DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

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LOGIC

1. a) Let L be a language whose non-logical symbols include the binary relation E. Let T be a theory of L such that $E^{\mathcal{A}}$ is an equivalence relation on A for every model $\mathcal{A}$ of T. Assume that in every model $\mathcal{A}$ of T there is exactly one infinite $E^{\mathcal{A}}$-class. Prove that there is some n in ω such that in every model $\mathcal{A}$ of T, all finite $E^{\mathcal{A}}$-classes have at most n elements.

b) Let T be a complete theory of some language L and let $\Phi(x)$ be an L-type consistent with T. Assume that Φ is omitted on some model of T. Prove that Φ is realized in some model of T by at least two different elements.

2. a) Let T be a complete theory in a countable language L and let $\mathcal{A}$ be the prime model of T. Let $\Phi(x)$ be any L-type. Prove that there is some L-type $\Psi(x)$ such that $\Psi^{\mathcal{A}} = A \setminus \Phi^{\mathcal{A}}$.

b) Let L be a countable language and let $L' = L \cup \{c_1, \dots, c_k\}$, where $c_1, \dots, c_k$ are individual constants not in L. Let T and T' be complete theories of L and L' respectively and assume $T \subseteq T'$. Prove that T has a countable universal model iff T' has a countable universal model.

3. a) Let L be a countable language. An L-structure $\mathcal{A}$ is said to be *locally finite* iff every element of A belongs to a finite L-definable subset of A. Let T be a complete L-theory and assume that no model of T is locally finite. Prove that there is some L-formula $\phi(x)$ consistent with T such that for every L-formula $\psi(x)$ and every model $\mathcal{A}$ of T, $\phi^{\mathcal{A}} \cap \psi^{\mathcal{A}}$ is infinite provided it is not empty.

b) Let T be a complete theory in a countable language L. Let $\mathcal{A}$ be a countable model of T which is not prime and let $\Phi(x)$ be a type omitted on $\mathcal{A}$. Prove that there is some countable model of T which also omits Φ but is not isomorphic to $\mathcal{A}$.

[Warning: You cannot assume that T has a prime model.]

4. a) Assume that A and B are r.e. subsets of ω such that $A \cup B$ is recursive. Prove that there are recursive subsets $A' \subseteq A$ and $B' \subseteq B$ such that $A \cup B = A' \cup B'$.

b) Recall that if $\phi(x)$ is a Σ-formula (in the language for arithmetic on the natural numbers) and if $Q \vdash \exists x \phi(x)$ then $Q \vdash \phi(\bar{n})$ for some n in ω. Prove that there is no total recursive function f such that whenever $\phi(x)$ is a Σ-formula and $Q \vdash \exists x \phi(x)$, then $Q \vdash \phi(f(\bar{k}))$, where $k = \ulcorner \phi \urcorner$.

[Hint: Let $\phi(x, y)$ be the Σ-formula representing in Q the relation "x is the Gödel number of a proof from Q of the sentence whose Gödel number is y" and consider the formulas $\phi_l(x) = \psi(x, \bar{l})$.]

5. a) Given a language L_1, let $L_2 = L_1 \cup \{c\}$, where c is an individual constant not in L_1. Let T_2 be a finitely axiomatizable, essentially undecidable theory of L_2 and let $T_1 = T_2 \cap Sn_{L_1}$. Prove that T_1 is also essentially undecidable.

b) Prove that $\{e : W_e \neq \omega\} \leq_m \{e : W_e \text{ is finite}\}$.

[Hint: First define a partial recursive function $f(e, x)$ which converges iff $\{e\}(y)$ converges for all $y < x$.]

6. a) Let A and B be subsets of ω. Prove that B is A-r.e. iff $B \leq_m A'$, where A' is the jump of A.

b) Let $C = \{\ulcorner \sigma \urcorner : \mathbf{N} \models \sigma\}$, where $\mathbf{N}$ is the standard model of arithmetic on the natural numbers. Prove that $A \leq_T C$ for all arithmetic sets A, and use this to conclude that C is not arithmetic.