy $\tau = 1 + \lambda^{-1}$. The condition that A, B, C preserve a conic (i.e., the representation is conjugate to a representation in $\mathbf{SO}(2, 1)$) is determined by checking that A, B, C preserve the bilinear form determined by the matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \tag{4-18}$$

(for which

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is an isotropic basis). Then A, B, C preserve a conic if and only if the boundary invariants satisfy $\tau_i = 1 + \lambda_i^{-1}$ and

$$\begin{aligned} &\alpha_1 = \lambda_1, && \alpha_2 = \sqrt{\frac{\lambda_3}{\lambda_1\lambda_2}}, && \alpha_3 = \sqrt{\frac{\lambda_2}{\lambda_3\lambda_1}}, \\ &\beta_1 = \sqrt{\frac{\lambda_3}{\lambda_1\lambda_2}}, && \beta_2 = \lambda_2, && \beta_3 = \sqrt{\frac{\lambda_1}{\lambda_2\lambda_3}}, \\ &\gamma_1 = \sqrt{\frac{\lambda_2}{\lambda_3\lambda_1}}, && \gamma_2 = \sqrt{\frac{\lambda_1}{\lambda_2\lambda_3}}, && \gamma_3 = \lambda_3, \end{aligned} \tag{4-19}$$

and

$$\begin{aligned} \rho_1 &= (1+\lambda_1)\left(1 + \sqrt{\frac{\lambda_3}{\lambda_1\lambda_2}}\right), \\ \rho_2 &= (1+\lambda_2)\left(1 + \sqrt{\frac{\lambda_1}{\lambda_2\lambda_3}}\right), \\ \rho_3 &= (1+\lambda_3)\left(1 + \sqrt{\frac{\lambda_2}{\lambda_3\lambda_1}}\right), \\ t &= 1 + \sqrt{\frac{\lambda_2}{\lambda_3\lambda_1}}. \end{aligned} \tag{4-20}$$

We claim that the solutions are parametrized by arbitrary pairs $s, t > 0$ of positive real numbers. By taking logarithms, (4–16) is equivalent to a system of six inhomogeneous linear equations in six real unknowns

$$\log\alpha_2, \log\alpha_3, \log\beta_1, \log\beta_3, \log\beta_3, \log\gamma_1, \log\gamma_2,$$

which has rank 5. Thus the solutions of (4–16) are parametrized by a positive real number $s > 0$ as follows:

$$
\begin{aligned}
&\alpha_1 = \lambda_1, && \alpha_2 = \sqrt{\frac{\lambda_3}{\lambda_1\lambda_2}}s^{-1}, && \alpha_3 = \sqrt{\frac{\lambda_2}{\lambda_3\lambda_1}}s, \\
&\beta_1 = \sqrt{\frac{\lambda_3}{\lambda_1\lambda_2}}s, && \beta_2 = \lambda_2, && \beta_3 = \sqrt{\frac{\lambda_1}{\lambda_2\lambda_3}}s^{-1}, \\
&\gamma_1 = \sqrt{\frac{\lambda_2}{\lambda_3\lambda_1}}s^{-1}, && \gamma_2 = \sqrt{\frac{\lambda_1}{\lambda_2\lambda_3}}s, && \gamma_3 = \lambda_3.
\end{aligned}
\tag{4-21}
$$

By (4–14) all $\rho_i > 1$ and, by (4–21) and (4–15), are given by quadratic polynomials in s:

$$
\begin{aligned}
\rho_1 &= 1 + \sqrt{\frac{\lambda_3\lambda_1}{\lambda_2}}\tau_1 s + \frac{\lambda_3}{\lambda_2}s^2, \\
\rho_2 &= 1 + \sqrt{\frac{\lambda_1\lambda_2}{\lambda_3}}\tau_2 s + \frac{\lambda_1}{\lambda_3}s^2, \\
\rho_3 &= 1 + \sqrt{\frac{\lambda_2\lambda_3}{\lambda_1}}\tau_3 s + \frac{\lambda_2}{\lambda_1}s^2.
\end{aligned}
\tag{4-22}
$$

