Three-Dimensional Affine Crystallographic Groups[‡]

DAVID FRIED*

Institut des Hautes Études Scientifiques,
Bures-sur-Yvette, France

AND

WILLIAM M. GOLDMAN[†]

Department of Mathematics,
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Those groups Γ which act properly discontinuously and affinely on $\mathbb{R}^3$ with compact fundamental domain are classified. First it is shown that such a group Γ contains a solvable subgroup of finite index, thus establishing a conjecture of Auslander in dimension three. Then unimodular simply transitive affine actions on $\mathbb{R}^3$ are classified; this leads to a classification of affine crystallographic groups acting on $\mathbb{R}^3$. A characterization of which abstract groups admit such an action is given; moreover it is proved that every isomorphism between virtually solvable affine crystallographic groups (respectively simply transitive affine groups) is induced by conjugation by a polynomial automorphism of the affine space. A characterization is given of which closed 3-manifolds can be represented as quotients of $\mathbb{R}^3$ by groups of affine transformations: a closed 3-manifold M admits a complete affine structure if and only if M has a finite covering homeomorphic (or homotopy-equivalent) to a 2-torus bundle over the circle.

0. INTRODUCTION

One of the most beautiful contributions of classical physics to mathematics is the theory of crystallographic groups. In studying a crystal it is important to understand the underlying symmetry group Γ. Here Γ is a discrete group of isometries of $\mathbb{R}^3$ with a compact fundamental domain and any such group is called *crystallographic*. To achieve a finite classification one identifies $\Gamma_1, \Gamma_2 \subset \mathrm{Isom}(\mathbb{R}^3)$ if they are conjugate in the larger group $\mathrm{Aff}(\mathbb{R}^3)$ of affine automorphisms of $\mathbb{R}^3$ (for instance, all groups generated by

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three independent translations are identified). Nearly a century ago it was shown by Fedorov in Russia, Schoenfliess in Germany, and Barlow in England, working independently, that there are 219 different crystallographic groups.

The corresponding question in higher dimensions is part of Hilbert's 18th Problem. Precisely, if one considers $\Gamma \subset \mathrm{Isom}(\mathbb{R}^n)$ with a compact fundamental domain up to conjugacy in $\mathrm{Aff}(\mathbb{R}^n)$, Hilbert asked whether a finite classification of n-dimensional crystallographic groups existed. In a series of theorems Bieberbach showed this was so. While the explicit classification is only known in dimension $n \leqslant 4$ and many detailed questions are unknown for $n > 4$, one knows a great deal about these groups [28].

There is another natural way to generalize the classical problem. Instead of raising the dimension one may broaden the class of allowed motions and consider those $\Gamma \subset \mathrm{Aff}(\mathbb{R}^3)$ with a compact fundamental domain. Such a Γ is called a *three-dimensional affine crystallographic group*. One still identifies Γ_1 and Γ_2 if they are conjugate in $\mathrm{Aff}(\mathbb{R}^3)$. The main content of this paper is the classification of this wider class of groups.

Some further motivation for this problem arises from another physically interesting problem, namely, the study of complete flat spacetimes. Such a spacetime is of the form $\mathbb{R}^n/\Gamma$ where Γ is a discrete group of pseudo-Riemannian isometries. Clearly $\Gamma \subset \mathrm{Aff}(\mathbb{R}^n)$. So our results classify the flat, complete compact spacetimes of dimension 3. This problem was first attacked by Auslander and Markus [6] who solved it under certain extra algebraic assumptions, most importantly that the linear holonomy $L(\Gamma) \subset GL(3, \mathbb{R})$ be abelian. We show that any spacetime as above is finitely covered by one of Auslander–Markus type, so that theirs is virtually a complete classification. We have recently obtained corresponding results for $n \geqslant 4$ as well [13, 19].

The raises the question of the group-theoretic conditions satisfied by affine crystallographic groups Γ. It was suggested by Auslander that such a Γ must be virtually solvable (i.e., have a subgroup of finite index that is solvable). The proof in [4] is unfortunately incorrect, but it is still an open and central problem; see Milnor [29]. We give in Section 2 a geometric proof that a crystallographic group of dimension $\leqslant 3$ is virtually solvable and this is the key in our classification.

The significance of virtual solvability is that it enables one to proceed, as Auslander did in [4], to associate certain Lie groups to Γ. This is analogous to the way one can embed a crystallographic group Γ generated by three independent translations in the Lie group of all translations. Our results in this direction hold in arbitrary dimensions and are placed in Section 1. The main result is Theorem 1.4, which associates a *crystallographic hull* $H \subset \mathrm{Aff}(\mathbb{R}^n)$ to a virtually solvable crystallographic group Γ. The identity component H_0 of H meets Γ is a subgroup of finite index, H_0 is solvable, and

H_0 acts simply transitively on $\mathbb{R}^n$. Thus H_0 plays the role of the group T of all translations in the 1st Bieberbach theorem. The proofs of Section 1 employ some basic results from algebraic groups and a geometrically oriented reader might well take Theorem 1.4 on fath on a first reading and begin reading in Section 2.

In Sections 3, 4 and 5 we classify the crystallographic hulls H for 3-dimensional crystallographic groups (H is uniquely determined by Γ in this dimension, but see 1.14). Section 3 and 4 are concerned with connected hulls (H_0 in the preceding paragraph). In Section 3 we also assume $H = H_0$ is nilpotent.

In general, our classification runs as follows. As abstract Lie groups, there are only finitely many crystallographic hulls H. Under the weaker relation of affine conjugacy, there are two one-parameter families of H's and finitely many other examples. By taking discrete cocompact subgroups of these Lie groups (which are quite well understood) we obtain a full set of representatives for the 3-dimensional affine crystallographic groups. One can readily decide when such groups are conjugate in $\mathrm{Aff}(\mathbb{R}^3)$ and we proceed reasonably far towards deciding this question as well. All these results are in Section 5.

There is some material covered in this paper that does not directly relate to our classification. We let Γ be a virtually solvable crystallographic group. The construction of a crystallographic hull for Γ is generalized by an analogous *syndetic hull* for any virtually solvable linear group in Theorem 1.6. Second, if Γ' is a crystallographic group isomorphic to Γ we prove that the two actions differ by a polynomial coordinate change on $\mathbb{R}^n$ in Theorem 1.20. Finally an appendix analyses the "internal structure" of a connected crystallographic hull in $\mathrm{Aff}(\mathbb{R}^3)$, that is, the invariant tensor fields and foliations. The use of 1-forms and vector fields in Section 2 shows that such information can be important.

The paper is organized as follows: 0. Introduction. 1. Complete affine manifolds with solvable holonomy (crystallographic/syndetic hulls, polynomial deformations). 2. Virtual solvability (linear holonomy of low dimensional crystallographic groups). 3. Simply transitive affine actions of low-dimensional nilpotent groups. 4. Simply transitive affine actions of unimodular Lie groups on $\mathbb{R}^3$. 5. Crystallographic hulls in dimension 3. Appendix.

1. COMPLETE AFFINE MANIFOLDS WITH SOLVABLE HOLONOMY

1.1. Let E be a real vector space. We denote the group of linear automorphisms of E by $GL(E)$. Its Lie algebra, consisting of all linear endomorphisms $E \to E$, will be denoted $\mathfrak{gl}(E)$. We let R^* denote the

subgroup of all scalar multiplications (homotheties). $SL(E)$ and $\mathscr{sl}(E)$ have their customary meaning. A transformation $x \to Ax + b$ where $A \in GL(E)$ and $b \in E$ is an *affine automorphism* of E. The group of all affine automorphisms of E will be denoted $\text{Aff}(E)$ and its Lie algebra by $a\!f\!f(E)$.

We denote the natural homomorphism by $L: \text{Aff}(E) \to GL(E)$ as well as the homomorphism of Lie algebras $\mathscr{L}: a\!f\!f(E) \to \mathscr{gl}(E)$.

1.2. A subgroup $\Gamma \subset \text{Aff}(E)$ which acts properly discontinuously on E such that E/Γ is compact is called an *affine crystallographic group*. If Γ acts freely in addition to properly discontinuously, then the quotient space E/Γ is a smooth manifold and the projection $E \to E/\Gamma$ is a covering space; such a manifold M is called a *complete affine manifold* or an *affine space form*, and Γ (which is sometimes called its *affine holonomy*) is isomorphic to the fundamental group $\pi_1(M)$.

The condition that Γ acts properly discontinuously on E is really two conditions on the action of Γ: First, the action of Γ is *proper*, that is, the map $\Gamma \times E \to E \times E$ defined by $(\gamma, x) \to (\gamma x, x)$ is proper. In other words, for every compact $K \subset E$ there is only a compact set of $\gamma \in \Gamma$ such that $\gamma K \cap K$ is nonempty. Second, $\Gamma \subset \text{Aff}(E)$ must be discrete. However, it is important to note that the converse does not hold, that most discrete subgroups of $\text{Aff}(E)$ do not act properly (discontinuously). Nor is it true for a properly discontinuous group Γ that $L(\Gamma)$ (sometimes called the *linear holonomy*) is always discrete in $GL(E)$; we shall see examples of both types. For other discussions of proper and properly discontinuous actions see Palais [32] and Thurston [39, sect. 8.1].

It follows from the definitions that every isotropy group of a proper (respectively properly discontinuous) action must be compact (resp. finite). In the converse direction it is easy to see that the barycenter of an orbit of a compact subgroup of $\text{Aff}(E)$ is a fixed point. Therefore for subgroups Γ of $\text{Aff}(E)$ which act properly there is a one-to-one correspondence between orbits of stationary points for elements of Γ and conjugacy classes of maximal compact subgroups of Γ. In particular a torsion-free group which acts properly discontinuously will act freely.

By a lemma of Selberg [36] (compare [33, 7.11]) every finitely generated subgroup $\Gamma \subset \text{Aff}(E)$ contains a normal torsion-free subgroup Γ_1 of finite index. It follows that if Γ acts properly discontinuously on E, then E/Γ is the quotient space of the complete affine manifold E/Γ_1 by the finite group Γ/Γ_1 of "affine automorphisms" of E/Γ_1.

1.3. In Milnor [29] the following conjecture is proposed:

Conjecture. If $\Gamma \subset \text{Aff}(E)$ acts properly discontinuously on E, then Γ has a solvable subgroup of finite index.

We shall say that a group is *virtually* solvable (resp. polycyclic, nilpotent, abelian, etc.) if it has a subgroup of finite index which is solvable (resp. polycyclic, etc.). In such a definition the subgroup can be chosen to be normal since every finite-index subgroup contains a normal finite-index subgroup. In [29] the following is given as partial "evidence" for the conjecture: if G is a connected subgroup of Aff(E) which acts properly on E, then G is a compact extension of a solvable group (i.e., G is amenable). Moreover, as Milnor shows in [28], every virtually polycyclic group Γ can act properly discontinuously and affinely on some vector space. The condition that Γ be virtually polycyclic is necessary since a discrete solvable subgroup of a Lie group with finitely many components must be polycyclic.

In Section 2 we shall see some special cases of this conjecture in low dimensions.

1.4. The principal tool for classifying affine crystallographic groups is the following result. A weaker version may be found in Auslander [4].

THEOREM. *Let $\Gamma \subset$ Aff(E) be virtually solvable and suppose Γ acts properly discontinuously on E. Then there exists at least one subgroup $H \subset$ Aff(E) containing Γ such that*:

(a) *H has finitely many components and each component of H meets Γ*;

(b) *H/Γ is compact*;

(c) *H and Γ have the same algebraic hull in Aff(E)*;

(d) *every isotropy group of H on E is finite*.

Moreover if Γ has a subgroup Γ_1 of finite index such that every element of $L(\Gamma_1)$ has all eigenvalues real, then H is uniquely determined by the above conditions.

Such a group H will be called a *crystallographic hull* for Γ.

Recall that the *rank* of a polycyclic group Γ is the number of infinite cyclic composition factors in any cyclic composition series of Γ. Moreover the rank of Γ equals the rank of any finite-index subgroup of Γ. Thus we can consistently extend the definition of rank to virtually polycyclic groups by defining the rank to be the rank of any polycyclic subgroup of finite index. (For the properties of polycyclic groups, the reader is referred to [29, sects. 3–4].) Furthermore it is well known that the rank of a virtually polycyclic group Γ equals the virtual cohomological dimension vcd Γ defined by Serre [37].

COROLLARY 1.5. *Suppose that $\Gamma \subset$ Aff(E) is a virtually polycyclic affine*

crystallographic group acting on E. Then there exists a simply transitive $G \subset \mathrm{Aff}(E)$ such that $\Gamma \cap G$ is a discrete cocompact subgroup of G and has finite index in Γ.

Proof. By 1.4 there exists a crystallographic hull H for Γ. Let G be the identity component of H. By 1.4(a), $\Gamma \cap G$ has finite index in Γ and by 1.4(b), $\Gamma \cap G$ is cocompact in G. It remains to show that G acts simply transitively on E. Passing to a torsion-free subgroup of finite index in Γ, we see $\dim G \geqslant \mathrm{rank}\, \Gamma = \dim E$ since both G/Γ and E/Γ are compact manifolds. By 1.4(d), $\dim G = \dim E$ so the orbits of G partition E into disjoint open sets. As E is connected G acts transitively on E. Thus any evaluation map $G \to E$ is a covering space. Since E is simply connected, G acts simply transitively on E.   Q.E.D.

If $G \subset \mathrm{Aff}(E)$ acts simply transitively on E, then we may define a *left-invariant complete affine structure on G* by pulling back the affine structure on E by any evaluation map $G \to E$. Given any affine structure on G invariant under left-multiplications, the space of left cosets $\Gamma \backslash G$ inherits an affine structure. If Γ is torsion-free, then the complete affine manifold E/Γ is affinely equivalent to the *complete affine solvmanifold $\Gamma \backslash G$.* Thus 1.5 asserts that *every compact complete affine manifold with virtually solvable holonomy is finitely covered by a complete affine solvmanifold.*

1.6. The principal tool we use to prove 1.4 is the following:

THEOREM. *Let V be a finite-dimensional real vector space and G a virtually solvable subgroup of $GL(V)$. Then there exists at least one closed virtually solvable subgroup $H \subset GL(V)$ containing G such that:*

(i) *H has finitely many components and each component meets G;*

(ii) *(syndeticity) there exists a compact set $K \subset H$ such that $H = K \cdot G$;*

(iii) *H and G have the same algebraic hull in $GL(V)$;*

(iv) *$\dim H \leqslant \mathrm{rank}\, G$.*

Moreover if G has a subgroup of finite index whose elements have all eigenvalues real, then G uniquely determines H satisfying the above conditions.

Such a group H will be called a *syndetic hull for G.* (This terminology is based on that of Gottschalk and Hedlund [18, sect. 2].)

We have stated Theorem 1.6 in more generality than is needed here since the more general form is useful in other contexts but not really any more difficult to prove. If G is a virtually solvable linear group, then the *rank* of G is defined as follows. Let $(\bar{G})^0$ denote the identity component of the closure of G; then its normalizer N is an algebraic group and has finitely many

components. The group $\tilde{G}/(\tilde{G})^0$ is a discrete virtually solvable subgroup of $N/(\tilde{G})^0$ and is thus virtually polycyclic, so its rank is defined. We thus define the rank of G to equal $\text{rank}(\tilde{G}/(\tilde{G})^0) + \dim(\tilde{G})^0$.

