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★**Geometric structures on manifolds.**
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The concept of a group, which dates back to the 1830s, is due to Évariste Galois and was used for the study of systems of polynomial equations. The profound fact is that any transformation of variables which preserves such a system $S$ transforms any solution of $S$ into another solution of $S$. The set of transformations which preserve $S$ is the group of symmetries of $S$. Another remarkable fact is that properties of the group of symmetries of $S$ can affect properties of the set of solutions of $S$. From this remark emerges another significant fact: one can study the properties of the set of solutions of $S$ without being able to explicitly give these solutions. This is a variation of the central ideas of Felix Klein's Erlangen program. The theory of infinite groups of Lie and Cartan is an extension to systems of partial differential equations [I. M. Singer and S. Sternberg, J. Analyse Math. **15** (1965), 1–114; MR0217822]. Via formalisms based on Koszul-Spencer homology, H. Lewy's counterexample informs us that there exist systems of partial differential equations which are formally integrable but which have no differentiable solution [Ann. of Math. (2) **66** (1957), 155–158; MR0088629]. This provides an illustration of links between properties of the groups of symmetries of a system $S$ and properties of the set of solutions of $S$.

Comparisons of mathematically structured sets provide other areas where groups reign. Before receiving a clear description, the comparison parameters of a structured mathematical set are called local charts. A local chart is used to compare a structured set $Y$ with another structured set $X$ which serves as a model. Roughly speaking, a local chart is a partial map $f$ of $Y$ into $X$.

The risk of prematurely jumping in with both feet is great if two local charts defined at $y \in Y$ are not compatible. The crucial problem is to specify the content of the notion of compatibility of two local charts. To progress in this crucial problem one considers the group $H$ formed of symmetries of the structured set $X$. By composition of maps, a local chart $f$ is associated with the set of partial maps

$$H(f) = H \circ f.$$

Let $f$ and $f'$ be two local charts defined at $y \in Y$; they are compatible if

$$f' \in H(f).$$

Now one relies on the Erlangen program to say that the pair $(H, X)$ is a *geometry*. To any subgroup

$$G \subset H$$

is associated the geometry $(G, X)$ which is a restriction of $(H, X)$.

Two local charts $f$ and $f'$ which are defined at $y \in Y$ are $G$-compatible if

$$f' \in G(f).$$

A $(G, X)$-structure in $Y$ is a *complete* family of local charts which are pairwise $G$-compatible. There arises the following fundamental question:

Given a (structured) geometry $(G, X)$, what are the (structured) sets $Y$ which admit a $(G, X)$-structure?

The book under review is devoted to this fundamental question. In addition to a rich foreword and introduction, the book is composed of 15 chapters which are grouped into three parts. The numbering of the chapters is independent of parts. Below is an overview.

The central object of Part 1 is formed of two geometries which are closely linked to linear algebra and which will serve as model structure for differential topological spaces (otherwise called differentiable manifolds). These two structures are the affine structure and the projective structure.

Chapter 1 is devoted to the affine spaces. An affine space is a pair $(T, X)$ formed of a vector space $T$ and a set $X$ subject to a simply transitive action

$$T \times X \to X.$$

An element of $T$ is called a translation on $X$. Two elements of $X$ are parallel if we pass from one to the other by means of a translation.

Thus the affine geometry is the geometry of parallelism. A straight line is an orbit of one parameter subgroup of $T$. The group of symmetries of $(T, X)$ is called the affine group $(T, X)$. The affine group of $(T, X)$ is denoted by $\mathrm{Aff}(T)$. The notion of affine connection is introduced. An important fact is that an affine structure can be defined in terms of both the affine connection and the notion of parallelism relative to an affine connection. Following the general principle, any subgroup of $\mathrm{Aff}(T)$ defines a sub-geometry of the affine geometry. This principle is implemented to introduce both new notions and the groups of their symmetries. Examples are the notions of distance, volume and angle which give rise to the group rigid affine transformations, the special linear group and the group of similarities.

Chapter 2 is devoted to the projective spaces. The relationships between the affine space and the sub-models treated as in Chapter 1 are called extension or restriction relationships.

On the other hand, the transition from affine spaces to projective spaces is based on processes of transition to quotient modulo equivalence relations in an affine space $(T, X)$ and in the affine group $\mathrm{Aff}(T)$. It is also a process of compactification (see Section 2.1).

It is appropriate to remember that the image a straight line by an affine map is either another straight line or a point. The second alternative gives rise to the notion of a singular projective mapping, which is the subject of Subsection 2.3.2.

There are two other notions which serve as a basis for introducing the axioms of projective geometry. These are the notion of alignment and the notion of cross-ratio of four aligned points. Section 2.5 deals with these notions.

Another useful key notion is the projective duality which transforms aligned pints into concurrent straight lines. This viewpoint is also an object of Chapter 2. Numerous examples and exercises are offered to readers.

Chapter 3 is devoted to duality and non-Euclidean geometry. Marking (axiomatic) points in the development of projective geometry are presented, with particular emphasis on the case of dimension 2.

A constant quality of this book is the abundance of exercises to arouse more curiosity in the reader.

Chapter 4 deals with the notion of convexity. A subset $C$ of an affine space is convex if any line segment whose extremities belong to $C$ is entirely included in $C$. Section 4.1 is devoted to convex domains with particular emphasis on convex cones. Convex cones and their geometry are the subject of Section 4.2.

It is appropriate here to emphasize that convex domains offer sumptuous frameworks for the convergence of algebra, analysis, geometry and information [F. Barbaresco, AIP Conference Proc. **1641** (2015), no. 1, 74–81, doi:10.1063/1.4905965]. What's more, this list is not exhaustive.

