Department of Mathematics
Graduate Written Examination in Partial Differential Equations
August, 2004

Instructions

- Answer all six questions. Each will be assigned a grade from 0 to 10.
- Begin your answer to each question on a separate answer sheet. Write your code number on each page of your answers. Do not use your name.
- Keep scratch work on separate sheets, which should not be submitted.
- Carefully explain your steps. If you invoke a "well known" theorem, it is your responsibility to make clear exactly which theorem you are using, and to justify its use.

1. Let $f \in C^1(\mathbb{R}^n)$ and suppose that for each open ball B that there exists a solution of the boundary value problem

$$-\Delta u = f \quad \text{in } B, \quad \partial u/\partial n = 0 \quad \text{on } \partial B.$$

Show that $f \equiv 0$.

2. Suppose that $u(x,t)$ is a smooth solution of

$$\begin{aligned} u_t + uu_x &= 0 \quad \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) &= f(x) \quad \text{for } x \in \mathbb{R}. \end{aligned}$$

Assume that f is a C^1 function such that

$$f(x) = \begin{cases} 0 & \text{for } x < -1 \\ 1 & \text{for } x > 1 \end{cases} \quad \text{and} \quad f'(x) > 0 \quad \text{for } |x| < 1.$$

Show that for $t > 0$,

$$\lim_{r\to\infty} u(rx, rt) = \begin{cases} 0 & \text{for } x < 0 \\ x/t & \text{for } 0 < x < t \\ 1 & \text{for } x > t \end{cases}$$

3. Let $D \subset \mathbb{R}^2$ be the open disk $D = \{x^2 + y^2 < 1\}$ and let $C = \{x^2 + y^2 = 1\}$. Using polar coordinates, let $u_r = \partial u/\partial r$ and $u_\theta = \partial u/\partial\theta$.

a) Find the weak formulation of the oblique derivative problem

$$-\Delta u + \lambda u = f \quad \text{in } D, \quad u_r + u_\theta = 0 \quad \text{on } C. \tag{1}$$

In other words, find a bilinear form $a_\lambda(u,v)$ continous on $H^1(D)\times H^1(D)$ such that

$$a_\lambda(u,v)=\int_D fv\;dxdy \quad \text{for all } v\in H^1(D) \tag{2}$$

is equivalent to (1) if $u\in C^2(\bar{D})$. You may assume the trace theorem

$$\int_C uv_\theta\;d\theta \le C\|u\|_{H^1(D)}\|v\|_{H^1(D)} \text{ for } u,v\in H^1(D).$$

b) Show that for $f\in L_2(D)$ and for $\lambda>0$, there is unique solution $u\in H^1(D)$ of (2).

4. For $x\in\mathbb{R},\ t\ge 0$, solve the problem

$$\begin{aligned} u_{tt}-u_{xx} &= 0 \quad \text{for} \quad |x|<t \\ u(-t,t) &= \alpha(t) \quad \text{for} \quad t\ge 0 \\ u_t(t,t) &= \beta(t) \quad \text{for} \quad t\ge 0, \end{aligned}$$

where α and β are smooth functions.

5. Let Ω be a bounded open set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $u(x,t)\in C^2(\Omega\times(0,\infty))$, with u continuous on $\bar{\Omega}\times[0,\infty)$, be a solution of the initial-boundary value problem

$$\begin{aligned} u_t-\Delta u &= \sin(u) \quad \text{for } x\in\Omega,\ t>0 \\ u(x,t) &= 0 \quad \text{for } x\in\partial\Omega \\ u(x,0) &= f(x) \quad \text{for } x\in\Omega. \end{aligned}$$

Show that if $f(x)\le 1$, then $u(x,t)\le e^t$ for all $x\in\Omega, t>0$. Hint: Note that $\sin x = x(\sin x)/x$ and that $(\sin x)/x$ is bounded. Look for a way to use the maximum principle.

6. Let $F:\mathbb{R}\to\mathbb{R}$ have two continuous derivatives, with $F(0)=0$ and suppose that there is constant M such that $|F(x)|+|F'(x)|+|F''(x)|\le M$ for all $x\in\mathbb{R}$. Let $u\in C^2(\mathbb{R})$ with compact support and set $v(x)=F(u(x))$.

Show that there is a constant C, independent of u such that

$$\|v\|_{H^2(\mathbb{R})}\le C\|u\|_{H^2(\mathbb{R})}(1+\|u\|_{H^1(\mathbb{R})}).$$