1.7. The proofs of 1.4 and 1.6 involve some of the structure theory of solvable linear algebraic groups (for which the reader is referred to the books of Borel [8] and Humphreys [22] for background). If G is any subgroup of $GL(V)$, then its Zariski closure in $GL(V)$ is a subgroup, which is called the *algebraic hull* of G and denoted $A(G)$. If G is already Zariski closed, it is called an *algebraic subgroup*. An example of an algebraic subgroup of $GL(E + R)$ is the *affine group* $\text{Aff}(E)$ consisting of all matrices of the form

$$\begin{bmatrix} A \\ 0 \cdots 0 \end{bmatrix} \begin{bmatrix} b \\ 1 \end{bmatrix},$$

where $A \in GL(E)$ and $b \in E$. It is easy to see that on the hyperplane $E \times \{1\}$ the above transformation acts by the affine transformation $x \to Ax + b$. Hence every group of affine transformations can be considered as a linear group and has a well-defined algebraic hull which is an algebraic subgroup of the affine group.

If G is an algebraic group, then G decomposes as a semidirect product $U \rtimes R$ where U is the *unipotent radical* of G (i.e., the maximal normal unipotent algebraic subgroup) and R is a maximal reductive subgroup. When G is Zariski connected and solvable, then R is abelian and is isomorphic (as a Lie group) to a product of copies of the group of nonzero reals $\mathbb{R}^*$ and the circle group $SO(2)$.

It is easy to prove that every connected unipotent subgroup H of $GL(V)$ is Zariski closed; moreover a subgroup $G \subset H$ is Zariski dense if and only if it is syndetic (see, e.g., [33, 2.6]).

We shall also need the *Jordan decomposition* of elements of $aff(E)$. If $g \in aff(E)$, we say that g is *semisimple* if g fixes some point in E and its linear part $L(g)$ is a semisimple linear map. (Note that g fixes a point in E if and only if it is conjugate by a translation to its linear part.) g is said to be *unipotent* (resp. *nilpotent*) if its linear part is unipotent (resp. nilpotent). The following theorem is a consequence of the usual Jordan decomposition of linear maps applied to the embedding of $\text{Aff}(E)$ in $GL(E + \mathbb{R})$:

PROPOSITION (*Jordan decomposition for nonsingular affine transformations*). *Suppose $g \in \text{Aff}(E)$. Then there exist unique affine transformations $g^{(u)}, g^{(s)} \in \text{Aff}(E)$ such that $g = g^{(u)}g^{(s)} = g^{(s)}g^{(u)}$ and $g^{(u)}$ is unipotent.*

We call $g^{(s)}$ (resp. $g^{(u)}$) the *semisimple* (resp. *unipotent*) part of g. If G is a connected solvable linear (or affine) group, then the unipotent parts of

elements of G form a group which in fact is the unipotent radical of the algebraic hull $A(G)$. One can give an inductive proof of this, along exactly the same lines as Lemma 4.36 of [33].

1.8. Now we show how to deduce Theorem 1.4 from Theorem 1.6. Note first of all that both of these statements are true for a group G once they are known for a subgroup of G of finite index. Thus we will pass to subgroups of finite index whenever convenient.

Let $\Gamma \subset \operatorname{Aff}(E)$ act properly discontinuously on E and suppose that Γ is virtually solvable. By 1.6 there exists a syndetic hull H for Γ. Then properties (a), (b), (c) of 1.4 follow immediately from (i), (ii), (iii) of 1.6. We now prove (d). It is an easy exercise in general topology that if $\Gamma \subset H$ is a discrete cocompact subgroup and H acts on a locally compact space, then H acts properly if Γ acts properly. Applying this to the current situation, we conclude that H acts properly on E; in particular every isotropy group $H_x = \{g \in H: gx = x\}$ is compact. We must show that every isotropy group is finite, i.e., that for all $x \in E$, $\dim H_x = 0$.

Passing to a subgroup of finite index we may assume that Γ acts freely and H is connected. Write $A(H) = A(\Gamma) = U \rtimes T$ where U is the unipotent radical and T is maximal reductive.

LEMMA 1.9. *Suppose that $H \subset \operatorname{Aff}(E)$ is a connected group which acts properly on E. Then the unipotent radical U of its algebraic hull $A(H)$ acts freely on E.*

Proof. Every element of U is the unipotent part $g^{(u)}$ in the Jordan decomposition of some $g \in H$. Thus there is a semisimple element $g^{(s)}$ commuting with $g^{(u)}$ with $g = g^{(u)} g^{(s)}$. If $g^{(u)}$ has a stationary point, then the set E_1 of stationary points of $g^{(u)}$ is a nonempty affine subspace invariant under $g^{(s)}$. Since $g^{(s)}$ is semisimple it must fix some point $y \in E_1$. It follows that $gy = y$. But the isotropy group H_y is compact, and the only affine transformations which lie in a compact group are semisimple. Hence $g = g^{(s)}$ so $g^{(u)} = 1$. Therefore U acts freely, concluding the proof of Lemma 1.7.   Q.E.D.

We have proved so far that U acts freely; we must show that H acts freely. Let $x \in E$ be a point fixed by T (such points exist because T is reductive). Its orbit Ux is invariant under $A(\Gamma)$ and hence under Γ. Since Γ acts properly discontinuously and freely, and because U is a connected unipotent group, the quotient Ux/Γ is a manifold homotopy-equivalent to an Eilenberg–MacLane space $K(\Gamma, 1)$. Consequently rank $\Gamma = \operatorname{cd} \Gamma \leqslant \dim U$. By Raghunathan [33], Lemma 4.36, $\dim U \leqslant \dim H$, and by 1.4(iv), $\dim H \leqslant$ rank Γ. Combining these inequalities we get rank $\Gamma \leqslant \dim U \leqslant \dim H \leqslant$ rank Γ. Thus rank $\Gamma = \dim H$.

To prove H acts freely on E, we consider the fibration sequence $\Gamma \to H \to H/\Gamma$. We obtain an exact sequence $\pi_1(H) \to \pi_1(H/\Gamma) \to \Gamma$, whence $\operatorname{rank}(\pi_1(H)) + \operatorname{rank} \Gamma = \operatorname{rank} \pi_1(H/\Gamma) = \dim H$ since H/Γ is compact. Since $\operatorname{rank} \Gamma = \dim H$ we may conclude $\operatorname{rank} \pi_1(H) = 0$, i.e., H has finite fundamental group. But H is a solvable Lie group, so this implies H is simply connected. It follows that the compact group H_x must be trivial. This concludes the proof of (d).

The uniqueness assertion in 1.4 follows from the uniqueness assertion in 1.6. This concludes the proof of 1.4.   Q.E.D.

Remark. When Γ is nilpotent, an easier argument is available using results from Malcev [26]. See [15, sect. 7] for this argument.

1.10. Now we begin the proof of 1.6. We initially assume that G *is abelian.* In that case the algebraic hull $A(G)$ is an abelian Lie group. Passing to a subgroup of finite index we may assume that G is a subgroup of a connected abelian Lie group which must be isomorphic to $\mathbb{R}^n/\Lambda$ where Λ is a discrete subgroup of the vector group $\mathbb{R}^n$.

It is clear that if G is a subgroup of a vector group, then the usual $\mathbb{R}$-linear span of G satisfies the conditions (1.6) of a syndetic hull of G, and is unique. When $G \subset \mathbb{R}^n/\Lambda$ consider the preimage $\pi^{-1}G \subset \mathbb{R}^n$ where $\pi: \mathbb{R}^n \to \mathbb{R}^n/\Lambda$ denotes projection. Let $\tilde{H} \subset \mathbb{R}^n$ be its $\mathbb{R}$-linear span. Then $H = \pi(\tilde{H})$ is a syndetic hull for G in $\mathbb{R}^n/\Lambda$.

1.11. For general nonabelian groups G, we proceed in two steps. Passing to a subgroup of finite index we may assume the commutator subgroup $G' = [G, G]$ is unipotent. For unipotent groups the algebraic hull is the unique syndetic hull [33, 2.6]. Now G' is Zariski dense in $A(G)'$ (Lemma 1.12) so G maps into the abelian group $A(G)/A(G')$ by the projection $p: A(G) \to A(G)/A(G')$. We form the syndetic hull Q for $p(G)$ in $A(G)/A(G')$; we claim that $H = p^{-1}Q$ is a syndetic hull for G.

The proofs of all these claims involve the fact that $A(G)' = A(G')$. The following proof of this fact was given us by Hochschild (see also Mostow [31, Lemma 6.3]).

LEMMA 1.12. *For any subgroup G of a linear algebraic group let $A(G)$ denote its Zariski closure and $[G, G]$ its commutator subgroup. Then $A([G, G]) = [A(G), A(G)]$.*

Proof. Clearly $G \subset A(G)$ so that $[G, G] \subset [A(G), A(G)]$. Since the commutator subgroup of an algebraic group is Zariski closed, we have $A([G, G]) \subset [A(G), A(G)]$.

It remains to prove $[A(G), A(G)] \subset A([G, G])$. We use the following result proved in Hochschild [21, sect. 5, exercise 2]: for every algebraic group H there exists a number $n > 0$ such that every element of the group $[H, H]$ can be written as an n-fold product of commutators $[a, b] = a^{-1}b^{-1}ab$. Hence $[H, H]$ is the image of the polynomial map $p: H \times \cdots \times H \to H$, $p(a_1, b_1, \dots, a_n, b_n) = [a_1, b_1] \cdots [a_n, b_n]$. Applying this to $A(G) = H$ we see that $[G, G]$ is generated by $p(G \times \cdots \times G)$. Now $A([G, G])$ is a Zariski closed subgroup containing $[G, G]$ and $p^{-1}(A([G, G]))$ is a Zariski closed subgroup containing $G \times \cdots \times G$; hence $p^{-1}(A([G, G])) \supset A(G) \times \cdots \times A(G)$. It follows that $A([G, G]) \supset [A(G), A(G)]$.   Q.E.D.

1.13. Lemma 1.12 implies that $A(G)/A(G')$ is abelian. As described in 1.10 there exists a syndetic hull Q for the projection $p(G)$ of G in $A(G)/A(G')$. We prove that $H = p^{-1}Q$ is a syndetic hull for G.

Condition (i) of 1.6 is clearly satisfied and allows the passage to subgroups of finite index. In the two special cases that G is abelian or unipotent, condition (ii) is satisfied and it is easy to deduce the general case from these two special cases (since every solvable matrix group is a finite extension of an abelian extension of a unipotent group).

To prove (iii), we may quickly reduce to the case when G is an abelian group since connected unipotent groups are automatically Zariski-closed. Furthermore we also assume that G has no nontrivial unipotent subgroups. This implies that G maps faithfully under the projection of $A(G)$ to a maximal reductive subgroup of $A(G)$. Thus it will suffice to consider the case that G is an abelian subgroup of $GL(V)$ which acts reductively. Passing to a subgroup of finite index, we may assume that G preserves a direct sum decomposition $V = (\oplus_{i=1}^{r} \mathbb{R}_i) \oplus (\oplus_{j=1}^{s} \mathbb{C}_j)$ where each $\mathbb{R}_i$ is a 1-dimensional subspace and each $\mathbb{C}_j$ is a 2-dimensional subspace with an invariant complex structure. By further passing to subgroups of finite index we may assume that the image of G in $GL(\mathbb{R}_i) \cong \mathbb{R}^*$ lies in the subgroup $\mathbb{R}^+$ with positive eigenvalue, and that the image of G in $GL(\mathbb{C}_j) \cong \mathbb{C}^*$ does not lie in a finite extension of the "real subgroup" $\mathbb{R}^* \subset \mathbb{C}^*$ (for otherwise we could reduce the number s of complex factors).

Given these reductions one can easily prove that if a nontrivial $g \in G$ lies on a one-parameter subgroup $\{\exp(tX): t \in \mathbb{R}\}$ where each $\exp(tX)$ preserves the decomposition of V into $\mathbb{R}_i$'s and $\mathbb{C}_j$'s, then the whole one-parameter subgroup lies in the Zariski closure of the cyclic group $\{g^n: n \in \mathbb{Z}\}$. If S is a generating set for G, then H is generated by one-parameter subgroups as above containing $g \in S$. It follows that $H \subset A(G)$, proving (iii).

We now prove (iv). Assume that G is closed. When G is abelian $\dim H \leqslant$ rank G is clear from 1.10. Suppose now that G is unipotent. Then G normalizes its identity component G^0 and G/G^0 is a discrete unipotent subgroup of the algebraic group Normalizer $(G^0)/G^0$. For discrete unipotent

groups, the dimension of the syndetic (= algebraic) hull equals the rank of the discrete group. It follows that for G a closed unipotent subgroup of $GL(V)$ the dimension of H equals the rank of G. Thus (iv) is true whenever G is unipotent or abelian.

In the general case, (iv) follows immediately from the *additive property of rank*: If $G \subset GL(V)$ may be written as an extension $U \to G \to A$ where U is unipotent and A is abelian, then rank $G = $ rank $U + $ rank A.

1.14. Now we turn to the discussion of uniqueness in 1.6 and 1.4. Assume that the elements of G have all eigenvalues real. Let H be a subgroup of $GL(V)$ containing G satisfying (i), (ii), (iii) of 1.6. If $G_i \supset G$ as a subgroup of finite index, then $H \cdot G_1$ is the unique syndetic hull for G_1 which contains H; conversely if G_1 is a subgroup of finite index in G, then the unique syndetic hull for G_1 which is contained in H is the union of components meeting G_1. For unipotent groups the syndetic hull is uniquely determined because it is the algebraic hull. By passing to quotients by unipotent groups we may reduce once again to closed abelian groups which act reductively. For abelian groups G the only ambiguity in defining the syndetic hull (1.10) arises when G lies in a group which is not simply connected; the only case in which this cannot be avoided occurs when some eigenvalues are not real. This concludes the proof of 1.6.

1.15. Unless the elements of $L(\Gamma)$ have all eigenvalues real (or at least up to subgroups of finite index), there will be infinitely many possible hulls $\mathcal{H}(\Gamma)$. For a simple example let $\Gamma \subset \mathrm{Aff}(R^3)$ be the cyclic group consisting of the affine transformations

$$\begin{bmatrix} r^n \cos n\theta & -r^n \sin n\theta & 0 \\ r^n \sin n\theta & r^n \cos n\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix},$$

for $h \in \mathbb{Z}$, where $\theta/2\pi$ is irrational and $r > 0$. Then for any integer k, the group of all

$$\begin{bmatrix} r^t \cos t(\theta + 2k\pi) & -r^t \sin t(\theta + 2k\pi) & 0 \\ r^t \sin t(\theta + 2k\pi) & r^t \cos t(\theta + 2k\pi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix},$$

for $t \in R$, is a possible H. It is easy to check that Γ acts properly discontinuously and freely but is not crystallographic. Note that if $r \neq 1$, then Γ does not preserve volume, in contrast to the situation (1.22) for crystallographic groups.