Using (4–7) the coordinates $a_2, a_3, b_1, b_3, c_1, c_2$ are

$$
\begin{aligned}
&a_2 = t, && a_3 = 2, \\
&b_1 = \frac{1}{t} + \sqrt{\frac{\lambda_2\lambda_3}{\lambda_1}}\tau_3\frac{s}{t} + \frac{\lambda_2}{\lambda_1}\frac{s^2}{t}, && b_3 = 2, \\
&c_1 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{\lambda_1\lambda_2}{\lambda_3}}\tau_2 s + \frac{\lambda_1}{2\lambda_3}s^2, && c_2 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{\lambda_3\lambda_1}{\lambda_2}}\tau_1 s + \frac{\lambda_3}{2\lambda_2}s^2,
\end{aligned}
\tag{4-23}
$$

polynomials in $s, s^{-1}, t, t^{-1}, (\lambda_i)^{1/2}, (\lambda_i)^{-1/2}, \tau_i$. It follows that the fiber of the boundary-invariant map $\mathscr{C} \to \mathfrak{R}^3$ is parametrized by arbitrary pairs

$$(s, t) \in \mathbf{R}_+ \times \mathbf{R}_+,$$

so that $\mathscr{C} \to \mathfrak{R}^3$ is 2-cell fibration over the open 6-cell $\mathfrak{R}^3$. This concludes the proof of Proposition 4.3 and hence of Theorem 4.1.

5. Assembling convex $\mathbb{RP}^2$-manifolds

5.1. In this section the results of the two preceding sections are used to prove Theorem 1. The proof is based on the Fenchel-Nielsen coordinate system on Teichmüller space (compare [1, 3.2], [10, exposé 6], [17], [18], [21], [38, §5]). Let M be a convex $\mathbb{RP}^2$-manifold representing a point in $\mathfrak{P}(S)$, and fix a universal covering $\mathbf{p}: \tilde{M} \to M$, a fundamental group $\pi = \pi_1(M)$ and a development pair $(\mathbf{dev}, h)$. To each simple closed curve $C \subset S$ there is a unique simple closed geodesic g_C representing it; g_C develops to a principal line segment, and by 2.8 each tubular neighborhood contains a principal annular neighborhood. It follows that the germ of the $\mathbb{RP}^2$-structure at g_C is completely determined by the conjugacy class of the holonomy $h(\gamma_C)$ where $\gamma_C \in \pi$ corresponds to C. This conjugacy class is recorded in the invariants $\theta(C) = (\lambda, \tau)(C): \mathfrak{P}(S) \to \mathfrak{P}(C) = \mathfrak{R}$ defined in 2.9. Recall that if $\partial S = \partial_1(S) \cup \cdots \cup \partial_n(S)$ the map

$$\theta_{\partial S}: \mathfrak{P}(S) \to \mathfrak{P}(\partial S) = \prod_{i=1}^{n} \mathfrak{P}(\partial_i S) = \mathfrak{R}^n,$$

which records the germ of a convex $\mathbb{RP}^2$-manifold M near ∂M is given by

$$M \mapsto (\theta(\partial_1 M), \cdots, \theta(\partial_n M)).$$

We shall prove Theorem 1 stated in the following form:

5.2. Theorem. *Let S be a compact surface with $\chi(S) < 0$ and having $n \geq 0$ boundary components. Then the map*

$$\Theta_{\partial S}: \mathfrak{P}(S) \to \mathfrak{P}(\partial S)$$

is a fibration over the $2n$-cell $\mathfrak{P}(\partial S)$ with fiber an open cell of dimension $-8\chi(S) - 2n$.

The proof of 5.2 will be based on the following two lemmas.

5.3. Lemma. *Suppose that $C \subset S$ is a two-sided simple closed curve such that each component of $S|C$ has negative Euler characteristic. Let*

$$\Pi_C: \mathfrak{P}(S) \to \mathfrak{P}(S|C)$$

be the map which arises from splitting a convex $\mathbb{RP}^2$-structure on S along the closed geodesic homotopic to C. Let $b_1, b_2 \subset \partial(S|C)$ be the two boundary components corresponding to C. Then there exists an $\mathbf{R}^2$-action Ψ on $\mathfrak{P}(S)$ such that Π_C is a Ψ-invariant fibration onto the subspace of $\mathfrak{P}(S|C)$ defined by the conditions

$$\theta(b_1) = \theta(b_2),$$

and Ψ is simply transitive on each fiber of Π_C. In particular each fiber of Π_C is an open 2-cell.

