WRITTEN QUALIFYING EXAMINATION
ORDINARY DIFFERENTIAL EQUATIONS
AUGUST 1996

Instructions: Answer each of the following six problems, giving careful statements of any theorems you invoke and giving careful explanations of your procedures. Your answer to each question will be assigned a grade from 0 to 10. Although some preblems are broken into parts, the parts need not have equal weight.
Begin each problem on a separate answer sheet. Write the problem number and your code number (NOT your name) on the answer sheets.
Clearly indicate the pages that are to be regarded as scratch work and the pages that are not part of the solution.

1. Let $f(x) = x\ln(1/x)$ when $0 < x < 1$ and $f(x) = 0$ when $x \leq 0$. Show that the scalar IVP $\{\dot{x} = f(x);\ x(0) = 0\}$ has a unique solution.

2. Consider the 2-dimensional system (1) $\dot{x} = f(x)$ where f is smooth, $f(0) = 0$, $x^t f(x) > 0$ in a punctured neighborhood of 0, and $f(x) \neq 0$ for all $x \neq 0$. If (1) does not possess a periodic orbit (except the fixed point at the origin), prove that all solutions of (1) except $x \equiv 0$ are unbounded on $[0, \infty)$.

3. Consider the n-dimensional system $\dot{x} = f(x)$, where f is smooth, and assume that there is a point x_0 whose positive semiorbit is compact. Prove that x_0 is either a fixed or a periodic point.

4. Consider the second order equation (1) $\ddot{y} + p(t)y = 0$, where p is continuous and T-periodic. Let y_1, y_2 be solutions of (1) which satisfy the initial conditions $y_1(0) = 1$, $\dot{y}_1(0) = 0$, $y_2(0) = 0$ and $\dot{y}_2(0) = 1$. Let ρ be a complex number.
Show that there is a nontrivial (possibly complex valued) solution y that satisfies $y(t+T) \equiv \rho y(t)$ if and only if ρ satisfies the equation $\rho^2 - (y_1(T) + \dot{y}_2(T))\rho + 1 = 0$.

5. Consider the second order equation $\ddot{y} + g(y, \dot{y}) = 0$, where g is smooth on $\mathbb{R} \times \mathbb{R}$, $g(0,0) = 0$ and $\dfrac{\partial g}{\partial y}(0,0)$, $\dfrac{\partial g}{\partial \dot{y}}(0,0) > 0$. Prove that $y(t) \equiv 0$ is asymptotically stable.

6. Consider the scalar second order IVP

$$\ddot{y} + \mu\dot{y} + h(y) = 0,\ y(0) = y_0 > 0,\ \dot{y}(0) = 0, \tag{1}$$

where $\mu > 0$, h is smooth and $yh(y) > 0$ for all $y \neq 0$.
a) Show that the solution of (1) is bounded above for $t \geq 0$.
b) Show that the solution of (1) is bounded below for $t \geq 0$.