Here is an example of a crystallographic group which does not have a unique crystallographic hull. This example will be 4-dimensional, for we

shall see that there are no such examples in lower dimensions. Let $A \in SL(3; \mathbb{Z})$ be an integral matrix having only one real eigenvalue and being of infinite order. Form the semidirect product $\Gamma = \mathbb{Z}^3 \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on the normal subgroup $\mathbb{Z}^3$ by the automorphism A. Define an affine action $\Gamma \to \mathrm{Aff}(R^4)$ as follows. Let $\mathbb{Z}^3$ act by translations in three linearly independent directions. Let the $\mathbb{Z}$ factor act by the linear transformation A in the 3-dimensional subspace spanned by $\mathbb{Z}^3$ and by a nontrivial translation in the fourth direction. It is straightforward to check that Γ is crystallographic (indeed it even acts freely). Just as there are infinitely many matrices $B \in \mathfrak{sl}(3, R)$ with $A = \exp B$, there will be infinitely many crystallographic hulls H.

1.16. Although the crystallographic hulls H are in general not unique, the algebraic hulls $A(\Gamma)$ do enjoy a rather strong uniqueness property:

THEOREM 1.17. *Let $\Gamma_i \subset \mathrm{Aff}(E)$ ($i = 1, 2$) be virtually solvable affine crystallographic groups. Then every isomorphism $\phi: \Gamma_1 \to \Gamma_2$ extends to an isomorphism $\hat{\phi}: A(\Gamma_1) \to A(\Gamma_2)$ of the algebraic hulls (as algebraic groups). In particular $\hat{\phi}$ defines an isomorphism between the unipotent radicals of the $A(\Gamma_i)$ and $\hat{\phi}$ takes every crystallographic hull of Γ_1 to a crystallographic hull of Γ_2.*

The proof of Theorem 1.17 is based upon the following general theorem in Raghunathan [33, Lemma 4.41]. Let Γ_i be polycyclic groups embedded in algebraic groups A_i such that the following conditions are satisfied:

   (i) Γ_i is Zariski dense in A_i;

   (ii)   the unipotent radical U_i of A_i has dimension equal to the rank of Γ_i;

   (iii) the centralizer of U_i in A_i equals the center of U_i.

Then every isomorphism $\phi: \Gamma_1 \to \Gamma_2$ extends to an isomorphism $\hat{\phi}: A_1 \to A_2$ of algebraic groups. (In the language of [33], A_i is an "$\mathbb{R}$-algebraic hull" of the abstract polycyclic group Γ_i.) Therefore to prove Theorem 1.17 it is simply necessary to verify conditions (ii) and (iii) above for $A_i = A(\Gamma_i)$. These are both consequences of the following beautiful result of Auslander:

PROPOSITION 1.18 (Auslander [5, Sect. 3]). *Let $G \subset \mathrm{Aff}(E)$ act simply transitively on E. Then the unipotent radical of $A(G)$ acts simply transitively on E as well.*

To verify (ii), let $G_i = H_i^0$. Passing to subgroups of finite index we may assume $A(G_i) = A(\Gamma_i)$. Since U_i acts simply transitively on E, $\dim U_i =$

dim $E =$ dim $G_i =$ rank Γ_i, proving (ii). To verify (iii), suppose $g \in A(G_i)$
centralizes U_i. The centralizer of U_i in $A(G_i)$ is an algebraic group and
therefore contains the semisimple part $g^{(s)}$ of any of its elements g. If g is not
unipotent, then $g^{(s)}$ is nontrivial and therefore fixes a nonempty proper affine
subspace E_1 of E. But since $g^{(s)}$ centralizes U_i, the subspace E_1 is U_i-
invariant, contradicting the transitivity of U_i. We conclude that g is
unipotent. But U_i is precisely the set of unipotent elements of $A(G_i)$, so
$g \in U_i$. But this means that g is in the center of U_i, proving (iii).

It follows from Lemma 4.41 of [33] that isomorphisms $\phi: \Gamma_1 \to \Gamma_2$ extend
to isomorphisms $\hat{\phi}: A(\Gamma_1) \to A(\Gamma_2)$. Clearly any isomorphism of algebraic
groups preserves unipotent radicals. Similarly it is obvious from the
construction of the crystallographic hulls that $\hat{\phi}$ preserves crystallographic
hulls.

This concludes the proof of 1.17.   Q.E.D.

Polynomial Deformations

1.19. We shall now apply 1.17 to understanding the various defor-
mations of a virtually solvable affine crystallographic group. To motivate
this discussion, we consider the case of crystallographic groups isomorphic
to $\mathbb{Z}^2$. (For a complete discussion of the 2-dimensional crystallographic
groups isomorphic to $\mathbb{Z}^2$, see Section 3.) We shall only quote results here, all
of which have been known previously; see 3.3 for references.)

Every crystallographic affine $\mathbb{Z}^2$-action embeds in a simply transitive
affine $\mathbb{R}^2$-action. Of course there is a "natural" simply transitive $\mathbb{R}^2$-action
by translations:

$$T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} : (s, t \in \mathbb{R}) \right\}.$$

Somewhat surprisingly, there is another simply transitive affine $\mathbb{R}^2$-action:

$$H = \left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s + t^2/2 \\ t \end{bmatrix} : (s, t \in \mathbb{R}) \right\}.$$

If f denotes the polynomial mapping $\mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x + y^2/2, y)$, then
it is easy to check that f conjugates the action T into the action H. From this
it is easy to prove that any two affine crystallographic actions of $\mathbb{Z}^2$ are
conjugate by a polynomial mapping (which will be quadratic).

We therefore define a *polynomial automorphism* of a vector space E to be
a diffeomorphism $f: E \to E$ such that both f and f^{-1} are polynomial
mappings. If G_1 and G_2 are groups and E_i is a G_i-space $(i = 1, 2)$, we say
that a homeomorphism $f: E_1 \to E_2$ *induces* an isomorphism $F: G_1 \to G_2$ if for
each $g \in G_1$ the following diagram commutes:

$$E_1 \xrightarrow{\ f\ } E_2$$
$$g \downarrow \qquad \downarrow F(g) \ .$$
$$E_1 \xrightarrow{\ f\ } E_2$$

THEOREM 1.20.   *Let* $\Gamma_i \subset \mathrm{Aff}(E)$ $(i = 1, 2)$, *be virtually solvable affine crystallographic groups. Let* $F: \Gamma_1 \to \Gamma_2$ *be an isomorphism. Then there exists a polynomial automorphism* f *of* E *which induces* F.

The proof of 1.20 uses the crystallographic and algebraic hulls of the affine crystallographic groups. The analogue of 1.20 for crystallographic hulls is the following:

THEOREM 1.21.   *Let* $G_i \subset \mathrm{Aff}(E)$ $(i = 1, 2)$ *act transitively on* E *with finite isotropy groups. Then every isomorphism* $F: G_1 \to G_2$ *is induced by a polynomial automorphism of* E.

An analogous statement is true when the groups G_i act transitively and properly (all isotropy groups compact), but this generalization will not concern us here.

Proof of 1.21.   First we assume that G_1 is a unipotent subgroup of $\mathrm{Aff}(E)$. Then G_1, and hence G_2, are nilpotent groups; by Scheuneman [35] or [15], this implies that G_2 is also unipotent. Let g_i be the Lie algebra of G_i and $\mathscr{F}: g_1 \to g_2$ be the isomorphism associated to F.

Fix an origin $0 \in E$ and let $u_i: G_i \to E$ be the evaluation map at 0. By [15, Lemma 8.2], each map $u_i \circ \exp: g_i \to E$ is a polynomial isomorphism with inverse $\log \circ w_i: E \to g_i$.

$$g_i \underset{\log}{\overset{\exp}{\rightleftarrows}} G_i \underset{w_i}{\overset{u_i}{\rightleftarrows}} E$$

Then clearly $f = u_2 \circ F \circ w_1 = (u_2 \circ \exp) \circ \mathscr{F} \circ (\log \circ w_1)$ is a polynomial automorphism of E. For any $g \in G_1$ and $x \in E$, we have

$$f(g(x)) = f \circ u_1(gh) \qquad \text{(writing } h = w_1(x))$$
$$= u_2 \circ F(gh) = u_2(F(g) F(h))$$
$$= F(g) u_2(F(h)) = F(g) f(x)$$

so that f induces F as desired.

Now we prove the general case of 1.21. By arguments similar to 1.17, F extends to an isomorphism $\bar{F}: A(G_1) \to A(G_2)$ of algebraic hulls. Decompose $A(G_i) = U_i \rtimes T_i$ where U_i is the unipotent radical and the T_i are reductive groups satisfying $\bar{F}(T_1) = T_2$. By Auslander's theorem (1.18) each U_i acts

simply transitively so by the argument above, there exists a polynomial automorphism $f: E \to E$ inducing $\bar{F}: U_1 \to U_2$. We claim that f induces $\bar{F}: A(G_1) \to A(G_2)$ as well, i.e., for all $g \in A(G_1)$, $x \in E$,

$$f(gx) = \bar{F}(g)f(x). \tag{*}$$

Clearly it suffices to prove (*) for $g \in T_1$ since (*) is known for $g \in U_1$ and $A(G_1) = U_1 T_1$. For each $g \in T_1$ let $J_g: E \to E$ be the map defined by $J_g(x) = f^{-1} \circ \bar{F}(g^{-1})f(gx)$. Since T_1 is reductive it fixes a point $x_0 \in E$; then $gx_0 = x_0$ and $\bar{F}(g)f(x_0) = f(x_0)$, so $J_g(x_0) = x_0$. Moreover if $u \in U_1$, then $J_g(ux) = f^{-1} \circ \bar{F}(g^{-1})f(gux) = f^{-1} \circ \bar{F}(g^{-1}) \circ \bar{F}(gug^{-1})f(gx) = f^{-1} \circ \bar{F}(u) \circ \bar{F}(g^{-1})f(gx) = uJ_g(x)$; thus J_g centralizes U. Taking $x = ux_0$ these two facts together imply $J_g(x) = uJ_g(x_0) = x$. Thus J_g is the identity and (*) is proved. Therefore f induces the isomorphism $\bar{F}: A(G_1) \to A(G_2)$ as desired. In particular f induces the isomorphism $F: G_1 \to G_2$.   Q.E.D.

Proof of 1.20. By 1.17, F extends to an isomorphism $\bar{F}: A(\Gamma_1) \to A(\Gamma_2)$. Taking $G_1 \subset A(\Gamma_1)$ to be a crystallographic hull of Γ_1, its image $G_2 = \bar{F}(G_1)$ is a crystallographic hull of Γ_2. By 1.21 $\bar{F}: G_1 \to G_2$ is induced by a polynomial automorphism f of E; thus f induces the given isomorphism $F: \Gamma_1 \to \Gamma_2$ by restriction.   Q.E.D.

Polynomial Cohomology

1.22. Theorem 1.15 implies that any general statement involving polynomials, etc., true for a virtually solvable affine crystallographic group Γ must be valid for any affine crystallographic group isomorphic to Γ. An example of such a statement is the following, proved in Goldman [16]:

THEOREM. *Let M be a compact complete affine manifold with virtually solvable fundamental group. Then the de Rham cohomology of M is computable from the complex of differential forms on M whose coefficients are polynomial functions of the affine coordinates.*

For some specific examples, see the Appendix.

Parallel Volume

1.21. Corollary 1.5 may be used to show that a virtually solvable affine crystallographic group which preserves orientation must preserve Euclidean volume. For a proof of this, see Goldman and Hirsch [17]. It follows that every compact complete affine manifold with virtually solvable fundamental group has a natural probability measure which is induced from Lebesgue measure on $\mathbb{R}^n$.

2. VIRTUAL SOLVABILITY

2.1. The goal of this section is to prove that certain properly discontinuous groups of affine transformations contain solvable subgroups of finite index. We summarize the results of this section as follows:

THEOREM 2.1. *Suppose that $\Gamma \subset \mathrm{Aff}(E)$ acts properly discontinuously on E, where E is a vector space. Then Γ has a solvable subgroup of finite index whenever one of the following cases occurs*:

(a) $\dim E = 2$;

(b) $\dim E = 3$ *and E/Γ is compact*;

(c) $\dim E = 3$ *and no subgroup of finite index of $L(\Gamma)$ preserves a Lorentzian inner product on E.*

Later, in Section 5, we shall see, in the above cases (a)–(c), that Γ is indeed solvable.

We begin with an elementary consequence of the definition of solvability and the structure of $\mathrm{Aff}(E)$.

LEMMA 2.2. *A subgroup $G \subset \mathrm{Aff}(E)$ is solvable if and only if $L(G) \cap SL(E)$ is solvable.*

The next fact was first observed in this context by Hirsch:

LEMMA 2.3. *If G is a subgroup of $\mathrm{Aff}(E)$ which acts freely on E, then every $L(g) \in L(G)$ has 1 as an eigenvalue.*

Proof. As G acts freely, $gx \neq x$ for all $x \in E$. But if $L(g)$ does not have 1 as an eigenvalue, then $L(g) - I$ is invertible and we may easily solve for a fixed point of g.   Q.E.D.

COROLLARY 2.4. *If $G \subset \mathrm{Aff}(E)$ acts freely on E, then for all $g \in A(G)$, the linear part $L(g)$ has 1 as an eigenvalue.*

Proof. The condition that $L(g)$ has 1 as an eigenvalue is the polynomial condition $\det(L(g) - I) = 0$. Hence it holds for $g \in A(G)$ as well.   Q.E.D.

This corollary is due to Kostant and Sullivan; see [24].

2.5. The next ingredient in the proof is the Levi decomposition (see, e.g., [22]). Recall that every Lie algebra $\mathscr{g}$ may be decomposed as a semidirect sum of its unique maximal solvable ideal (its radical) and any maximal semisimple subalgebra, called its *Levi factor*. If G is a Lie group with Lie algebra $\mathscr{g}$, then the subgroup corresponding to the Levi factor of $\mathscr{g}$

will be called the *Levi factor* of *G*. If *G* is a subgroup of Aff(*E*) then its Levi factor is conjugate to a subgroup of *SL*(*E*).

LEMMA 2.6. (i) *There is no proper semisimple connected subgroup of* $SL(2, R)$.

(ii) *The only proper semisimple connected subgroups of* $SL(3, R)$ *are the orthogonal groups* $SO(3)$ *and* $SO(2, 1)^0$ *and the subgroup* $SL(2, R) \times \{1\}$.

The proof is an easy consequence of the classification of simple Lie algebras.

Proof of 2.1(a). We must show that if $\Gamma \subset \text{Aff}(\mathbb{R}^2)$ acts properly discontinuously, then Γ has a solvable subgroup of finite index. By 1.2 we may replace Γ by a subgroup of finite index in order to assume that Γ acts freely. Since the identity component $A(\Gamma)^0$ has finite index in $A(\Gamma)$, we may further pass to a subgroup of finite index to assume that Γ lies in the identity component of its algebraic hull.

Let *S* be the Levi factor of $A(\Gamma)^0$. By 2.6 either *S* is trivial (in which case $A(\Gamma)^0$ is solvable) or $S = SL(2, R)$. But 2.4 implies that every element of $S \subset A(\Gamma)^0$ has 1 as an eigenvalue. Thus $S \neq SL(2, R)$ and Γ is solvable.

<div align="right">Q.E.D.</div>

2.7. Now we turn to properly discontinuous groups of affine motions of $\mathbb{R}^3$. For the remainder of this section *E* will denote $\mathbb{R}^3$ and Γ will denote a properly discontinuous subgroup of Aff(*E*). Since we shall in this section only be concerned with Γ *up to subgroups of finite index*, we henceforth assume that Γ acts freely on *E*, and also that $\Gamma \subset A(\Gamma)^0$. We will show that $A(\Gamma)^0$ must be solvable.