Chapter 4 is dense and rich in points of history of the geometry of convex domains as well as in still-promising perspectives.

In Part 2 the frameworks are topological spaces enriched with differential structure, which makes them differentiable manifolds.

The four chapters of Part 1 are devoted to geometries which will serve as (local) models for geometric structures in differentiable manifolds. There are several ways to provide a topological space $Y$ with a differential structure. The method used in Section 5.1 is that of complete atlas of local charts.

Section 5.2 is devoted to the notion of development of an $(H, X)$-structure in a differentiable manifold $Y$. A relevant by-product is the notion of holonomy, which is a representation of the fundamental group of $Y$ in $H$. These objects are presented in a manner accessible to non-experts. For instance, Section 5.4 is devoted to geometric structures on manifolds of dimension 1.

Chapter 6 is of central interest to non-geometers. It is essentially devoted to examples.

The subject of Chapter 7 is the classification of geometric structures.

It must be emphasized that from the point of view of global analysis on differentiable manifolds, the structures studied in this book are almost all of type 1. Chapter 7 deals with what in global analysis is called the equivalence problem [B. Malgrange, Ann. Fac. Sci. Toulouse Math. (6) **26** (2017), no. 5, 1087–1136; MR3746623]. It is known that in the category of structures of finite type formal equivalence leads to differentiable equivalence. Thus for Chapter 7, only algebraic formalisms are necessarily expected.

Sections 7.1 to 7.4 are devoted to geometric structures in compact manifolds while Section 7.5 is devoted to the cases of open manifolds.

The main subject of Chapter 8 is the completeness of geometric structures. As was mentioned in Part 1, an affine structure can be defined by means of gauge theory (i.e., study of linear connections). In this context based on gauge theory, the development mapping is defined using parallel transport [J.-L. Koszul, Ann. Inst. Fourier (Grenoble) **18** (1968), fasc. 1, 103–114; MR0239529]. So, the word "completeness" conveys three possible understandings, two of which each correspond to one of two notions of developing map, the third being geodesic completeness. These alternatives and their interrelations are at the heart of Chapter 8. A central question is under what sufficient conditions an affine structure is complete. The author recalls conjectures including the conjecture of Markus which links completeness to a structural property holonomy. An important and large part of Chapter 8 is devoted to an survey of works on the conjecture of Markus.

As a reminder, Part 1 was largely dedicated to model structures strongly linked to the geometry of affine spaces and that of projective spaces. Part 3 is devoted to differentiable manifolds which admit these structures.

Among the pioneering works in affine geometry, there are those of Benzécri. Benzécri's works and their extensions by other authors largely occupy Chapter 9. According to one of the most striking classics of Benzécri, the flat torus is the only orientable closed surface which admits a complete affine structure.

Nowadays, there exists a category of differentiable manifolds whose power affects an immense part of the mathematical universe. This is the category of Lie groups.

Any group $H$ is a domain of two canonical dynamics, $(L_H, H)$ and $(\mathrm{Ad}_H, H)$; $L_H$ is the group of left translations on $H$ and $\mathrm{Ad}_H$ is the group of inner automorphisms of $H$. Let $G$ be an $m$-dimensional real Lie group. An $(\mathrm{Aff}(\mathbb{R}^m), \mathbb{R}^m)$-structure on $G$ is defined by a complete $\mathrm{Aff}(\mathbb{R}^m)$-compatible atlas

$$A = \{(U_j, \phi_j)\}\,.$$

What does it mean that an affine structure on $G$ is invariant by $L_G$?

Another challenge is the question of knowing which finite-dimensional Lie groups admit an affine structure which is invariant under left translations.

In Part 1 it is established that affine geometry has a *versus linear connection*. A linear connection defines a multiplicative structure in the vector space of vector fields. Thus, as for the studies of linear connections invariant by one or the other of the two dynamics $L_G$ and $\mathrm{Ad}_G$, they are reduced to the studies of certain types of linear representations of the Lie algebra of vector fields which are invariant by $L_G$. These studies and their subsidiaries constitute the content of Chapter 10.

Chapter 11 is a more extensive return to relations between volume invariance and completeness.

Chapter 12 is devoted to hyperbolicity. Here one needs to be vigilant regarding the meaning. In Riemannian geometry hyperbolicity characterizes a property of the curvature. In affine geometry hyperbolicity characterizes a (topological) property of the development mapping. Chapter 12 is concerned with the second configuration where, however, Riemannian structures stand out in an unavoidable way (Hessian structure, and Kobayashi metric).

Chapter 13 is devoted to the second issue of Part 3, i.e., the projective structures. The challenge is centered on the projective geometry of surfaces.

Chapter 14 is devoted to complex projective structures. Both Chapters 13 and 14 are not long, but they are dense.

The category of differentiable manifolds of dimension 3 is a universe of great geometric and topological challenges (remember the long resistance of the Poincaré conjecture). Chapter 15 is devoted to geometric structures in differentiable manifolds of dimension 3 and 4. Significant space is devoted to Margulis spacetimes and to deformations of hyperbolic surfaces.

The last section of the book is 15.4. It is devoted to Dupont's classification of hyperbolic torus bundles.

The final part of the book is devoted to appendices. Here, with a pedagogical concern worthy of praise, the author brings together tools, many of which are only familiar to specialists. The bibliography is of relevance that commands admiration.

This book should be of great interest to a wide spectrum of readers, ranging from curious readers to researchers through non-geometric teachers.

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