

**ANALYSIS QUALIFYING  
EXAMINATION  
JANUARY 7, 2011  
MATHEMATICS DEPARTMENT  
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Unless otherwise stated, you may appeal to a “well known theorem” in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified. You may use any given hint without proving it. In problems with multiple parts, be sure to go on to the rest of the problem even if there is some part you cannot do. In working on any part, you may assume the answer to any previous part, even if you have not proved it.

1. Let  $f \in AC[0, 1]$  be an absolutely continuous function on  $[0, 1]$  with  $f > 0$ . Prove that  $1/f \in AC[0, 1]$ .

2. The functions  $\{f_n\}$  are holomorphic on a domain  $D$  and converge uniformly on compact subsets to a function  $f$ . Show that either  $f$  is identically zero or for each open subset  $U$  of  $D$  with compact closure in  $D$  and with  $f$  having no zeros on the boundary of  $U$  that: there is an integer  $n_U$  such that for  $n \geq n_U$ ,  $f_n$  and  $f$  have the same number of zeros on  $U$ .

3. Let  $f \in L^p(\mathbb{R})$ ,  $1 \leq p < \infty$ ,  $\alpha > 0$ , and define

$$E_\alpha(f) = \{x \in \mathbb{R} : |f(x)| > \alpha\}.$$

(i) Show that  $E_\alpha$  has finite Lebesgue measure.

(ii) Use (i) to show that every  $f \in L^p(\mathbb{R})$ ,  $1 \leq p \leq 2$ , can be decomposed as  $f_1 + f_2$  where  $f_1 \in L^1(\mathbb{R})$  and  $f_2 \in L^2(\mathbb{R})$ .

4. Let  $\mathcal{F}$  be the set of holomorphic maps  $f$  with  $f(0) = i$  and domain the unit disc  $\mathbb{D}$  and range contained in the upper half plane  $\mathbb{H}$ . Show that the supremum of the imaginary parts  $\sup_{f \in \mathcal{F}} \Im f(i/2)$  is bounded (3 points). Find the supremum and justify your answer (7 points).

5. Let  $\{f_n\}$  be a sequence of measurable functions which converges *a. e.* to  $f$  on  $\mathbb{R}$ , and suppose there exists  $g \in L^1(\mathbb{R})$  such that for all  $n \geq 1$ ,  $|f_n| \leq g$  *a. e.* on  $\mathbb{R}$ . Given  $\epsilon > 0$ , prove that there is a measurable subset  $A \subset \mathbb{R}$  such that  $m(A) < \epsilon$  and  $f_n \rightarrow f$  uniformly on  $A^c$ .

6. Suppose  $R$  is a region with  $f$  a non constant holomorphic function on  $R$ . Suppose  $D$  is an open disc with  $\overline{D} \subset R$  and with  $|f|$  constant on the boundary of  $D$ . Prove that  $f$  has zeros in  $D$ .