Let *S* denote the Levi factor of $A(\Gamma)^0$. By 2.4, *S* must be a proper subgroup of *SL*(*E*). If *S* is conjugate to *SO*(3), it is not hard to see that the only connected subgroups of *SL*(*E*) which has *S* as its Levi factor are *S* and the product $S \cdot \mathbb{R}_+$ of *S* with the group of (positive) scalar matrices. The case $S \cdot \mathbb{R}_+$ is ruled out by 2.4. Thus $L(A(\Gamma)^0) = SO(3)$, up to conjugacy. In that case Γ is a discrete group of Euclidean isometries, and the classical Bieberbach theorems (see [2] or [42, sect. 3]) imply that Γ contains an abelian subgroup of finite index.

2.8. We next consider the case $S = SL(2, \mathbb{R}) \times \{1\}$. If *G* is a connected subgroup of $SL(3, \mathbb{R})$ whose Levi factor is *S*, then it is not hard to see that *G* may be conjugated to lie in one of the following groups such that projection to $GL(2, \mathbb{R})$ maps onto at least $SL(2, \mathbb{R})$:

$$\left(\begin{array}{cc|c} & & * \\ GL(2,\mathbb{R}) & & * \\ \hline 0 & 0 & * \end{array}\right), \qquad \left(\begin{array}{cc|c} & & 0 \\ GL(2,\mathbb{R}) & & 0 \\ \hline * & * & * \end{array}\right).$$

Moreover if every element of G is to have 1 as an eigenvalue, then as G contains $SL(2,\mathbb{R})$ the last diagonal entry must be identically 1. It follows that $L(A(\Gamma)^0)$ must lie in one of the groups:

$$\left(\begin{array}{cc|c} & & * \\ GL(2,\mathbb{R}) & & * \\ \hline 0 & 0 & 1 \end{array}\right), \qquad \left(\begin{array}{cc|c} & & 0 \\ GL(2,\mathbb{R}) & & 0 \\ \hline * & * & 1 \end{array}\right).$$

In the first case the linear functional $(x, y, z) \to z$ is invariant under $L(\Gamma)$ so the parallel 1-form dz on E is invariant under Γ. We define a homomorphism $\psi: \Gamma \to R$ by the rule

$$\psi(\gamma) = z(\gamma(p)) - z(p) \qquad \text{for} \quad p \in E;$$

since dz is Γ-invariant ψ is independent of the choice of p and indeed defines a homomorphism. The kernel of ψ is precisely the set of all $\gamma \in \Gamma$ which leave some (and hence every) plane $z(p) = z_0$ invariant. Therefore Ker ψ acts on such planes, and clearly the action is properly discontinuous and affine. It therefore follows from part (a) of Theorem 2.1 that Ker ψ has a solvable subgroup of finite index. Since Γ is an extension of Ker ψ by a subgroup of $\mathbb{R}$, it must itself contain a solvable subgroup of finite index.

2.9. Now we consider the second case, i.e. when the last column of $L(\Gamma)$ is

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

In that case the vector

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is invariant under $L(\Gamma)$, and the vector field $\partial/\partial z$ parallel to it is invariant under Γ. Let M denote the quotient E/Γ and let ξ denote the parallel vector on M induced by $\partial/\partial z$. Since $\partial/\partial z$ is completely integrable on E the vector field ξ is completely integrable on M. Let $\{\xi_t\}_{t \in \mathbb{R}}$ denote the corresponding flow. We shall need to make use of the following elementary fact:

LEMMA 2.10. *Suppose that G is a subgroup of $GL(2, \mathbb{R})$ which is not solvable. Then there exist elements $g_i \in G \cap SL(2, \mathbb{R})$ $(i = 1, 2)$, such that g_1 and g_2 are hyperbolic (i.e., have real distinct eigenvalues) and such that they share no common eigenspace.*

Proof. We may easily reduce to the case $G \subset SL(2, \mathbb{R})$. The condition on hyperbolic elements g_i that they share no common eigenspace means that they generate a nonsolvable group. The representation of G on $\mathbb{R}^2$ is irreducible (otherwise G would be solvable), so an argument using the Burnside theorem as in Conze–Guivarch [10] implies that G contains some hyperbolic element g_1. It also follows from irreducibility that there exists some $h \in G$ which is either elliptic (has complex conjugate eigenvalues) or whose set of eigenspaces is disjoint from that of g_1. If h is hyperbolic, take $g_2 = h$. Otherwise for sufficiently large n, $g_2 = (g_1)^n h$ will be hyperbolic satisfying the desired property.   Q.E.D.

We now return to studying the affine manifold M and the parallel vector firld ξ. Suppose that Γ is not solvable; then by Lemma 2.2 and Lemma 2.10 there exist $\gamma_i \in \Gamma$ $(i = 1, 2)$ such that the elements $g_i \in GL(2, \mathbb{R})$ defined by

$$L(\gamma_i) = \begin{pmatrix} & & 0 \\ g_i & & 0 \\ * & * & 1 \end{pmatrix}$$

satisfy the conclusion of Lemma 2.9.

As g_i is hyperbolic, γ_i leaves invariant a unique line ℓ_i in E which is parallel to $\partial/\partial z$. Thus the image of ℓ_i in M is a closed orbit σ_i of ξ. As the Poincaré map around σ_i is $g_i^{\pm 1}$, the closed orbit is hyperbolic (in the sense of a smooth dynamical system—see [38]) for ξ. Let W_i denote the stable manifold $\{m \in M : \xi_t(m) \to \sigma_i$ as $t \to +\infty\}$. By first passing to the covering space $E/\{\gamma_i^n : n \in \mathbb{Z}\}$ we see that the W_i are covered by planes P_i in E parallel to the $(e_i, \partial/\partial z)$-plane where e_i is an eigenvector of $L(\gamma_i)$ with eigenvalue not equal to 1. Since e_1 and e_2 are not parallel, $P_1 \cap P_2$ is a line m parallel to $\partial/\partial z$.

Choose a complete Riemannian metric on M and lift it to E. Let N_i be the ε-tubular neighborhood of ℓ_i in E. Clearly N_i is invariant under γ_i. It follows that N_i contains the positive rays of m (see Fig. 1). Thus $\text{dist}(\sigma_1, \sigma_2) \leqslant \text{dist}(\ell_1, \ell_2) \leqslant 2\varepsilon$. As ε is arbitrary and the σ_i are compact, we have $\sigma_1 = \sigma_2$.
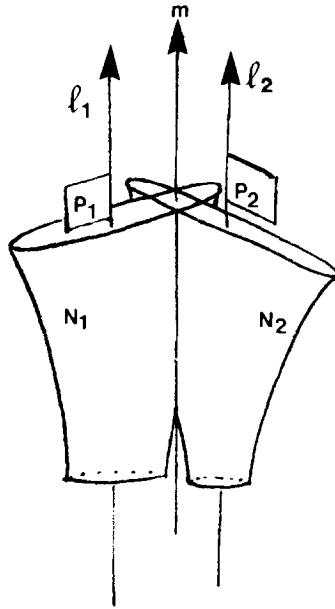
FIG. 1.   Stable manifolds of an impossible parallel flow.

As the lifts ℓ_1, ℓ_2 of σ_1 are zero distance apart, they are equal. The subgroup S generated by γ_1 and γ_2 acts on ℓ_1 by translation and is properly discontinuous. Hence S is cyclic, violating our condition on the eigenspaces of the γ_i.

We have now eliminated the case $S = SL(2, \mathbb{R}) \times \{1\}$.

2.11. By Lemma 2.6 we are reduced to considering $S = SO(2, 1)^0$. The only connected proper subgroup of $SL(3, \mathbb{R})$ containing $SO(2, 1)^0$ is the product $SO(2, 1)^0 \cdot \mathbb{R}_+$, which contains elements not having 1 as an eigenvalue. Therefore we may assume that $L(\Gamma) \subset SO(2, 1)$, for the last remaining case. We have, however, completed the proof of part (c) of Theorem 2.1.

2.12. At this point we shall require that E/Γ be compact. This assumption, however, will only be used in case (iii) below. Let $\Gamma \subset \mathrm{Aff}(E)$ act properly discontinuously and freely on E and suppose that $L(\Gamma) \subset SO(2, 1)$. We shall analyze the homomorphism $L: \Gamma \to SO(2, 1)$. The analysis splits naturally into three separate cases:

   (i) $L(\Gamma)$ is not discrete in $SO(2, 1)$;

   (ii) $L: \Gamma \to SO(2, 1)$ is not injective;

   (iii) $L: \Gamma \to SO(2, 1)$ is an isomorphism onto a discrete subgroup.

In case (i) the identity component $\overline{(L(\Gamma))}^0$ of the closure of $L(\Gamma)$ is nontrivial and evidently normalized by $L(\Gamma)$. However a result of Auslander [3] (see also Raghunathan [33, Theorem 8.24]) states that for *any* discrete subgroup $\Gamma \subset \mathrm{Aff}(E)$ the group $\overline{(L(\Gamma))}^0$ is solvable. Hence $L(\Gamma)$ normalizes a nontrivial solvable subgroup of $SO(2, 1)$. It is easy to prove that such a group must be solvable.

In case (ii) we use the fact that any subgroup of $SO(2, 1)$ which leaves invariant a 1- or 2-dimensional linear subspace must be solvable. Applying this to the subspace spanned by $T = \mathrm{Ker}\, L = \{\text{translations in } \Gamma\}$ we see that if $L(\Gamma)$ is not solvable, then Γ contains three linearly independent translations. It follows that E/Γ is covered by the 3-torus E/T. Since E/T is compact it follows that T has finite index in Γ.

In case (iii) we note that since Γ is the fundamental group of a closed aspherical 3-manifold the (virtual) cohomological dimension of Γ equals 3. On the other hand, $H/L(\Gamma)$ is a smooth surface of constant negative curvature, where H is the hyperbolic plane upon which $SO(2, 1)$ acts by isometries. Thus the cohomological dimension of Γ is strictly less than 3. This contradiction completes the proof of Theorem 2.1.

2.13. Combining Theorem 2.1(b) with 1.4 we obtain:

THEOREM. *Let* $\Gamma \subset \mathrm{Aff}(\mathbb{R}^3)$ *be an affine crystallographic group. Then there exists a crystallographic hull* $G = H(\Gamma)$, *i.e., a Lie group with finitely many components, each component meeting* Γ, *and the identity component* G^0 *acting simply transitively on* E.

Geometrically, this says that *every compact complete affine 3-manifold is finitely covered by a compact complete affine solvmanifold* $\Gamma \backslash G$.

3. Simply Transitive Affine Actions of Low-Dimensional Nilpotent Groups

Let G be a 1-connected nilpotent Lie group of dimension 2 or 3. In this section we classify all simply transitive affine actions of G. We shall view a simply transitive affine action of G as a *left-invariant complete affine structure on* G. The classification of such structures on G can then be reduced to a problem involving the Lie algebra $\mathscr{g}$ of G, namely, the classification of *complete affine structures on the Lie algebra* $\mathscr{g}$.

Our initial restriction to nilpotent G is quite natural for two reasons. First, the results of this section will enable us to classify (in Section 5) the many affine crystallographic groups in these dimensions which are virtually nilpotent. Second, the general case of a simply transitive $G \subset \mathrm{Aff}(E)$ can best

be understood in terms of the action of the unipotent radical $UA(G)$ of the algebraic hull of G, which by a theorem of Auslander's ([5], see also the discussion in 1.18) is simply transitive.

3.1. We begin by linearizing the problem. Suppose $G \subset \text{Aff}(E)$ is a simply transitive affine action of G on E. Thus for every $x \in E$, the orbit $Gx = E$ is a diffeomorphic copy of G, under the evaluation map $G \to E$ at x. When $x = 0$, the evaluation map is just the translational part of the action of G on E. In general this evaluation map may be replaced by the mapping $u_x(g) = g(x) - x$; this map represents the "parallel displacement" of g acting on x, and has the useful property that $u_x(e) = 0$.

It is easy to see that G acts simply transitively of E if and only if for all $x \in E$ the map $u_x \colon G \to E$ has nonsingular derivative at e (because all orbits are then open implying G acts transitively). We can identify the tangent space $T_e G$ with the Lie algebra $\mathscr{g}$ of *right-invariant* vector fields on G. By identifying the tangent spaces $T_x E$ with E by parallel translations, the differential of u_x gives rise to a map $\alpha_x \colon \mathscr{g} \to E$. If $\alpha \colon \mathscr{g} \to a\!f\!f(E)$ is the inclusion of Lie algebras associated to $G \subset \text{Aff}(E)$, then α_0 is the translational part of α. It is not difficult to see that, for $Y \in \mathscr{g}$,

$$\alpha_x(Y) = \alpha_0(Y) + L\alpha(Y)(x),$$

where $L\alpha \colon \mathscr{g} \to \mathscr{gl}(E)$ denotes the linear part of α.

Remark. One may understand the Lie algebra $\alpha(\mathscr{g})$ as right-invariant vector fields in the following way. Any evaluation map $u_x \colon G \to E$ is a diffeomorphism, and thus defines on G an affine structure coming from E. The map u_x is G-equivariant where G acts on itself by left-translations and G acts on E by $G \subset \text{Aff}(E)$. For the elements of $\alpha(\mathscr{g})$ are the infinitesimal generators of $G \subset \text{Aff}(E)$; right-invariant vector fields are infinitesimal left-multiplications. Since the action of G by left-translations on itself is u_x-related to a group of affine transformations, this affine structure is *left-invariant*. (See [15, 17, 20, 27, 29] for more information.)

This suggests the following definition. Let $\mathscr{g}$ be a Lie algebra and E a vector space of the same dimension as $\mathscr{g}$. An *affine structure* on $\mathscr{g}$ is defined by a representation $\alpha \colon \mathscr{g} \to a\!f\!f(E)$ of Lie algebras, such that the translational part of α is a linear isomorphism $\mathscr{g} \to E$ of vector spaces. Two such representations define the same affine structure if they are conjugate under some affine transformation $P \in \text{Aff}(E)$, i.e., $\alpha' = (\text{Ad } P) \circ \alpha$. We say the affine structure is *complete* if for a fixed α, and any $x \in E$, the map $\alpha_x \colon \mathscr{g} \to E$ defined by $\alpha_x(Y) = \alpha(Y) + L\alpha(Y)x$ (where α and $L\alpha$ are the translational and linear parts, respectively, of α) is a linear isomorphism for all $x \in E$.

In summary, there are canonical bijections between the following classes of objects:

$$\left\{\begin{array}{l}\text{left-invariant affine structures}\\ \text{on } G, \text{up to affine equivalence}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{affine}\\ \text{structures on } \mathscr{g}\end{array}\right\}$$

$$\leftrightarrow \left\{\begin{array}{c}\text{locally simply transitive affine actions}\\ \text{of } G, \text{up to conjugation in Aff}(E)\end{array}\right\},$$

$$\left\{\begin{array}{c}\text{left-invariant complete}\\ \text{affine structures on } G\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{complete affine}\\ \text{structures on } \mathscr{g}\end{array}\right\}$$

$$\leftrightarrow \left\{\begin{array}{c}\text{simply transitive}\\ \text{affine actions of } G\end{array}\right\}.$$

For the definition of affine structures (in general), and many more details, the reader is referred to [15, 29, 20 and 27].