5.4. Lemma. *Suppose that $C \subset S$ is a one-sided simple closed curve. Then the map*

$$\Pi_C \colon \mathfrak{P}(S) \to \mathfrak{P}(S|C)$$

which arises from splitting a convex $\mathbb{RP}^2$-structure on S along the closed geodesic homotopic to C is a diffeomorphism.

Proof of 5.2 *assuming* 5.3 *and* 5.4. We claim that 5.2 follows for a compact connected surface S with $n \geq 0$ boundary components satisfying $\chi(S) < 0$ once it is known for each component of $S|C$ where $C \subset S$ is a simple closed curve such that each component of $S|C$ has negative Euler characteristic. For every compact surface of negative Euler characteristic can be split along such curves into pairs-of-pants as follows: If S is not already a planar surface, then S can be successively split along simple closed curves to obtain a planar surface of the same Euler characteristic. The resulting planar surface can then be split along a family of $3 - \chi(S)$ disjoint simple closed curves into $-\chi(S)$ pairs-of-pants. For each cut, the Euler characteristic of each component of the complement is negative and (by additivity of χ under identification along boundary components) cannot be decreased. Thus the proof of 5.2 will be a double induction on $-\chi(S)$ and $2-\chi(S)+n$ where n is the number of boundary components. The initial stage of the induction occurs when S is a pair-of-pants; that special case of 5.2 was proved as Theorem 4.1. Thus we consider a simple closed curve $C \subset S$ such that each component $S^{(i)}$ of $S|C$ has negative Euler characteristic, and we inductively assume that 5.2 is valid for each $S^{(i)}$.

We begin with the case that C separates S into two components $S^{(1)}$ and $S^{(2)}$. Write the components of $\partial S^{(i)}$ as $\partial_1 S^{(i)}, \cdots, \partial_{n_i} S^{(i)}$ where n_i is the number of components of $\partial S^{(i)}$, and $\partial_1 S^{(i)}$ is the boundary component corresponding to C, for $i = 1, 2$. We denote the invariant of $\partial_j S^{(i)}$ by $\theta_j^{(i)} \in \mathfrak{P}(\partial_j S^{(i)})$. By 5.2 applied to $S^{(i)}$, the map

$$\Theta_{\partial S^{(i)}} \colon \mathfrak{P}(S^{(i)}) \to \mathfrak{P}(\partial S^{(i)})$$

is a fibration with fiber a cell of dimension $-8\chi(S^{(i)}) - 2n_i$. Thus their Cartesian product $\Theta_{\partial(S|C)} \colon \mathfrak{B}(S|C) \to \mathfrak{P}(\partial(S|C))$ is a fibration with fiber a cell of dimension

$$(-8\chi(S^{(1)}) - 2n_1) + (-8\chi(S^{(2)}) - 2n_2) = -8\chi(S) - 2(n+2)$$

(since $n_1 + n_2 = n+2$) over a cell of dimension $2(n+2)$. Now 5.3 implies that the image $\Pi_C(\mathfrak{P}(S))$ equals the inverse image under $\Theta_{\partial(S|C)}$ of the subset of $\mathfrak{P}(\partial(S|C))$ consisting of all

$$(\theta_1^{(1)}, \cdots, \theta_{n_1}^{(1)}, \theta_1^{(2)}, \cdots, \theta_{n_2}^{(2)})$$

satisfying $\theta_1^{(1)} = \theta_1^{(2)}$; evidently this subset of $\mathfrak{P}(\partial(S|C))$ is a $2(n+1)$-cell, and $\Pi_C(\mathfrak{P}(S))$ is a cell of dimension

$$-8\chi(S) - 2(n+2) + 2(n+1) = -8\chi(S) - 2.$$

But 5.3 implies also that Π_C is a 2-cell fibration over its image and hence $\mathfrak{P}(S)$ is a cell of dimension $-8\chi(S)$ as desired. Furthermore $\Theta_{\partial S}$ is the composition of $\Theta_{\partial(S|C)} \circ \Pi_C$ with the projection

$$(\theta_1^{(1)}, \cdots, \theta_{n_1}^{(1)}, \theta_1^{(2)}, \cdots, \theta_{n_2}^{(2)}) \mapsto (\theta_2^{(1)}, \cdots, \theta_{n_1}^{(1)}, \theta_2^{(2)}, \cdots, \theta_{n_2}^{(2)})$$

and is thus a $(-8\chi(S) + 2n)$-cell fibration as desired.

