DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
JANUARY 1996

MATH/MAPL 673-674

Instructions to the student

a. Answer all six questions. Each will be assigned a grade from 1 to 10.
b. Use a different booklet for each question. Complete the top of the first page of each booklet. Write your **code number** on each page of the booklet. DO NOT USE YOUR NAME.
c. Keep scratch work on separate pages of the booklet.
d. If you use a "well-known" theorem in your solution to any problem, it is your responsibility to make clear exactly which theorem you are using and to justify its use.

****

1. Let $\Omega \subset R^2$ be a bounded domain with smooth boundary. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$u_{xx} + u^2 u_{yy} + 1 + u - u^3 = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Prove that $u(x, y) < 2$ for $(x, y) \in \Omega$.

2. Consider the Cauchy problem

$$u_{tt} = c^2 \Delta u - q(x)u, \quad x \in R^3,\ t > 0,$$
$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x),$$

where f and g are smooth with compact support and $q(x) \geq 0$ and $c > 0$ is a constant. Use an energy method to show that this problem has at most one C^2 solution. (You may assume finite speed of propagation.)

3. Consider the following boundary value problem for Laplace's equation in the infinite strip $\Omega = \{(x, y) \in R^2 \mid -\infty < x < \infty,\ 0 < y < 1\}$,

$$u_{xx} + u_{yy} = 0 \quad \text{in } \Omega,$$
$$u(x, 1) = f(x), \quad \frac{\partial u}{\partial n}(x, 0) = 0, \qquad (-\infty < x < \infty).$$

Here n denotes the outward unit normal on the boundary $\partial\Omega$. The *Dirichlet-Neumann* map T is defined (when possible) by

$$(Tf)(x) = \frac{\partial u}{\partial n}(x, 1) \quad -\infty < x < \infty.$$

Show that if $f \in \mathcal{S}$ is a function in the Schwartz class $\mathcal{S}$ of smooth, rapidly decreasing functions on R, then $Tf \in \mathcal{S}$. Show that T may be extended by continuity to be a bounded map $T : H^1(R) \mapsto L^2(R)$. (Hint: Use the Fourier transform.)