3.2. Now we specialize to the case when G is a nilpotent Lie group. Then we have the following basic theorem first proved in Scheuneman [35] (see also [15]):

PROPOSITION 3.3. *If G is a group of affine automorphisms of E which acts simply transitively on E and G is nilpotent, then G acts unipotently. In particular there exists a basis of E in which $L(G)$ is represented by upper unitriangular matrices.*

In terms of the Lie algebra $\mathscr{g}$, the condition that $G \subset \text{Aff}(E)$ is unipotent means that $\mathscr{L}(\mathscr{g}) \subset \mathscr{g\ell}(E)$ consists of nilpotent endomorphisms of E. It follows from Engel's theorem that $\mathscr{L}(\mathscr{g})$ generates a nilpotent associative subalgebra of $\mathscr{g\ell}(E)$ and that in some basis $\mathscr{L}(\mathscr{g})$ is represented by upper triangular nilpotent matrices.

It follows from the fact that orbits of connected unipotent subgroups G of $\text{Aff}(G)$ are closed (see [7, 34]) that G acts transitively on E if and only if some orbit is open (this fact also follows immediately from Theorem 6.8 of [15]). This simplifies the criterion for an action $G \subset \text{Aff}(E)$ to be simply transitive when G is nilpotent: G acts simply transitively if and only if $L(\mathscr{g})$ generates a nilpotent subalgebra of $\mathscr{g\ell}(E)$ and the translational part $u: \mathscr{g} \to E$ is a linear isomorphism. Thus *a complete affine structure on a nilpotent Lie algebra $\mathscr{g}$ consists of an affine structure on $\mathscr{g}$ such that the corresponding linear representation $\mathscr{g} \to \mathscr{g\ell}(E)$ is nilpotent* (Scheuneman [35, sect. 1; 15, Theorem 6.8]).

THEOREM 3.4 (Kuiper [25]). *Let $E = \mathbb{R}^2$. Suppose that $G \subset \text{Aff}(E)$ is a*

simply transitive unipotent action on $\mathbb{R}^2$. *Then* G *is conjugate in* $\mathrm{Aff}(E)$ *to one of the two subgroups*:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \right\}, \qquad \left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s + t^2/2 \\ t \end{bmatrix} \right\}$$

where $(s, t) \in \mathbb{R}^2$.

In the first case G is the group of translations and preserves a Euclidean metric on E. In the other case G does not preserve any Riemannian metric which has the same set of geodesics as the underlying affine structure of E. Thus we call the first case *Euclidean* and the second case *non-Riemannian*.

Proof. The corresponding 2-dimensional Lie algebra $\mathscr{g}$ is a subalgebra of the algebra of all "upper triangular nilpotent affine endomorphisms"

$$\left\{ \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} : (r, x, y \in \mathbb{R}) \right\}$$

of $\mathbb{R}^2$. Under the projection $(r, x, y) \to (x, y)$, $\mathscr{g}$ maps isomorphically onto $\mathbb{R}^2$. Thus $r = r(x, y)$ is a linear function $r: \mathbb{R}^2 \to \mathbb{R}$. By rechoosing coordinates in $\mathscr{g}$ by $(x, y) \to (x + ky, y)$ we may assume that r is independent of x. Then $r = cy$ where $c \in \mathbb{R}$ is a constant. Therefore $\mathscr{g}$ has the form

$$\mathscr{h}_c = \left\{ \begin{bmatrix} 0 & cy \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\}.$$

All nonzero c give conjugate $\mathscr{h}_c$, so there are two types, depending on whether $c = 0$ or $c \neq 0$. Thus for the non-Riemannian structure $c \neq 0$ we can normalize with $c = 1$, obtaining

$$\mathscr{H}_1 = \exp \mathscr{h}_1 = \left\{ \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x + y^2/2 \\ y \end{bmatrix} \right\}. \qquad \text{Q.E.D.}$$

THEOREM 3.5. *Let* $E = \mathbb{R}^3$ *and suppose that* G *is a nilpotent Lie group which acts simply transitively by affine transformations on* E. *Then* G *is conjugate to one of the groups in the following classes*:

$(\mathscr{H}_\alpha)$ *Here* α *is a bilinear map* $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, *defined by a* 2×2 *matrix* $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. $\mathscr{H}_\alpha$ *is the group*

$$\left\{ \begin{bmatrix} 1 & a_{11}t + a_{12}u & a_{21}t + a_{22}u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s + (a_{11}t^2 + a_{22}s^2) \\ \quad + (a_{12} + a_{21})st)/2 \\ t \\ u \end{bmatrix} \right\}.$$

($\mathcal{H}_{(b,c)}$)   Here (b, c) is an ordered pair of real numbers. $\mathcal{H}_{(b,c)}$ is the group

$$\left\{ \begin{bmatrix} 1 & cu & bt + cu^2/2 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s + (b+c)tu/2 + u^3/6 \\ t + u^2/2 \\ u \end{bmatrix} \right\}.$$

In both cases (s, t, u) varies over $\mathbb{R}^3$. The conjugacy class of $\mathcal{H}_\alpha$ corresponds to the $GL(\mathbb{R}^2) \times GL(\mathbb{R})$-class of the bilinear map $\alpha: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$. The conjugacy class of $\mathcal{H}_{(b,c)}$ is determined by $(b, c) \in \mathbb{R}^2$ up to multiplication by positive real numbers. The only pairs of conjugate subgroups on this list are:

(1) $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ where β is obtained from α by changing bases in $\mathbb{R}^2$ and $\mathbb{R}$;

(2) $\mathcal{H}_{(b,c)}$ and $\mathcal{H}_{(ab,ac)}$ where $a > 0$.

$\mathcal{H}_\alpha$ is abelian if and only if α is a symmetric bilinear form. $\mathcal{H}_{(b,c)}$ is abelian if and only if $b = c$. If the group is not abelian, then it is isomorphic to the real 3-dimensional Heisenberg group.

3.6. The first step in the proof of 3.5 is the following result:

PROPOSITION.   If $G \subset \text{Aff}(\mathbb{R}^3)$ is nilpotent and acts simply transitively, then the center of G contains a nontrivial translation.

Remark.   This answers affirmatively an old conjecture of Auslander in dimension 3. In Scheuneman [35], an incorrect proof of this proposition is given, wich claims the result in all dimensions. In Fried [12] an example is given of a simply transitive unipotent affine action on $\mathbb{R}^4$ which contains no central translation.

Proof.   Suppose that G contains no nontrivial translations. Then $L: G \to GL(E)$ is an isomorphism onto a unipotent subgroup of $GL(E)$. By conjugation we may assume that $L(G) \subset U(3)$ where $U(3)$ denotes the group of upper-triangular unipotent 3×3 matrices. Since $U(3)$ is connected and 3-dimensional, and so is $L(G)$, we must have $L(G) = U(3)$. The center of G contains Z with

$$L(Z) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As Z has no fixed point we can choose a basis for $\mathbb{R}^3$ (that respects the flag determined by $L(G)$) in which the translational part of Z is either

$$\varepsilon_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{or} \quad \varepsilon_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The centralizer of $L(Z) + \varepsilon_3$ does not map onto $U(3)$, so that the case $Z = L(Z) + \varepsilon_3$ is ruled out. Let $U \text{Aff}(3)$ denote the subgroup of $\text{Aff}(\mathbb{R}^3)$ generated by $U(3)$ and the group of translations. The centralizer of $L(Z) + \varepsilon_2$ in $U \text{Aff}(3) \subset GL(4, \mathbb{R})$ consists of all matrices $\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}$ where A is a 2×2 matrix $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ and B is a 2×2 matrix. As Z is the commutator of $X = \begin{bmatrix} A_X & B_X \\ 0 & A_X \end{bmatrix}$ and $Y = \begin{bmatrix} A_Y & B_Y \\ 0 & A_Y \end{bmatrix}$ we obtain

$$\begin{bmatrix} A_X & B_X \\ 0 & A_X \end{bmatrix} \begin{bmatrix} A_Y & B_Y \\ 0 & A_Y \end{bmatrix} = \begin{bmatrix} A_Y & B_Y \\ 0 & A_Y \end{bmatrix} \begin{bmatrix} A_X & B_X \\ 0 & A_X \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$$

from which $A_X B_Y + B_X A_Y = A_Y A_X + A_Y B_X + B_Y A_X$. Taking traces gives trace $(A_Y A_X) = 0$, but $A_Y A_X = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$ which is a contradiction.

Thus G contains nontrivial translations, so if G is abelian, the proposition is proved. In any case, $L(G)$ is a proper connected subgroup of $U(3)$ and therefore $L(G)$ is abelian. It follows that the commutator subgroup $[G, G]$ consists of translations. Since every 3-dimensional nilpotent nonabelian Lie group satisfies $[G, G] = $ center (G), this concludes the proof of 3.6.   Q.E.D.

3.7. We break the proof of 3.5 into the consideration of several different cases. Let G be as above and let $\zeta(G)$ denote the subgroup of G consisting of central translations.

Case I: dim $\zeta(G) = 3$. Here G is a group of translations

$$\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \quad (s, t, u \in \mathbb{R}) \right\rangle.$$

This group is of the type $\mathcal{H}_\alpha$ where α is identically zero.

Case II: dim $\zeta(G) = 2$. We may choose a basis of E such that translations parallel to the first and second directions span $\zeta(G)$. This means the Lie algebra $\mathcal{g}$ is a subalgebra of

$$\left\langle \begin{bmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \right\rangle$$

which centralizes translations in the first two directions. Moreover since the translational part $\mathscr{g} \to E$ is an isomorphism, a_1 and a_2 must be functions of (s, t, u); furthermore (s, t, u) ranges over all $\mathbb{R}^3$. Since $\mathscr{g}$ contains translations in s and t, the variables a_1 and a_2 must be functions of u alone; since $\mathscr{g}$ is to be a Lie algebra, the functions $a_1(u)$ and $a_2(u)$ are linear. Thus we have a linear map $(a_1, a_2) \colon \mathbb{R} \to \mathbb{R}^2$, which is nonzero since $\zeta(G) < 3$. Thus by a change of basis we may assume that $a_1 = 0$ and $a_2 = u$. Then G is represented as the group of all

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t + u^2/2 \\ u \end{bmatrix} = \exp \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & u \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix},
$$

where $(s, t, u) \in \mathbb{R}^3$. It is easy to see that G is isomorphic to $\mathbb{R}^3$; in fact G decomposes as the direct product of $\mathbb{R}$ acting by translations and $\mathbb{R}^2$ acting simply transitively and affinely in the "non-Riemannian" manner (see Theorem 3.4).

This is of type $\mathscr{H}_{(b,c)}$ when $(b, c) = (0, 0)$ and of type $\mathscr{H}_\alpha$ when α is a nonzero symmetric degenerate form.

3.8. For the last case dim $\zeta(G) = 1$ we use the basic fact that if G acts simply transitively on E, and if E_1 is the subspace whose cosets are the $\zeta(G)$-orbits, then the action of $G/\zeta(G)$ on E/E_1 is a simply transitive affine action. When G is 3-dimensional and nilpotent, $G/\zeta(G)$ is one of the two simply transitive unipotent affine actions discussed in 3.4: it must either be "Euclidean" or "non-Riemannian." We thus distinguish two cases.

Case III. dim $\zeta(G) = 1$ and $G/\zeta(G)$ is Euclidean. We choose a basis of E such that translation by the first basis vector spans $\zeta(G)$. The condition that this translation centralizes G, together with the fact that $G/\zeta(G)$ is Euclidean (this means that the associated simply transitive affine action on $\mathbb{R}^2$ has trivial linear part) implies that $\mathscr{g}$ is conjugate to a Lie subalgebra of

$$
\left\{ \begin{bmatrix} 0 & \alpha_1 & \alpha_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \right\}.
$$

Since G acts simply transitively, α_1 and α_2 are linear functions of the (s, t, u) which range over all $\mathbb{R}^3$; since G contains translations in s, α_1 and α_2 are linear functions of t and u. It follows that $\mathscr{g}$ has the form

$$\left\{ \begin{bmatrix} 0 & a_{11}t+a_{12}u & a_{21}t+a_{22}u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} : (s,t,u) \in \mathbb{R}^3 \right\}.$$

The matrix $\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$ determines a bilinear form α on $\mathbb{R}^2 = G/\zeta(G)$; see the Appendix for other interpretations of α. When α is symmetric, then it is easy to compute that G is abelian, and hence isomorphic to $\mathbb{R}^3$. Otherwise G is isomorphic to the Heisenberg group, the unique nonabelian nilpotent 3-dimensional 1-connected Lie group. The cases when α is symmetric and degenerate have been treated as cases I and II.

When α is symmetric there are two possibilities for G up to conjugacy:

definite: $\quad \exp \left\{ \begin{bmatrix} 0 & t & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \right\}$

$$= \left\{ \begin{bmatrix} 1 & t & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s+(t^2+u^2)/2 \\ t \\ u \end{bmatrix} \right\},$$

α indefinite: $\quad \exp \left\{ \begin{bmatrix} 0 & u & t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \right\}$

$$= \left\{ \begin{bmatrix} 1 & u & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s+tu \\ t \\ u \end{bmatrix} \right\}.$$

(The cases when α is positive definite and negative definite are conjugate not only in $\mathrm{Aff}(E)$ but also the group $\mathrm{Aff}_+(E)$ of *orientation-preserving* affine transformations.)

3.9. For the case when α is not symmetric, we decompose α as the sum of its symmetric part S and its alternating part A. Since $A \neq 0$ we may find a basis in which $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and S is in diagonal canonical form. When $S = 0$, so α is alternating, we obtain the action

$$\exp \left\{ \begin{bmatrix} 0 & -u & t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 & -u & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \right\}.$$

This structure gives a left-invariant affine structure which is actually bi-invariant (see Appendix). This affine structure was known to Cartan [9]. When $S \neq 0$, but is degenerate, we obtain the action

$$
\exp \left\{ \begin{bmatrix} 0 & -u \pm t & t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 & \pm t - u & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s \pm t^2/2 \\ t \\ u \end{bmatrix} \right\}.
$$

Here there are two cases depending on the sign $\pm t$, under $\mathrm{Aff}_+(E)$-conjugacy; both cases are conjugate under the full affine group $\mathrm{Aff}(E)$.