Next consider the case that $C \subset S$ is a nonseparating two-sided simple closed curve. In that case $S|C$ is connected, has two more boundary components than S, and $\chi(S|C) = \chi(S)$. By 5.2 applied to $S|C$, the map

$$\Theta_{\partial(S|C)}: \mathfrak{P}(S|C) \to \mathfrak{P}(\partial(S|C))$$

expresses $\mathfrak{P}(S|C)$ as a $-8\chi(S)-2(n+2)$-cell fibration over the $2(n+2)$-cell $\mathfrak{P}(\partial(S|C))$. Write the components of $\partial(S|C)$ as $\partial_1(S|C), \cdots, \partial_{n+2}(S|C)$ where $\partial_1(S|C)$ and $\partial_2(S|C)$ are the boundary components corresponding to C. We denote the invariant of $\partial_j(S|C)$ by $\theta_j \in \mathfrak{P}(\partial_j(S|C))$. By 5.3 the image $\Pi_C(\mathfrak{P}(S))$ equals the inverse image under $\Theta_{\partial(S|C)}$ of the subset of $\mathfrak{P}(\partial(S|C))$ consisting of all

$$(\theta_1, \theta_2, \cdots, \theta_{n+2})$$

satisfying

$$\theta_1 = \theta_2.$$

Evidently this subset of $\mathfrak{P}(\partial(S|C))$ is a $2(n+1)$-cell and thus $\Pi_C(\mathfrak{P}(S))$ is a cell of dimension

$$-8\chi(S) - 2(n+2) + 2(n+1) = -8\chi(S) - 2.$$

As in the above case, 5.3 implies that Π_C is a 2-cell fibration over its image and hence $\mathfrak{P}(S)$ is a cell of dimension $-8\chi(S)$ as desired. Furthermore $\Theta_{\partial S}$ is the composition of $\Theta_{\partial(S|C)} \circ \Pi_C$ with the projection

$$(\theta_1, \theta_2, \cdots, \theta_{n+2}) \mapsto (\theta_3, \cdots, \theta_{n+2})$$

and is thus a $(-8\chi(S) + 2n)$-cell fibration is desired.

Finally consider the case that C is a one-sided simple closed curve in S. Then $S|C$ has one more boundary component than S and $\chi(S|C) = \chi(S)$. By 5.2 applied to $S|C$, the map

$$\Theta_{\partial(S|C)}: \mathfrak{P}(S|C) \to \mathfrak{P}(\partial(S|C))$$

expresses $\mathfrak{P}(S|C)$ as a $(-8\chi(S)-2(n+1))$-cell fibration over the $2(n+1)$-cell $\mathfrak{P}(\partial(S|C))$. By 5.4 the map

$$\Pi_C: \mathfrak{P}(S) \to \mathfrak{P}(S|C)$$

is a diffeomorphism. Now $\Theta_{\partial S}$ is the composition of $\Theta_{\partial(S|C)} \circ \Pi_C$ with the projection $\mathfrak{P}(\partial(S|C)) \to \mathfrak{P}(\partial S)$, which is a 2-cell fibration over a $2n$-cell; it follows that $\Theta_{\partial S}$ is a $(-8\chi(S)+2n)$-cell fibration over a $2n$-cell as desired. This concludes the proof of 5.2 assuming 5.3 and 5.4.

5.5. We now define the $\mathbb{R}^2$ action Ψ on $\mathfrak{P}(S)$. This action generalizes the Fenchel-Nielsen twist flows (also known as "earthquakes"—see [10], [29], [41], [9, 3.5]) on the Teichmüller space. We define the action of an element $(u, v) \in \mathbb{R}^2$ on a point in $\mathfrak{P}(S)$ represented by a convex $\mathbb{RP}^2$-manifold M. Thus for $(u, v) \in \mathbb{R}^2$ we construct a new convex $\mathbb{RP}^2$-manifold $\Psi_{(u,v)}(M)$ representing a point in $\mathfrak{P}(s)$. Choose a universal covering $\mathbf{p}: \tilde{M} \to M$ and a development pair $(\mathbf{dev}, h)$. We assume that C is a simple closed geodesic on M and that a representative element $\gamma \in \pi$ has been chosen so that $h(\gamma) \in \mathscr{A}$ is represented by the diagonal matrix (1–1) satisfying (1–2). Clearly the centralizer of $h(\gamma)$ in $\mathbf{SL}(3, \mathbb{R})$ equals $\mathscr{A}$ whose identity component A_+ is the direct product of the two one-parameter groups

