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TWO EXAMPLES OF AFFINE MANIFOLDS

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An affine manifold is a manifold with a distinguished system of affine coordinates, namely, an open covering by charts which map homeomorphically onto open sets in an affine space E such that on overlapping charts the homeomorphisms differ by an affine automorphism of E. Some, but certainly not all, affine manifolds arise as quotients Ω/Γ of a domain in E by a discrete group Γ of affine transformations acting properly and freely. In that case we identify Ω with a covering space of the affine manifold. If $\Omega=E$, then we say the affine manifold is complete. In general, however, there is only a *local* homeomorphism of the universal covering into E, which is equivariant with respect to a certain affine representation of the fundamental group. The image of this representation is a certain subgroup of the affine group on E, is called the affine holonomy and is well defined up to conjugacy in the affine group. See Fried, Goldman, and Hirsch.

THEOREM. *There exists a compact affine manifold M satisfying the following conditions*:

(1) *M is the quotient of a proper convex domain which is not a convex cone.*

(2) *M is incomplete and the affine holonomy group leaves invariant no proper affine subspace.*

The example we give will be three-dimensional (it is a torus bundle over the circle); by taking products with compact complete manifolds, we obtain similar examples in all dimensions greater than two. The affine holonomy group Γ will be solvable; by Fried, Goldman, and Hirsch [3] condition (2) cannot occur if Γ contains a nilpotent subgroup of finite index—hence there cannot be any two-dimensional examples. There was some hope that the affine holonomy of a compact incomplete affine manifold would necessarily leave invariant an affine subspace; verifying this for similarity structures is the key step in the recent classification of closed similarity manifolds by David Fried [2].

There has also been some work proving, under various hypotheses, that the universal covering of a compact affine manifold, if it is convex, must be a convex cone. If in the universal cover, *no* geodesic extends infinitely in both directions, it follows from result of Jacques Vey, that the universal cover must be a cone.

Finally, the above example leads immediately to another somewhat unusual example: if Ω is the convex universal covering of M above, then the holonomy group Γ acts properly and freely on the interior of the complement of Ω as well; thus we obtain a new affine manifold $\operatorname{int}(E - \Omega)/\Gamma$, which is compact, and has for its universal covering a *concave* domain which is not the complement of an affine subspace.

THE EXAMPLES. Consider a parabola in the plane R^2, say the one given by $y^2 - 2x = 0$. Then the group G' of orientation-preserving affine transformations preserving it is the two-dimensional group of affine maps

$$\begin{pmatrix} e^{2s} & e^s t \\ 0 & e^s \end{pmatrix} \begin{pmatrix} \frac{1}{2} t^2 \\ t \end{pmatrix} \qquad (s, t \in R) .$$

(Here the square matrix denotes the linear part and the collumn vector denotes the translational part of the specified affine map). It is easy to check that G' is nonabelian, and acts simply transitively on each of the two open sets

$$0_+ = \{(x, y) | y^2 - 2x > 0\}$$
$$0_- = \{(x, y) | y^2 - 2x < 0\} .$$

The set 0_+ is convex although it is not a cone. This should be contrasted to the result of J. L. Koszul [4] that a convex domain in affine space upon which a *unimodular* group of affine transformations acts transitively must be a cone.

To obtain a compact affine manifold it is necessary to consider the domains $0_+ \times R$ and $0_- \times R$ in R^3. To do this we add a one-parameter group of translations in the new direction, as well as modify G' so as to make the nonunipotent one-parameter subgroup $(t = 0)$ exponentially contract in the new direction. Specifically, consider the group G of affine transformations of R^3 given by

$$\begin{pmatrix} e^{2s} & e^s t & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-s} \end{pmatrix} \begin{pmatrix} \frac{1}{2} t^2 \\ t \\ u \end{pmatrix} \qquad (s, t, u \in R) .$$

Clearly G acts simply transitively on $0_+ \times R$ and $0_- \times R$. It is easily checked that G is isomorphic to the Lorentz group $E(1, 1)$ which admits discrete cocompact subgroups Γ, none of which have nilpotent subgroups of finite index.