When $A \neq 0$ and S is nondegenerate we obtain an invariant of the affine structure which varies continuously with the affine structure. The ratio $\lambda = \det A / \det S$ is clearly an invariant of the bilinear form α and is nonzero unless H_α is abelian ($A = 0$). We thus obtain six one-parameter families of H_α when $A \neq 0$ and S is nondegenerate:

S positive definite:
$$
\exp \left\{ \begin{bmatrix} 0 & t - \lambda u & t + u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \right\}
$$
$$
= \left\{ \begin{bmatrix} 1 & t - \lambda u & t + u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s + (t^2 + u^2)/2 \\ t \\ u \end{bmatrix} \right\} ;
$$

S indefinite:
$$
\exp \left\{ \begin{bmatrix} 0 & t - \lambda u & t - u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \right\}
$$
$$
= \left\{ \begin{bmatrix} 1 & t - \lambda u & t - u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s + (t^2 - u^2)/2 \\ t \\ u \end{bmatrix} \right\} ;
$$

S negative definite:
$$
\exp \left\{ \begin{bmatrix} 0 & -t - \lambda u & t - u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \right\}
$$
$$
= \left\{ \begin{bmatrix} 1 & -t - \lambda u & t - u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s + (-t^2 - u^2)/2 \\ t \\ u \end{bmatrix} \right\} .
$$

3.10. Case IV. dim $\zeta(G) = 1$ and $G/\zeta(G)$ is non-Riemannian. In this case $\mathcal{g}$ must be conjugate to a subalgebra of

$$\left\{ \begin{bmatrix} 0 & at+bu & ct+du \\ 0 & 0 & u \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \right\}$$

and again we see that (s, t, u) must range over all of $\mathbb{R}^3$. The commutation relations give $a = 0$. Conjugating by the linear transformation

$$\begin{bmatrix} 1 & -d & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

reduces to the case $d = 0$ and gives the desired form

$$\mathcal{h}_{(b,c)} = \begin{bmatrix} 0 & bu & ct \\ 0 & 0 & u \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix}$$

which exponentiates to $\mathcal{H}_{(b,c)}$. The case when $b = c = 0$ is conjugate to case II. When $b = c$ the group is easily seen to be abelian; otherwise it is isomorphic to the Heisenberg group. In the Heisenberg case we obtain a continuously varying modulus $\lambda = b/b - c$. In that case $\mathcal{H}_{(b,c)}$ is conjugate to

$$\left\{ \begin{bmatrix} 1 & (\lambda-1)u & t+(\lambda-1)u^2/2 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{bmatrix} \right.$$
$$\left. \times \begin{bmatrix} s+(2\lambda-1)tu/2+(\lambda-1)u^3/6 \\ t+u^2/2 \\ u \end{bmatrix} \right\}.$$

3.11. To see the uniqueness statements in Theorem 3.5, and also for future reference, we compute the automorphism groups of these affine structures. If G has a left-invariant complete affine structure (recall this is equivalent to an affine conjugacy class of simply transitive affine actions of G), then the affine automorphism group of G is the smallest subgroup of Aff(E) which preserves G, i.e., the normalizer Norm(G) of G in Aff(E). Of these, we distinguish the "inner" affine automorphism, coming from left-translations in G. The quotient group Norm(G)/G is then denoted by Aff(G)

and called the *outer affine automorphism group* of the left-invariant affine structure on G. Since G acts simply transitively, we may write

$$\mathrm{Norm}(G) \cong G \rtimes (\mathrm{Norm}(G) \cap GL(E))$$

and take $\mathrm{Aff}(G) \cong (\mathrm{Norm}(G) \cap GL(E))$.

There is a canonical homomorphism $\mathrm{Aff}(G) \to \mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$. This homomorphism is injective because the centralizer of G in $\mathrm{Aff}(E)$ lies in G. (To see this, suppose that $g \in \mathrm{Aff}(E)$ centralizes G. Then g acts freely since otherwise G would leave invariant a proper affine subspace—the fixed point set of g. Since G acts transitively, any coset gG contains an element which fixes a point, so $g \in G$.)

If $\mathrm{Aff}(G)$ is all of $\mathrm{Out}(G)$, we say that the affine structure on G is *fully symmetric*. Clearly the action of E on itself is fully symmetric. In the Appendix we give a description of the Cartan bi-invariant affine structure on the Heisenberg group, which is a fortiori fully symmetric. The idea is that the commutation relations on a 2-step nilpotent group give rise to covariant differentiation relations corresponding to a left-invariant flat affine connection. When the group is no longer 2-step nilpotent the same construction replaces a left-invariant affine structure on a Lie group by a certain kind of algebra structure on the Lie algebra. In this way we can understand the affine automorphism group $\mathrm{Aff}(G)$ as the automorphism group of the algebra associated to G. For more details see the Appendix and the references quoted there.

The following result can be proved by direct computation, or alternatively by the algebraic techniques described in the Appendix.

PROPOSITION 3.12.   (i)

$$\mathrm{Aff}(\mathscr{H}_a) = \left\{ \begin{bmatrix} c & w \\ 0 & P \end{bmatrix} : \begin{array}{l} \text{where } c \in \mathbb{R}, \ w \text{ is a } 1 \times 2 \text{ real row matrix,} \\ \text{and } P \in GL(\mathbb{R}^2) \text{ satisfies } P^* a = c \cdot a \end{array} \right\}$$

unless $a = 0$, *in which case* $\mathscr{H}_\alpha$ *is normal in* $\mathrm{Aff}(E)$.

(ii)

$$\mathrm{Aff}(\mathscr{H}_{(b,c)}) = \left\{ \begin{bmatrix} a^3 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{bmatrix} \cdot \begin{bmatrix} 1 & (b+c)v & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{bmatrix} \right\}$$
$$\text{where } 0 \neq a, v, w \in \mathbb{R}$$

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unless $b = c = 0$, in which case

$$\text{Aff}(\mathcal{H}_{(b,c)}) = \left\{ \begin{bmatrix} a^3 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{bmatrix} \cdot \begin{bmatrix} f & 0 & g \\ d & 1 & h \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$
$$\text{where } a \neq 0, f \neq 0, d, g, h \in \mathbb{R}$$

Abelian ($\mathbb{R}^3$)

Nonabelian (Real Heisenberg group)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \qquad \begin{bmatrix} 1 & -u & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix}$$

$$\begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s + t^2/2 \\ t \\ u \end{bmatrix} \qquad \begin{bmatrix} 1 & t-u & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s + t^2/2 \\ t \\ u \end{bmatrix}$$

$|\epsilon_1| = |\epsilon_2| = 1;$

$\lambda \neq 0$

$$\begin{bmatrix} 1 & \epsilon_1 t & \epsilon_2 u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s + (\epsilon_1 t^2 + \epsilon_2 u^2)/2 \\ t \\ u \end{bmatrix} \qquad \begin{bmatrix} 1 & \epsilon_1 - \lambda u & \epsilon_2 + \lambda t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s + (\epsilon_1 t^2 + \epsilon_2 u^2)/2 \\ t \\ u \end{bmatrix}$$

$\lambda \in \mathbb{R}$

$$\begin{bmatrix} 1 & u & t+u^2/2 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s + t^2/2 + u^3/6 \\ t + u^2/2 \\ u \end{bmatrix} \qquad \begin{bmatrix} 1 & (\lambda-1)u & t+(\lambda-1)u^2/2 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s+(2\lambda-1)tu/2+(\lambda-1)u^3/6 \\ t + u^2/2 \\ u \end{bmatrix}$$

FIG. 2.   Summary of unipotent simply transitive affine actions on $\mathbb{R}^3$.

4. SIMPLY TRANSITIVE AFFINE ACTIONS OF UNIMODULAR LIE GROUPS ON $\mathbb{R}^3$

In this section we conclude the classification of simply transitive $G \subset \text{Aff}(E)$ at least when E has dimension 3 and G is unimodular. (The identity component of a crystallographic hull of an affine crystallographic group is unimodular since it admits a discrete uniform subgroup.) One can use the same method to classify all simply transitive affine actions on $\mathbb{R}^3$ but for the sake of brevity we only consider the unimodular case here. We shall prove the following result:

THEOREM 4.1. *Suppose G is unimodular and acts simply transitively on $\mathbb{R}^3$ by affine transformations. Suppose G is not nilpotent (otherwise see 3.5). Then G is conjugate to one of the subgroups*

$$
I_\lambda = \begin{bmatrix} 1 & \lambda e^s u & \lambda e^{-s} t \\ 0 & e^s & 0 \\ 0 & 0 & e^{-s} \end{bmatrix} \begin{bmatrix} s + \lambda t u \\ t \\ u \end{bmatrix} : (s, t, u \in \mathbb{R}) \quad,
$$

$$
D_\lambda = \left\{ \begin{bmatrix} 1 & \lambda(t \cos s - u \sin s) & \lambda(t \sin s + u \cos s) \\ 0 & \cos s & -\sin s \\ 0 & \sin s & \cos s \end{bmatrix} \right.
$$

$$
\left. \times \begin{bmatrix} s + \lambda(t^2 + u^2)/2 \\ t \\ u \end{bmatrix} : (s, t, u \in \mathbb{R} \right\} .
$$

There are four distinct conjugacy classes according to whether $\lambda = 0$ or not.

In Milnor [29] and Auslander [5] (see also Helmstetter [20]) it is proved that if $G \subset \mathrm{Aff}(E)$ acts simply transitively on E, then G is solvable (however it is not known whether every 1-connected solvable Lie group admits a simply transitive affine action). Thus for the purposes of classification of the simply transitive affine actions we may restrict attention to solvable 1-connected groups. Moreover since the nilpotent cases have already been treated in the preceding section we also assume that G is not nilpotent. Up to isomorphism there are in dimension three only two 1-connected such Lie groups, represented as the identity components of the universal covers of the isometry groups $E(2)$ and $E(1, 1)$ of flat pseudo-Riemannian metrics on the plane. We denote these groups by D and I respectively. Each is a semidirect product $\mathbb{R}^2 \rtimes \mathbb{R}$ where $\mathbb{R}$ acts on $\mathbb{R}^2$ by one of the actions:

$$
u \to \begin{bmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{bmatrix} \quad \text{(where } G = E(2) \text{)}
$$

or

$$
u \to \begin{bmatrix} e^u & 0 \\ 0 & e^{-u} \end{bmatrix} \quad \text{(where } G = E(1, 1) \text{)}.
$$

In what follows G will denote either one of these groups and $\mathscr{g}$ will denote its Lie algebra. We choose a basis $\{X, Y, Z\}$ of $\mathscr{g}$ such that $[X, Y] = 0$; this implies that X and Y span the derived algebra of $\mathscr{g}$.

If $A \in a\!f\!f(E)$ denotes an affine map $E \to E$, there is a unique Jordan decomposition (see, e.g., [8, 22]) of A as $A = \mathrm{nil}(A) + s(A)$ where $\mathrm{nil}(A)$ and

$s(A)$ are respectively nilpotent and semisimple affine maps which commute. (An affine map is nilpotent if its linear part is nilpotent; an affine map is semisimple if it is a semisimple linear map.)

LEMMA 4.2. *Let $\mathscr{g}$ be as above and let $\mathscr{g} \subset a\!f\!f(E)$ define a complete affine structure on $\mathscr{g}$. Let n denote the span of X, Y, and nil(Z) in $a\!f\!f(E)$. Then n is an abelian Lie algebra and the inclusion $n \subset a\!f\!f(E)$ defines a complete affine structure on n.*

Remark. This lemma is a special case of Auslander's theorem III.1 of [5]. Compare the general discussion in 1.18.

Proof. Since Z lies in the algebraic Lie algebra which preserves $[\mathscr{g}, \mathscr{g}]$ so does its nilpotent part nil(Z). Then ad nil(Z) acts on $[\mathscr{g}, \mathscr{g}] = \text{span}(X, Y)$ by the nilpotent part of ad Z acting on span(X, Y) (Borel [8],). As ad Z is semisimple, nil(Z) commutes with X and Y. Thus X, Y, and nil(Z) span an abelian Lie algebra n.

By the Lie–Kolchin theorem there exists a basis of E such that in that basis $L(\mathscr{g})$ is represented by upper-triangular matrices. It follows that $[\mathscr{g}, \mathscr{g}]$ consists of nilpotent affine transformations. On the other hand nil(Z) is already nilpotent and from the upper-triangularizability of $\mathscr{g}$ it follows that n actually consists of nilpotent affine endomorphisms. By the remarks in 3.3 it will suffices to prove that the translational part map $n \to E$ is a linear isomorphism in order to prove that n has a complete affine structure. But $s(Z)(0) = 0$ so $s(Z)$ has zero translational part. Thus $n \to E$ has the same image as $\mathscr{g} \to E$, which is onto since $\mathscr{g} \subset a\!f\!f(E)$ defines an affine structure. This proves 4.2.   Q.E.D.

Combining 3.5 and 4.2 we see that the subgroup $N = \exp n$ of $\text{Aff}(E)$ corresponding to n must be conjugate to one of the groups $\mathscr{H}_\alpha$ or $\mathscr{H}_{(b,c)}$ where α is symmetric or $b = c$, respectively. We show first of all that type $\mathscr{H}_{(b,c)}$ cannot occur if $b = c \neq 0$.

LEMMA 4.4. *$s(Z)$ is singular.*

Proof. The one-parameter subgroup $\exp(tZ)$ acts freely, so $\exp L(tZ)$ has one has an eigenvalue (compare 2.4). Therefore Z, and hence also $s(Z)$, has zero as an eigenvalue.   Q.E.D.

Now clearly $s(Z)$ normalizes n. It follows from 3.12 that any semisimple affine endomorphism which normalizes $\mathscr{H}_{(b,c)}$ with $b \neq 0$ must be nonsingular. Thus N is not conjugate to $\mathscr{H}_{(b,c)}$ with $b = c \neq 0$. When $b = c = 0$, then N is conjugate to an $\mathscr{H}_\alpha$, the case we shall treat next.

4.5. It follows from the classification 3.12 that N is conjugate to a

group of type $\mathscr{H}_\alpha$ specified by a symmetric bilinear form α on $\mathbb{R}^2$. Recall from Section 3 that

$$n = \left\{ \begin{bmatrix} 0 & a_{11}t + a_{12}u & a_{12}t + a_{22}u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} : (s, t, u \in \mathbb{R}) \right\}$$

so that $\exp s(Z)$ must be of the form

$$\begin{bmatrix} c & w \\ 0 & P \end{bmatrix}$$

where w is a 1×2 row vector and $P \in GL(\mathbb{R}^2)$ satisfies $P^*\alpha = c \cdot \alpha$. It follows from the commutation relations in $\mathscr{g}$ that $s(Z)$ is $\mathrm{Aff}(E)$-conjugate to a multiple of either

$$s_I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad s_D = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In particular the set of eigenvalues of $s(Z)$ is either $\{0, -1, 1\}$ or $\{0, -\sqrt{-1}, \sqrt{-1}\}$.

In case the bilinear form α is identically zero then N is a group of translations and we readily see that there are only the two possibilities for $\mathscr{g}$:

$$\left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & -s \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \right\}, \quad \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -s \\ 0 & s & 0 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \right\}.$$

We thus obtain the algebras I_0 and D_0, respectively.

4.6. In case α is nonzero but degenerate, then any $P \in GL(\mathbb{R}^2)$ satisfying $P^*\alpha = c \cdot \alpha$ must have an eigenvalue equal to $c^{1/2}$. It follows that $s(Z)$ must have two different eigenvalues λ_1, λ_2 satisfying $2\lambda_1 = \lambda_2$. Clearly this is not the case for either s_I or s_D.