$$T^u = \begin{pmatrix} e^{-u} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^u \end{pmatrix}, \qquad U^v = \begin{pmatrix} e^{-v} & 0 & 0 \\ 0 & e^{2v} & 0 \\ 0 & 0 & e^{-v} \end{pmatrix},$$

where $u, v \in \mathbb{R}$. Consider the split $\mathbb{RP}^2$-manifold $M|C$; let $b_1, b_2 \subset \partial(M|C)$ be the two boundary components corresponding to C. For any $(u, v) \in \mathbb{R}^2$, there exist principal collar neighborhoods $N(b_i) \subset M|C$ of b_i for $i = 1, 2$ and a projective isomorphism $f: N(b_1) \to N(b_2)$ such that f is related by $\mathbf{dev}$ to the projective transformation $T^u U^v$. As in 3.6 there is a corresponding $\mathbb{RP}^2$-manifold $(M|C)/f$ representing a point $\Psi_{(u,v)}(M)$ in $\mathfrak{P}(S)$. It is easy to check that $(M|C)/f$ is independent of the choices of collar neighborhoods and that Ψ determines an $\mathbb{R}^2$-action on $\mathfrak{P}(S)$. (The flows $\Psi_{(u,0)}$ and $\Psi_{(0,v)}$ on $\mathfrak{P}(S)$ are actually special cases of the "generalized twist flows" discussed in Goldman [15] on $\mathbf{Hom}(\pi, \mathbf{SL}(3, \mathbb{R}))$. The potential functions for these twist flows are the invariant functions $\ell(C)$ and $m(C)$, respectively, defined in 1.8. A generalization of these flows to real projective structures on hyperbolic manifolds of higher dimension is discussed in Johnson-Millson [22].)

Proof of 5.3. An $\mathbb{RP}^2$-structure on $S|C$ arises from an $\mathbb{RP}^2$-structure on S if and only if there exist collar neighborhoods of the two boundary components of $S|C$ which are projectively isomorphic; the various structures on S which give the same structure on $S|C$ correspond to the possible identifications between these collars. Suppose that M is a convex $\mathbb{RP}^2$-manifold representing a point in $\mathfrak{P}(S)$, and let $b_1, b_2 \subset \partial(S|C)$ be the two boundary components which correspond to C. By the Collar Lemma 3.8 and the remarks in 2.6 there are collar neighborhoods $N(b_1), N(b_2) \subset M|g_C$ which are projectively isomorphic, and the projective isomorphism class depends solely on the invariants of the holonomy. Thus the image $\Pi_c(\mathfrak{P}(S))$ consists of all convex $\mathbb{RP}^2$-manifolds in $\mathfrak{P}(S|C)$ such that $\theta(b_1) = \theta(b_2)$.

Clearly $M|C \cong (\Psi_{(u,v)}(M))|C$, so that Π_C is invariant under the action Ψ. We now identify the fiber of Π_c over the point in $\mathfrak{P}(S|C)$ corresponding to a convex $\mathbb{RP}^2$-manifold M_0 diffeomorphic to $S|C$. Let M be a convex $\mathbb{RP}^2$-manifold corresponding to a point in $\mathfrak{P}(S)$ such that $M|g_C \cong M_0$. The fiber of Π_C may then be identified with all germs of projective isomorphisms $N(b_1) \to N(b_2)$. By 2.8 this centralizer acts simply transitively on the set of such germs; evidently the action of this centralizer determines the action Ψ on $\mathfrak{P}(S)$. The proof of 5.3 is complete.

Proof of 5.4. Suppose that $C \subset S$ is a one-sided simple closed curve; then topologically S is obtained from $S|C$ by attaching a cross-cap to the boundary component C_0 of $S|C$ corresponding to C. Let M_0 be an $\mathbb{RP}^2$-manifold representing a point in $\mathfrak{P}(S|C)$. Then the germ of the $\mathbb{RP}^2$-structure at $C_0 \subset M_0$ determines a unique principal cross-cap neighborhood, and by 3.9 there is a unique $\mathbb{RP}^2$-manifold M obtained by attaching this cross-cap. Thus Π_C is bijective and the proof of 5.4 is complete.