Since G acts simply transitively on $0_+ \times R$ (resp. $0_- \times R$) subgroups Γ as above act properly and freely; the quotient $0_+ \times R/\Gamma$ (resp. $0_- \times R/\Gamma$) is a compact affine manifold of dimension three having solvable fundamental group. The rest of the assertions in the theorem follow immediately, taking $M = (0_+ \times R)/\Gamma$.

The affine manifold M should be contrasted with the following theorem of J. Vey [5]:

THEOREM (*Vey*). *Let Ω be a convex domain in E and let Γ be a group of affine transformations preserving Ω such that Ω/Γ is compact* (*but not necessarily Hausdorff*). *Suppose*

(A) *Ω contains no complete line*

and one of

(B) *Ω/Γ is Hausdorff and Γ is discrete*

(C) *Ω contains an open cone.*

Then Ω is an open cone.

The domain $\Omega = 0_+$ satisfies (A) but neither (B) nor (C) (taking $\Gamma = G'$) and is not an open cone; similarly the domain $\Omega = 0_+ \times R$ satisfies (B) but neither (A) or (C) and is not an open cone. Hence Vey's result is best possible. (Actually, the conclusion of Vey's theorem follows from just hypothesis (C)—for every convex domain is a product $\Omega' \times R^k$ where Ω' satisfies (A); the result for Ω' then implies the result for $\Omega' \times R^k$.)

It is interesting to vary to group G with a parameter p. By considering the parabola $py^2 - 2x = 0$ instead, $p \neq 0$, we may replace the group G by the group G_p consisting of the affine maps

$$\begin{pmatrix} e^{2s} & pe^s t & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-s} \end{pmatrix} \begin{pmatrix} \frac{p}{2} t^2 \\ t \\ u \end{pmatrix} \qquad (s, t, u \in R) .$$

Since all parabolas are affinely equivalent, the groups $G_p (p \neq 0)$ are all conjugate subgroups of the affine group. However, as $p \to 0$, the groups G_p converge to an isomorphic group G_0 which acts simply transitively on each of the half-spaces which compose the complement of the affine subspace $x = 0$. Choosing a discrete subgroup Γ as above, we obtain still another affine structure on the compact 3-manifold M. Notice as p varies from positive to negative, the affine manifold $0_+ \times R/\Gamma$ continuously deforms to the affine manifold $0_- \times R/\Gamma$, passing through the structure obtained from G_0.

It is possible to deform the group G_0 in another way. Let $G_{0,\lambda} (\lambda \in R)$ be the group of affine transformations

$$\begin{pmatrix} e^{\lambda s} & 0 & 0 \\ 0 & e^{s} & 0 \\ 0 & 0 & e^{-s} \end{pmatrix} \begin{pmatrix} 0 \\ t \\ u \end{pmatrix}.$$

For $\lambda \neq 0$, this group acts simply transitively on a half-space Ω. For $\lambda = 2$, the group $G_{0,\lambda}$ is just the group G_0 above. By taking discrete cocompact subgroups Γ, we obtain more affine manifolds (all of which are homeomorphic) Ω/Γ. For different values of λ these affine structures are distinct.

For more examples of affine structures on these manifolds, the reader is referred to the introduction to Auslander [1], and Fried, Goldman, and Hirsch [3].

Finally we note that most of these examples are "topologically conjugate," i.e., there is a homeomorphism of R^3 conjugating one group to another. For example, $G_{0,\lambda}$ and $G_{0,\mu}$ are conjugate under the Holder continuous map $(x, y, z) \to (x^{\mu/\lambda}, y, z)$, at least if λ and μ have the same sign. All the groups G_p are conjugate to G_0 under the algebraic morphism $(x, y, z) \to (x - p/2\, y^2, y, z)$. Thus the affine manifolds $0_+/\Gamma$ and $0_-/\Gamma$ are algebraically equivalent although the behavior of geodesics on them is quite different.

References

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