This brings us to the case that α is nondegenerate. Suppose first that α is indefinite; then every $P \in GL(\mathbb{R}^2)$ satisfying $P^*\alpha = c \cdot \alpha$ must have eigenvalues satisfying $\lambda_1 \lambda_2 = c$. Thus if $\exp s(Z)$ is of the form

$$\begin{bmatrix} c & w \\ 0 & P \end{bmatrix},$$

then it follows from the fact that $s(Z)$ has as eigenvalues $\{0, 1, -1\}$ or $\{0, \sqrt{-1}, -\sqrt{-1}\}$ that $c = 1$ and P lies in the isometry group $O(1, 1)$ of the

form α. Since $[\mathrm{nil}(Z), s(Z)] = 0$, we see that $\mathrm{nil}(Z)$ must be a translation in the first direction (i.e., $\mathrm{nil}(Z) \in \zeta(N)$). It follows that $\mathscr{g}$ must be of the form

$$
\left\{ \begin{bmatrix} 0 & u + b_1 s & t + b_2 s \\ 0 & s & 0 \\ 0 & 0 & -s \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \right\}
$$

where b_1 and b_2 are real constants. Unless $b_1 = b_2 = 0$, however, the corresponding group will not act freely. Thus G is conjugate to

$$
\exp \left\{ \begin{bmatrix} 0 & u & t \\ 0 & s & 0 \\ 0 & 0 & -s \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \right\}
$$

$$
= \left\{ \begin{bmatrix} 1 & e^s u & e^{-s} t \\ 0 & e^s & 0 \\ 0 & 0 & e^{-s} \end{bmatrix} \begin{bmatrix} s + tu \\ t \\ u \end{bmatrix} : s, t, u \in \mathbb{R} \right\}.
$$

4.7. We next consider the case when α is definite. Then $P*\alpha = c \cdot \alpha$ implies that both the eigenvalues of P have norm $|c|$. If $\exp s(Z)$ is to be of the form

$$
\begin{bmatrix} c & w \\ 0 & P \end{bmatrix},
$$

then it follows from the calculation of the eigenvalues of $s(Z)$ that $c = 1$ and P must preserve α, i.e., $P \in O(2)$. The rest of the argument is completely analogous to the case when α is indefinite and proves that G must be conjugate to

$$
\exp \left\{ \begin{bmatrix} 0 & t & u \\ 0 & 0 & -s \\ 0 & s & 0 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} \right\}
$$

$$
= \left\{ \begin{bmatrix} 1 & t \cos s - u \sin s & u \cos s + t \sin s \\ 0 & \cos s & -\sin s \\ 0 & \sin s & \cos s \end{bmatrix} \times \begin{bmatrix} s + (t^2 + u^2)/2 \\ t \\ u \end{bmatrix} : (s, t, u \in \mathbb{R}) \right\}.
$$

This completes the proof of 4.1.   Q.E.D.

4.8. In 3.11 we saw how the "outer" affine automorphisms of a left-invariant complete affine structure could be effectively identified with the group of linear automorphisms of E which normalize the simply transitive group $G \subset \mathrm{Aff}(E)$. By a fairly elementary calculation we obtain the following:

THEOREM. (I)

$$\mathrm{Aff}(I_\lambda) = \left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^N \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} : N \in \mathbb{Z} \pmod 2 \text{ and } (a, b) \right.$$

ranges over all of $\mathbb{R}^* \times \mathbb{R}^*$ if $\lambda = 0$;
otherwise (a, b) lies in the subgroup $ab = 1\}$;

(D)

$$\mathrm{Aff}(D_\lambda) = \left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^N \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{bmatrix} : N \in \mathbb{Z} \pmod 2 \text{ and } (a, b) \right.$$

ranges over all of
$\mathbb{C}^* = \{(a, b) \in \mathbb{R} \times \mathbb{R} : a^2 + b^2 \neq 0\}$
if $\lambda = 0$; otherwise (a, b)
is restricted to lie in the circle $a^2 + b^2 = 1\}$.

4.9. For the remainder of the paper we will not need to consider the groups D_λ since these groups do not arise as crystallographic hulls of affine crystallographic groups. It is easy to prove that every lattice subgroup of D is virtually abelian. It follows easily from 1.4 that the crystallographic hull of a virtually abelian affine crystallographic group is itself virtually abelian. Therefore the connected 3-dimensional crystallographic hulls (i.e., the identity components of 3-dimensional crystallographic hulls) fall into exactly three isomorphism classes: abelian $\mathbb{R}^3$; nonabelian nilpotent $N = 3$-dimensional Heisenberg group; nonnilpotent solvable $I = E(1, 1)^0$.

5. CRYSTALLOGRAPHIC HULLS IN DIMENSION THREE

In previous sections we showed that every three-dimensional affine crystallographic group Γ admits a unique crystallographic hull $G = H(\Gamma)$. We have listed (up to conjugacy) all the connected Lie subgroups of $\mathrm{Aff}(3)$ that arise as crystallographic hulls, namely, the nilpotent families H_α and $H_{(b,c)}$ and the nonnilpotent examples I_0 and I_1. In this section we determine

the disconnected crystallographic hulls in Aff(3), up to conjugacy, by examining the finite extensions of the connected ones.

We will say that a discrete co-compact subgroup Γ of a solvable Lie group G is *snug* in G if Γ meets every coset of G and dim $G = $ rank Γ. Thus an affine crystallographic group is snug in its crystallographic hull.

We begin with the case $H(\Gamma)_0 = H_\alpha$. Since $\alpha = 0$ gives the well-known classical crystallographic groups, we will restrict it to $\alpha \neq 0$.

THEOREM 5.1. *Suppose* $\Gamma \subset \text{Aff}(3)$ *is an affine crystallographic group with* $H(\Gamma)^0 = H_\alpha$, $\alpha \neq 0$. *Then there is a finite subgroup* $F \subset GL(3, \mathbb{R}) \subset \text{Aff}(3)$ *in the table below and an element* $M \in \text{Aff}(H_\alpha)$ *for which* $MH(\Gamma)M^{-1} \subset H_\alpha \cdot F$.

	$S = 0$	$S \neq 0$ S Singular	S Nonsingular Definite	S Nonsingular Indefinite
$A = 0$	—	$\left\{ \begin{pmatrix} 1 & & \\ & \varepsilon_1 & \\ & & \varepsilon_2 \end{pmatrix} \right\}$	$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} \right\}$ $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \right\}$	$\left\{ \begin{pmatrix} \tau(P) & 0 \\ 0 & P \end{pmatrix} \right\}$
$A \neq 0$	$\left\{ \begin{pmatrix} b & 0 \\ 0 & P \end{pmatrix} \middle| b = \det P \right\}$ $\left\{ \begin{pmatrix} b & 0 \\ 0 & Q \end{pmatrix} \middle| b = \det Q \right\}$	$\left(\begin{matrix} 1 & & \\ & \varepsilon_1 & \\ & & \varepsilon_1 \end{matrix} \right)$	$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} \middle| \det P = 1 \right\}$ $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \middle| \det Q = 1 \right\}$	$\left\{ \begin{pmatrix} \tau(P) & 0 \\ 0 & P \end{pmatrix} \middle| \tau(P) = \det P \right\}$

Here $\alpha = A + S$, $A^t = -A$, $S^t = S$,

$\varepsilon_i \in \{\pm 1\}$,

$P \in D_4 = \{i^k z, i^k \bar{z}\} \subset GL(2, \mathbb{R})$,

$\tau(P) = (-1)^k$ *for* $P = i^k z$ *or* $P = i^k \bar{z}$

$Q \in D_6 = \{\omega^j z, \omega^j \bar{z}\} \subset GL(2, \mathbb{R})$, $\omega = e^{2\pi i/6}$.

Proof. As $G = H(\Gamma)$ is a finite extension of H_α, it has the form $H_\alpha \cdot F$ where $F \cap H_\alpha = 1$, F is a finite subgroup of $\text{Aff}(H_\alpha)$. Recall that $\text{Aff}(H_\alpha) = \{\begin{pmatrix} c & w \\ 0 & R \end{pmatrix} \mid R^* \alpha = c\alpha \}$. Choose an inner product on $\mathbb{R}^3$ invariant under F and rechoose the y and z axes so that the standard basis is orthogonal in this metric. In this basis, $F \subset \{\begin{pmatrix} c & 0 \\ 0 & R \end{pmatrix} \mid R^* \alpha = c\alpha \}$.

We now consider various possibilities for $\alpha = S + A$ and determine the R's and c's in each instance.

Column 1. $S = 0$: As $a \neq 0$ we have $A \neq 0$. From $R*a = ca$ follows $c = \det R$. H_α acts on $\mathbb{R}^2 = \mathbb{R}^3/\mathbb{R}$ by translations. Elements of $H_\alpha \cdot F$ act on $\mathbb{R}^3/\mathbb{R}$ by affine motions and the linear part is given by R. As the action of Γ_0 on $\mathbb{R}^2$ is by a pair of linearly independent translations and $\Gamma_0 \lhd \Gamma$, we see that the R, s preserve a lattice in $\mathbb{R}^2$.

Every finite subgroup of $Gl(2, \mathbb{Z})$ is conjugate to a subgroup of D_4 or D_6 in $Gl(2, \mathbb{R})$, these groups arising as the isometries of a square lattice and hexagonal lattice, respectively. So choosing a basis for the y-z plane appropriately, we may put F in the desired form.

Column 2. $S \neq 0$, S singular: We may choose the y and z coordinates so that $S = \left(\begin{smallmatrix} 0 & 0 \\ 0 & * \end{smallmatrix}\right)$. Then F preserves the axis $z = 0$ of the y-z plane. As S is semidefinite, $c > 0$. Since F is finite, $c = 1$.

Using that F is finite, one finds a complementary line to the y-axis that is preserved by F. Choosing this as the new z-axis gives $R = \left(\begin{smallmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{smallmatrix}\right)$. If $A \neq 0$, then $R*A = cA \Rightarrow \det R = c = 1$, so $\varepsilon_2 = \varepsilon_1$.

Column 3. S nonsingular, definite: As for column 2, we have $c = 1$. Assume that Γ_0 contains a nontrivial translation in the x-direction (this holds automatically if $A \neq 0$). Then Γ_0 acts on $\mathbb{R}^2$ by two independent translations. Reasoning as in column 1 gives $R \in D_n$, $n = 4$ or 6. If $A \neq 0$, then, as for column 2, $\det R = 1$.

If Γ_0 does not have such a translation, then H_α is abelian. The action of $\Gamma/\Gamma_0 \cong F$ on H_α by conjugation fixes the 1-dimensional subspace of translations. Pass from $\Gamma_0 \subset H_\alpha$ to the isomorphic pair $\mathbb{Z}^3 \subset \mathbb{R}^3$. The action of F on $\mathbb{R}^3$ fixes an irrational line and preserves $\mathbb{Z}^3$. The fixed set of F is a rational subspace of dimension $\geqslant 2$. One thus can rotate Γ_0 to a new lattice, also preserved by F, that meets this irrational line. This reduces to the case of the preceding paragraph.

Column 4. S nonsingular and indefinite: Choose the y- and z-axes to be the lightcone $S(y, z) = 0$. One sees then that $R \in D_4$ (possibly after scaling these axes correctly). For $P \in D_4$, $P*S = \tau(P)S$ so $c = \tau(P)$. If $A \neq 0$, then, as for column 1, one has $c = \det P$ as well. Q.E.D.

We now suppose $H(\Gamma)_0 = H_{(b,c)}$. As $H_{(0,0)}$ is conjugate to an H_α, we may suppose $(b, c) \neq 0$.

THEOREM 5.2. *Suppose $\Gamma \subset \mathrm{Aff}(3)$ is an affine crystallographic group with $H(\Gamma)_0 = H_{(b,c)}$, $(b, c) \neq 0$. Then for some $M \in \mathrm{Aff}(H_{(b,c)})$, $MH(\Gamma)M^{-1}$ is either $H_{(b,c)}$ or $H_{(b,c)} \cdot F$, where*

$$F = \left\{ \begin{pmatrix} \varepsilon & & \\ & 1 & \\ & & \varepsilon \end{pmatrix} \middle| \varepsilon = \pm 1 \right\} \cong \mathbb{Z}_2.$$

Proof. Assume $H(\Gamma)$ is disconnected and write $H(\Gamma) = H_{(b,c)} \cdot F$, $F \subset Gl(3, \mathbb{R})$, F finite. Consider the group $\mathrm{Aff}(H_{(b,c)})$ given in Proposition 3.12. Its only finite subgroups have one or two elements and any two-element subgroup is conjugate to the F in this theorem.   Q.E.D.

Finally, we consider $H(\Gamma)_0 = I_\lambda$.

THEOREM 5.3. *Assume $\Gamma \subset \mathrm{Aff}(3)$ is an affine crystallographic group and $H(\Gamma)_0 = I_\lambda$. Then $H(\Gamma) \subset I_\lambda \cdot F$, where*

$$\lambda = 0 : F = \left\{ \begin{pmatrix} \tau(P) & 0 \\ 0 & P \end{pmatrix} \Big| P \in D_4 \right\},$$

$$\lambda \neq 0 : F = \left\{ \begin{pmatrix} \tau(P) & 0 \\ 0 & P \end{pmatrix} \Big| P \in D_4, \tau(P) = \det P = 1 \right\} \cong \mathbb{Z}_2,$$

where $\tau(P)$ is as in Theorem 5.1.

Proof. These are the only elements of finite order in the normalizer of I_λ, as can be seen by direct calculation.   Q.E.D.

5.4. One can check that the groups $H_\alpha \cdot F$, $H_{(b,c)} \cdot F$ and $I_\lambda \cdot F$, listed in the preceding theorems, are actually the crystallographic hulls of appropriate lattices Γ. Thus crystallographic hulls in dimension three are (up to conjugacy) of the form $H_\alpha \cdot F'$, $H_{(b,c)} \cdot F'$ or $I_\lambda \cdot F'$ where F' is some subgroup of the group F in the above theorems. The crystallographic groups are obtained by taking the snug subgroups of these Lie groups. This depends only on the abstract Lie groups and no longer on the affine strucutre.

If Γ and Γ' are snug in $H(\Gamma) = H(\Gamma') = G$ and Γ and Γ' are not conjugate in G, they may still be conjugate in $\mathrm{Aff}(E)$. But the conjugating map must normalize G. As the normalizer of G is easily computed (cf. Proposition 3.12) this is a routine calculation. If one is not concerned with such duplications, then one can say that the problem of finding crystallographic groups in dimension 3 has been reduced to computing the snug subgroups of finitely many Lie groups G.

This problem is essentially a finite one since the snug subgroups of G_0 are easily listed. G_0 is either abelian, the Heisenberg group N or the nonnilpotent group S with Lie algebra $[x, y] = 0$, $[x, z] = x$, $[y, z] = -y$. In G_0 the snug subgroups are just the usual lattice $\mathbb{Z}^3$. In N the snug subgroups are the extensions of a $\mathbb{Z} \subset Z(N)$ by a $\mathbb{Z}^2 \subset N/Z(N)$. In S the snug subgroups are the extensions of a $\mathbb{Z}^2 \subset [S, S]$ by a $\mathbb{Z} \subset S/[S, S]$.

Thus the groups Γ_0 are precisely the groups $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ where $A \in GL(2, \mathbb{Z})$ has positive eigenvalues. If A is hyperbolic, then $H(\Gamma_0) \cong S$; if A is the identity, then $H(\Gamma_0)$ is abelian; if A is neither, then $H(\Gamma_0) \cong N$. This enables

one to find all the crystallographic actions of a given abstract group, on affine 3 space.

5.5. Just as we classify which groups admit crystallographic affine actions on $\mathbb{R}^3$, we can easily describe the class of closed 3-manifolds which admit complete affine structures.

