DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

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LOGIC (Ph.D./M.A. version)

1. (a) Let T be any L-theory and suppose that $\{\varphi_n(x) : n \in \omega\}$ are L-formulas such that $T \models \forall x(\varphi_n(x) \rightarrow \varphi_{n+1}(x))$ for all $n \in \omega$. Suppose further that every element of every model of T realizes some φ_n. Prove that $T \models \forall x \varphi_n(x)$ for some $n \in \omega$.

   (b) Let $\mathfrak{A}$ be an L-structure, let $a \in A$, and assume that a satisfies some complete L-formula in $\mathfrak{A}$. Let $L' = L \cup \{c\}$, and let $\mathfrak{A}'$ be the expansion of $\mathfrak{A}$ to an L'-structure in which $c^{\mathfrak{A}'} = a$. Suppose that $b \in A$ and that b satisfies a complete L'-formula in $\mathfrak{A}'$. Prove that the pair ab satisfies a complete L-formula in $\mathfrak{A}$.

2. A theory T is called *model complete* if every embedding of models of T is an elementary embedding.

   (a) Suppose that $L = \{E\}$ and T is the L-theory asserting that E is an equivalence relation with infinitely many classes, and each class is infinite. Prove that T is model complete.

   (b) Prove that if T is model complete, then for every L-formula $\varphi(x_1, \ldots, x_n)$, there is an existential L-formula $\psi(x_1, \ldots, x_n)$ such that

$$T \models \forall \overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$$

3. (a) Suppose $L = \{U, \leq\}$, where U is a unary predicate and $\leq$ is binary. Let $\mathfrak{A}$ be the L-structure with universe $\mathbb{R}$ (the real numbers), where $U^{\mathfrak{A}} = \mathbb{Q}$ (the rationals) and $\leq^{\mathfrak{A}}$ is the usual ordering on $\mathbb{R}$. Find, with proof, all countable models of $Th(\mathfrak{A})$, up to isomorphism.

   (b) Prove that if T is ω-categorical and $\mathfrak{A}$ is the infinite, countable model, then there is $\mathfrak{B} \preceq \mathfrak{A}$ with $\mathfrak{B} \neq \mathfrak{A}$.

4. (a) Prove that $Th(\mathfrak{N})$, where $\mathfrak{N} = (\omega, +, \cdot, 0, s)$, is not model complete (see Problem #2).

   (b) Assume that $PA + Con(PA)$ is consistent. Use Gödel's Second Incompleteness Theorem to conclude that $PA + \neg Con(PA)$ is consistent.

5. (a) Prove that there is an integer m so that $W_m = \{m\}$.

   (b) Let $Z = \{e : W_e \neq \emptyset\}$. Prove that Z is a many-one complete, recursively enumerable subset of ω.

6. (a) Determine (with proof) whether or not $\mathbf{TOT} = \{e : \{e\} \text{ is total}\}$ is Turing equivalent to $\mathbf{FIN} = \{e : W_e \text{ is finite}\}$.

   (b) Demonstrate that $\{e : W_e \text{ is recursive}\}$ is an arithmetic subset of ω.