DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
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MATH/MAPL 673-674

Instructions to the student

a. Answer all six questions. Each will be assigned a grade from 1 to 10.
b. Use a different booklet for each question. Complete the top of the first page of each booklet. Write your **code number** on each page of the booklet. DO NOT USE YOUR NAME.
c. Keep scratch work on separate pages of the booklet.

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1. Find the appropriate solution, at $t = 1$ and $t = 2$, of the equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

with initial data

$$u(x,0) = \begin{cases} 1 & x < -1, \\ -x & -1 < x < 0, \\ 1 & x > 0. \end{cases}$$

2. Find the characteristic system for the equation

$$u^2(1 + u_x^2 + u_y^2) = 1, \tag{2}$$

and show that the system can be solved in closed form by quadrature (I.e., show that the solution can be reduced to the evaluation of certain integrals. You need not evaluate the integrals.) Next, show that

$$u(x,y) = (1 - (x-a)^2 - (y-b)^2)^{1/2}$$

is a complete integral. Finally, solve (2) subject to the data

$$u(x,0) = 1.$$

3. Formulate Duhamel's principle for the solution to the inhomogeneous wave equation in three dimensions with homogeneous initial data:

$$\begin{aligned} &u_{tt} - \Delta u = f, \quad \text{in } R^3 \times (0,\infty), \\ &u = 0 \quad \text{and} \quad u_t = 0 \quad \text{in } R^3 \text{ at } t = 0. \end{aligned} \tag{3}$$

Given Kirchhoff's formula for the solution of

$$\begin{aligned} u_{tt} - \Delta u = 0, \quad &\text{in } R^3 \times (0,\infty), \\ u = g \quad \text{and} \quad u_t = h \quad &\text{in } R^3 \text{ at } t = 0, \end{aligned}$$

namely

$$u(x,t) = \frac{1}{4\pi t^2}\int_{|y-x|=t} th(y) + g(y) + DG(y)\cdot(y-x)\,dS(y),$$

derive the retarded potential solution to (3),

$$u(x,t) = \frac{1}{4\pi}\int_{|y-x|\le t} \frac{f(y, t-|y-x|)}{|y-x|}\,dy.$$

4. The following nonlinear parabolic PDE has a solution $u(x,t) = \tanh x$.

$$N(u) \equiv u_t - u_{xx} - 2(u - u^3) = 0. \tag{1}$$

If $u(x,t)$ is C^2, we say that u is a *subsolution* of (1) if $N(u) \le 0$ for all (x,t), and u is a *supersolution* if $N(u) \ge 0$.

(a) Suppose $v(x,t)$ is a C^2 supersolution of (1) and $w(x,t)$ is a C^2 subsolution, and that for any $T > 0$, there exists $L > 0$ such that

$$w(x,t) < v(x,t) \quad \text{if } 0 \le t \le T \text{ and } |x| \ge L.$$

Suppose that $w(x,0) < v(x,0)$ for all x. Prove that $w < v$ for all $(x,t) \in R \times [0,\infty)$. (Hint: find a subsolution of the heat equation in the form $z = e^{ct}(w - v)$.)

(b) Find positive functions a and b, with $a(0) = 1$, $\lim_{t\to\infty} a(t) = 0$, $b(0) = 0$, $0 < \lim_{t\to\infty} b(t) < \infty$, such that

$$v(x,t) = \delta\, a(t) + \tanh(x + \delta\, b(t)),$$

is a supersolution for (1), if $\delta > 0$ is sufficiently small.

(Remark: These results can be used to establish that the tanh solution is stable.)

5. Given $p < n$, define $p^* = np/(n-p)$.

(a) Assume $1 \le p < n$ and $q \ge 1$, with $q \ne p^*$. Let A be the set of C^1 functions u defined on R^n having compact support, with u not identically zero. Using a scaling argument, show that

$$\inf_{u \in A} \frac{||Du||_{L^p}}{||u||_{L^q}} = 0.$$

(b) Show that there is a positive constant C such that

$$\int_0^\infty |u(r)|^2 r\, dr \le C \left(\int_0^\infty |u_r(r)|\, r\, dr \right)^2$$

for every C^1 function $u : [0,\infty) \mapsto R$ with compact support. (Hint: establish a more general inequality for functions defined on R^2.)

6. Let U be a bounded open subset of R^2 with smooth boundary, and let $\rho \in C^1(\overline{U})$ be a positive function. Let $\vec{\zeta}$ be an L^2-vector-field on U (i.e., $\vec{\zeta} \in L^2(U, R^2)$), and define a functional F on $H = H_0^1(U, R)$ as follows: For $\phi \in H$,

$$F(\phi) = \int_U \frac{1}{2\rho} |\vec{\zeta} - \nabla\phi|^2 .$$

(a) Suppose $\phi \in H$ *minimizes* F on H, i.e., suppose $F(\phi) = \min_{u \in H} F(u)$. Find an elliptic boundary value problem which ϕ satisfies in the weak sense.

(b) Prove that this elliptic boundary value problem has a unique weak solution. Does it follow that a unique minimizing ϕ for F exists?

(c) Suppose $\vec{\zeta} \in C^1(U, R^2) \cap C(\overline{U}, R^2)$ and F has a minimizer $\phi \in C^2(U, R)$. Prove that $\vec{\zeta} = \nabla\phi + \rho\vec{v}$, where $\vec{v} \in C^1(U, R^2)$ with

$$\operatorname{div} \vec{v} = 0 \quad \text{in } U.$$