THEOREM. *Let M^3 be a closed 3-manifold. The following conditions are equivalent*:

   (i)   *M admits a complete affine structure*;

  (ii)   *M is finitely covered by a 2-torus bundle over the circle*;

 (iii)   *$\pi_1(M)$ is solvable and M is aspherical.*

Proof. (i) $\Rightarrow$ (iii) By 2.13, $\pi_1(M)$ is virtually solvable and there exists a crystallographic hull H for the affine holonomy group (isomorphic to $\pi_1(M)$) of some complete affine structure on M. The classification of crystallographic hulls in 5.1–5.2 shows that the group of components of H is a solvable group. Since the identity component of H is solvable, it follows that H is solvable. Since $\pi_1(M)$ admits a faithful representation in H, $\pi_1(M)$ is solvable. Clearly M is aspherical as it is covered by $\mathbb{R}^3$.

(iii) $\Rightarrow$ (ii) follows from Evans–Moser [11].

(ii) $\Rightarrow$ (i) We may assume that M has a finite regular covering space homeomorphic to a T^2-bundle over S^1 whose attaching map has positive eigenvalues. By 5.3 every such T^2-bundle N has a complete affine structure, which we may assume comes from a fully symmetric left-invariant complete affine structure. In that case every finite group action on the affine manifold N is homotopic to an affine action, which is free if the original action is free. Applying this to the group of deck transformations of N whose quotient is M, we conclude that M is *homotopy-equivalent* to a complete affine manifold. If M is Haken, it follows from Waldhausen's results (see, e.g., [23]) that such a homotopy-equivalence is homotopic to a homeomorphism. Otherwise M must be Seifert-fibered over S^2 with three exceptional fibers, and it follows from a recent result of Scott [43] that a homotopy-equivalence between such manifolds is homotopic to a homeomorphism. Thus M is homeomorphic to a complete affine manifold.   Q.E.D.

5.6. We may relate this classification to the "eight geometries" which Thurston has used to "uniformize" 3-manifolds [40]. Namely, if M is a closed 3-manifold, Thurston has conjectured (and proved, if M is Haken) that M may be canonically decomposed into "geometric" pieces each of which admits a complete Riemannian metric of finite volume which is locally isometric to one of eight homogeneous Riemannian 3-manifolds.

Three of these homogeneous Riemannian spaces are solvable Lie groups with left-invariant Riemannian metric: specifically Euclidean space $\mathbb{R}^3$, the Heisenberg group, and the exponential solvable unimodular non-nilpotent Lie group $E(1, 1)^0$ which we denoted by I in Section 4. The closed 3-manifolds which are uniformized by these geometries are precisely those closed 3-manifolds which are finitely covered by a T^2-bundle over S^1. Thus we may add the following condition to the equivalent statements in theorem in 5.4:

(iv) M has a Riemannian metric locally isometric to a left-invariant metric on a 3-dimensional solvable Lie group.

APPENDIX:
GEOMETRY OF SIMPLE TRANSITIVE AFFINE ACTIONS ON $\mathbb{R}^3$

A.1. In this appendix we discuss in more detail some natural geometric objects occurring on a compact complete affine 3-manifold. Thus we are led to consider the flows, foliations, tensor fields, etc., on $E = \mathbb{R}^3$ which are invariant under a simply transitive unimodular subgroup G of Aff(E). (Equivalently G is a simply transitive group of unimodular, i.e., volume-preserving, affine transformations.)

We may regard a simply transitive affine action as defining a left-invariant complete affine structure on a (1-connected) Lie group G. Using this interpretation we study left-invariant geometric objects on G—as they appear in affine coordinates.

We regard these examples as the heart of the theory developed in Sections 1, 3, 4, and 5. It is for this reason that we give here a brief summary of the theory as applied to these examples.

For example, many of the invariant tensor fields will turn out to be *polynomial tensor fields*. In fact, sufficiently many polynomial exterior differential forms exist to represent the full de Rham cohomology of a compact complete affine 3-manifold (1.22); see [16]. This may be traced to the fact that for simply transitive $G \subset$ Aff(E), the evaluation map at 0, dev: $G \to E$ has the property that the composite $\mathscr{g} \to^{\exp} G \to^{\text{dev}} E$ is a *polynomial isomorphism* of vector spaces. In view of the polynomial deformation theorem (1.20–1.21) these polynomial isomorphisms may be used to conjugate simply transitive (respectively, crystallographic) affine actions of isomorphic groups.

A.2. We briefly recall ([15, sect. 8]) how left-invariant tensors on a Lie group with left-invariant complete affine structure are computed in affine coordinates. Let dev: $G \to E$ be a developing map, e.g., evaluation at $0 \in E$ of the corresponding simply transitive $G \subset$ Aff(E) and let $f: E \to G$ be its

inverse map. Let $L: G \to GL(E)$ be the linear part of this action; if $g \in G$, we denote by $L(g)_*$ the map induced by $L(g)$ on the full tensor algebra on the vector space E. It is easy to see that if ω_0 is a tensor defined on $E \simeq T_0 E$, the unique G-invariant tensor field ω on E extending ω_0 is given by

$$\omega(x) = (L \circ f(x))_* \omega_0$$

for every $x \in E$.

A.3. Our first examples will be the groups H_α determined by a bilinear function $\alpha: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$. Recall that these groups are characterized as unipotent simply transitive $G \subset \mathrm{Aff}(E)$ having a central subgroup of translations T such that the induced affine action of G/T on the orbit space E/T is by translations.

We first suppose that α is symmetric; then in some basis α is associated to the quadratic form $q(y, z) = \varepsilon_1 y^2 + \varepsilon_2 z^2$ where $\varepsilon_i = -1, 0,$ or $+1$. Then

$$G = \left\{ \begin{bmatrix} 1 & \varepsilon_1 t & \varepsilon_2 u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s + (\varepsilon_1 t^2 + \varepsilon_2 u^2)/2 \\ t \\ u \end{bmatrix} \right\} : s, t, y \in \mathbb{R}$$

and the developing map for the corresponding left-invariant affine structure is

$$\mathrm{dev}(s, t, u) = (s + \tfrac{1}{2} q(t, u), t, u)$$

with inverse map

$$f(z, x, y) = (x - \tfrac{1}{2} q(y, z), y, z);$$

these are to be regarded as the transformations between exponential coorinates on G and the affine coordinates induced from E. A basis for the left-invariant vector fields (respectively, 1-forms) is given by

$$\left\{ \frac{\partial}{\partial x}, \varepsilon_1 y \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \varepsilon_2 z \frac{\partial}{\partial x} + \frac{\partial}{\partial z} \right\}$$

(resp. $\{dx - \varepsilon_1 y \, dy - \varepsilon_2 z \, dz, \, dy, \, dz\}$). Note that the foliation by the level sets of $q(y, z)$ defines a G-invariant measured foliation on E, which defines a foliation on every affine manifold $\Gamma \backslash G$.

A.4. In all of these cases G is abelian so the notions of right-invariant and left-invariant coincide. Now the right-invariant vector fields on a Lie group G with a left-invariant affine structure are the infinitesimal generators of the affine action $G \subset \mathrm{Aff}(E)$ corresponding to left-

multiplication. Thus the right-invariant vector fields are considerably easier to compute than the left-invariant vector fields for which one must solve a system of polynomial equations dev $\cdot f = \mathrm{id}_E$.

With this in mind, we consider the groups H_α where α is given by the matrix $\left[\begin{smallmatrix} 1 & -\lambda \\ \lambda & \varepsilon_2 \end{smallmatrix}\right]$. The case when $\lambda = 0$ is the symmetric case already treated; again we write $q(y, z) = \varepsilon_1 y^2 + \varepsilon_2 z^2$ for the quadratic form associated to the symmetric part of α. Now

$$
G = \left\{ \begin{bmatrix} 1 & -\lambda u + \varepsilon_1 t & \varepsilon_2 u + \lambda t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s + \tfrac{1}{2} q(t, u) \\ t \\ u \end{bmatrix} : s, t, u \in \mathbb{R} \right\}
$$

so that dev and f are as before. A basis for the right-invariant vector fields is

$$
\left\{ \frac{\partial}{\partial x}, (\varepsilon_1 y + \lambda z) \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, (\varepsilon_2 z - \lambda y) \frac{\partial}{\partial x} + \frac{\partial}{\partial z} \right\}
$$

and a basis for the left-invariant vector fields is

$$
\left\{ \frac{\partial}{\partial x}, (\varepsilon_1 y - \lambda z) \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, (\varepsilon_2 z + \lambda y) \frac{\partial}{\partial x} + \frac{\partial}{\partial z} \right\} .
$$

A.5. For these examples, we see the left-invariant vector fields are affine vector fields. This means that the left-invariant affine structure on G is also right-invariant. One can show that such *bi-invariant* complete affine structures on a Lie group G correspond to nilpotent associative algebra structures on the Lie algebra $\mathscr{g}$ such that $XY - YX = [X, Y]$ for $X, Y \in \mathscr{g}$ (see Vey [41], Helmstetter [20], Medina [27], or Milnor [29]). Namely, if $X, Y \in \mathscr{g}$ are left-invariant vector fields on a Lie group $\mathscr{g}$ with a left-invariant affine structure, we may define the product XY to be the covariant derivative $\nabla_X Y$ which is also a left-invariant vector field. It can be shown that the affine structure is complete if and only if the resulting algebra is nilpotent and the affine structure is bi-invariant if and only if the algebra is associative. Conversely, if A is a nilpotent associative algebra, the rule $\theta(x): y \to x + y + xy$ defines a homomorphism of Lie algebras $A \to \mathrm{End}(A)$ tangent to a simply transitive affine action defining a bi-invariant affine structure.

For example the associative algebras corresponding to the structures H_α are precisely those 3-dimensional associative algebras A satisfying $A^3 = 0$. The cases when α is symmetric and degenerate correspond to $\dim A^2 \geqslant 2$; otherwise α appears as the bilinear map $A/A^2 \times A/A^2 \to A^2$ induced by multiplication.

When α is nonzero and alternating, the associative product on $\mathscr{g}$ is defined

by $XY = \frac{1}{2}[X, Y]$. As observed by Cartan, this works whenever $\mathcal{g}$ is 2-step nilpotent. It follows that any Lie algebra automorphism of $\mathcal{g}$ is an automorphism of this associative algebra structure of $\mathcal{g}$ as well, i.e., this structure is "fully symmetric."

A.6. Now we turn our discussion to the groups $G = H_{(b,c)}$, where $(b, c) \in \mathbb{R}^2$. Here

$$G = \left\{ \begin{bmatrix} 1 & cu & bt + cu^2/2 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s + (b + c)tu/2 + cu^3/6 \\ t + u^2/2 \\ u \end{bmatrix} : s, t, u \in \mathbb{R} \right\}.$$

The inverse of $\mathrm{dev}(s, t, u) = (s + (b + c)\, tu/2 + cu^3/6,\ t + u^2 \cdot 2, u)$ is easily seen to be $g(x, y, z) = (x - (b + c)(y - z^2/2)z/2 - cz^3/6,\ y - z^2/2, z)$. We may use f to write the elements of G using affine coordinates:

$$\begin{bmatrix} 1 & cz & b(y - z^2/2) + cz^2/2 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The columns (resp. rows) of the square matrix (resp. its inverse) determine a basis of the left-invariant vector fields (resp. 1-forms) as they appear in affine coordinates.

Left-invariant:

vector fields	1-forms

$\dfrac{\partial}{\partial x}, cz \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y},$ $\qquad dx - cz\, dy - \left(b\left(\dfrac{y - z^2}{2} \right) - \dfrac{cz^2}{2} \right) dz,$

$\left(b(y - z^2 2) + \dfrac{cz^2}{2} \right) \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}$ $dy - z\, dz, dz;$

Right-invariant:

$\dfrac{\partial}{\partial x}, bz \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y},$ $\qquad dx - bz\, dx - cy\, dz,$

$cy + z \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z},$ $\qquad dz - z\, dz, dz.$

Notice that the quadratic term $(c - b)z^2/2$ disappears precisely when G is abelian.

A.7. Finally we discuss the case (I) when $G = \mathbb{R}^2 \rtimes \mathbb{R}_+$ where $\mathbb{R}_+$ acts by hyperbolic automorphisms of $\mathbb{R}^2$. (The case when $\mathbb{R}_+$ acts by elliptic automorphisms is completely analogous and is omitted.) We first consider the "standard" structure, where the unipotent radical of the algebraic hull acts by translations. Equivalently, G preserves a parallel Lorentz metric. We shall derive the other structure on G (due to Auslander) as a polynomial deformation of the Lorentzian affine structure on G.

When G is "Lorentzian" we have:

$$G = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-s} \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} : s, t, u \in \mathbb{R} \right\}$$

and some invariant tensor fields are tabulated:

Left-invariant:

vector fields	1-forms
$\dfrac{\partial}{\partial x}, e^x \dfrac{\partial}{\partial y}, e^{-x} \dfrac{\partial}{\partial z}$	$dx, e^{-x}\, dy, e^x\, dz;$

Right-invariant:

$\dfrac{\partial}{\partial x} + y \dfrac{\partial}{\partial y} - z \dfrac{\partial}{\partial z}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z},$	$dx, dy - y\, dx, dz + z\, dx.$

Now for each $\lambda \in \mathbb{R}$ consider the polynomial automorphism of $\mathbb{R}^3$ defined by $h_\lambda(x, y, z) = (x + \lambda yz, y, z)$. One can check that h_λ conjugates G to a group $G_\lambda = h_\lambda G h_\lambda^{-1}$ of affine transformations which can be computed to be

$$G_\lambda = \left\{ \begin{bmatrix} 1 & \lambda u e^s & \lambda t e^{-s} \\ 0 & e^s & 0 \\ 0 & 0 & e^{-s} \end{bmatrix} \begin{bmatrix} s + \lambda tu \\ t \\ u \end{bmatrix} = s, t, u \in \mathbb{R} \right\}.$$

Using h_λ one can compute the invariant tensors:

Left-invariant:

vector fields	1-forms
$\dfrac{\partial}{\partial x}, e^{x-\lambda yz} \left(\lambda z \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} \right),$	$dx - \lambda z\, dy - \lambda y\, dz,$
$e^{\lambda yz - x} \left(\lambda y \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial z} \right),$	$e^{x-\lambda yz}\, dy, e^{\lambda yz - x}\, dx;$

Right-invariant:

vector fields	1-forms

$$\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z},$$

$$dx - \lambda z\, dy - \lambda y\, dz,$$

$$z\, dx - \lambda z^2\, dy + (1 - \lambda yz)\, dz,$$

$$\frac{\partial}{\partial y} + \lambda z\frac{\partial}{\partial x}, \frac{\partial}{\partial z} + \lambda y\frac{\partial}{\partial x},$$

$$-y\, dx + (1 + \lambda yz)\, dy + \lambda y^2\, dz.$$

Notice that the group G_λ leaves invariant the polynomial Lorentz metric $(dx - \lambda z\, dy - \lambda z\, dz)^2 + dy\, dz$ which becomes parallel when $\lambda = 0$. If $\Gamma \subset G$ is a lattice subgroup, then the cohomology is represented by the invariant polynomial differential forms:

1	in degree 0,
$dx - \lambda z\, dy - \lambda y\, dx$	in degree 1,
$dy \wedge dz$	in degree 2,
$dx \wedge dy \wedge dz$	in degree 3.

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