5.6. Using these techniques we can give explicit coordinates for the space $\mathfrak{P}(S)$ in the spirit of Fenchel-Nielsen. These coordinates depend on a decomposition of S as a union of pairs-of-pants P_l; since $\chi(P) = -1$, there will be $-\chi(S)$ pants in the decomposition. Let M be a convex $\mathbb{RP}^2$-manifold representing a point in $\mathfrak{P}(S)$. To a simple closed curve $C \subset S$, there are invariants

$$\theta(C) = (\lambda(C), \tau(C)) \in \mathfrak{R} = \mathfrak{P}(C)$$

or alternately

$$\theta(C) = (\ell(C), m(C)) \in \{(\ell, m) \in \mathbb{R}_+ \times \mathbb{R} \mid |m| < \ell\} \cong \mathfrak{R} = \mathfrak{B}(C).$$

The (ℓ, m) invariants may be preferable here because of their more geometric interpretation as the generalization of the length coordinate in the Fenchel-Neilsen description. For a given P_l, write its boundary as $\partial P_l = \partial_1(P_l) \cup \partial_2(P_l) \cup \partial_3(P_l)$. Then given boundary invariants

$$(\theta(\partial_i(P_l)))_{i=1}^3 \in \mathfrak{P}(\partial P_l) = \mathfrak{R}^3$$

it follows from Theorem 4.1 that the $\mathbb{RP}^2$-structures on P_l are parametrized by $\mathbb{R}_+ \times \mathbb{R}_+$. Suppose that $C \subset S$ is a curve which lies on the common boundary of two adjacent pants P_{l_1} and P_{l_2}. Then the $\mathbb{RP}^2$-structure on P_{l_2} can be deformed so that the new $\mathbb{RP}^2$-structure on P_{l_2} has the same invariants $\Theta_{\partial P_2}$ and the same invariants in $\mathfrak{P}(P_{l_1}) \cong \mathbb{R}_+ \times \mathbb{R}_+$ as P_{l_1}. This deformation extends to a deformation of the $\mathbb{RP}^2$-structure on $P_{l_1} \cup_C P_{l_2}$ to an $\mathbb{RP}^2$-manifold Q. Let $2_C P_{l_1}$ denote the $\mathbb{RP}^2$-manifold obtained by doubling $P_{l_1} \subset M$ along the boundary component C; then by Lemma 5.3 there exists a unique $(u, v) \in \mathbb{R}^2$ such that $Q = \Psi_{(u,v)}(2_C P_{l_1})$. We regard $(u, v)(C) = (u, v) \in \mathbb{R}^2$ as another pair of coordinates depending on the curve C; these are the analogues of the twist parameter in the Fenchel-Nielsen coordinates. Just as for the classical twist parameter these coordinates are much less canonical than the (ℓ, m)-coordinates associated to C; the choice of such coordinates is equivalent to finding a slice for the $\mathbb{R}^2$-action Ψ associated with C.

We are now ready to define an explicit diffeomorphism to a cell. Let S have boundary components $b_1, \cdots, b_n$, and cut S along disjoint one-sided simple closed curves $a_1, \cdots, a_m$ (actually at most one such curve will suffice). Decompose S into pants P_l for $l = 1, \cdots, -\chi(S)$ along simple closed curves $c_1, \cdots, c_p$; then it is easy to see that $n + m + 2p = -3\chi(S)$. It follows from 4.1, 5.3 and 5.4 that the map

$$\mathfrak{P}(S) \to \mathfrak{R}^n \times \mathfrak{R}^m \times (\mathfrak{R} \times \mathbb{R}^2)^p \times (\mathbb{R}_+ \times \mathbb{R}_+)^{(-\chi(S))}$$

defined by

$$M \mapsto \{(\lambda(b_i), \tau(b_i))\}_{i=1}^n \times \{(\lambda(a_j), \tau(a_j))\}_{j=1}^m$$
$$\times \{(\lambda(c_k), \tau(c_k), (u, v)(c_k))\}_{k=1}^p \times \{((s, t)(P_l))\}_{l=1}^{-\chi(S)}$$

is a diffeomorphism of $\mathfrak{P}(S)$ onto a $-8\chi(S)$-dimensional cell.

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