

UMCP Department of Mathematics Qualifying Exam
Partial Differential Equations, January 2011

- (1) Solve all six problems. Each main problem will be assigned a grade from zero to ten.
 - (2) Begin your answer to each question on a separate sheet.
 - (3) Write your code number on each page of your answer sheets. Do *not* use your name.
 - (4) Keep any scratch work on separate sheets, which should not be submitted.
 - (5) Carefully explain all your steps. If you invoke a “well-known” theorem, you must make clear which theorem you are using and justify its use.
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1. Consider the conservation law

$$u_t + f(u)_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty), \quad (1)$$

where $f \in C^1(\mathbb{R})$.

- (a) Define an integral solution of (1).
 - (b) Derive the jump (Rankine-Hugoniot) condition satisfied by a piecewise smooth integral solution u across a C^1 curve where this u has a discontinuity
 - (c) Find an integral solution to (1) when $f(u) = u^2 + u$ with $u(x, 0) = 1$ if $x < 0$, $u(x, 0) = -3$ if $x > 0$.
2. Prove or disprove: Let $U \subset \mathbb{R}^n$ ($n \geq 2$) be an open, bounded set with C^1 boundary. Suppose $u \in W^{1,n}(U)$. Then, $u \in L^\infty(U)$ and there exists a constant $C > 0$ such that

$$\|u\|_{L^\infty(U)} \leq C \|u\|_{W^{1,n}(U)}$$

3. Consider $u : \bar{U} \rightarrow \mathbb{R}$ where $u \in C^2(U) \cap C(\bar{U})$ and the set $U \subset \mathbb{R}^n$ is open, bounded. This u satisfies $Lu = 0$ in U where

$$Lu := - \sum_{i,j=1}^n a^{ij}(x) u_{x_i x_j} + \sum_{i=1}^n b^i(x) u_{x_i} + c(x) u$$

Assume that the operator L is uniformly elliptic; and $a^{ij}(x)$, $b^i(x)$ and $c(x)$ are smooth with $c(x) \geq 0$ in U .

Let $v := \phi(u) \geq 0$ where $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$ is C^2 and convex with $\phi(0) = 0$. Show that

$$\max_{\bar{U}} v = \max_{\partial U} v$$

4. Show that there is at most one smooth (up to the boundary) solution to the equation $u_{tt} - u_{xx} = u_t - u^3$ in the domain $(0, 1) \times (0, \infty)$ with initial conditions $u(x, 0) = g(x)$, $u_t(x, 0) = h(x)$ and boundary conditions $u(0, t) = u(1, t) = 0$.

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5. Let \mathcal{D} be the unit disc in \mathbb{R}^2 , i.e., $\mathcal{D} = B^0(0, 1) \subset \mathbb{R}^2$. Given $f, g \in L^2(\mathcal{D})$, let $\begin{pmatrix} u \\ v \end{pmatrix} \in H^1(\mathcal{D}) \times H^1(\mathcal{D})$ satisfy the boundary value problem (BVP)

$$\begin{cases} -\Delta u = f, & -\Delta v = g & \text{for } x = (x_1, x_2) \in \mathcal{D}, \\ u_{x_2} = v_{x_1}, & u_{x_1} = -v_{x_2} & \text{on } \partial\mathcal{D}. \end{cases} \quad (*)$$

The weak formulation for this problem is defined by the relation

$$B \left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} U \\ V \end{pmatrix} \right] = \int_{\mathcal{D}} (fU + gV) dx \quad \text{for all } \begin{pmatrix} U \\ V \end{pmatrix} \in H^1(\mathcal{D}) \times H^1(\mathcal{D}), \quad (**)$$

where the bilinear form $B[\cdot, \cdot]$ is

$$B \left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} U \\ V \end{pmatrix} \right] := \int_{\mathcal{D}} ((u_{x_2} - v_{x_1})(U_{x_2} - V_{x_1}) + (u_{x_1} + v_{x_2})(U_{x_1} + V_{x_2})) dx.$$

- (a) Check that if $u, v \in C^\infty(\overline{\mathcal{D}})$ and satisfy BVP (*), then relation (**) is true for all $U, V \in C^\infty(\overline{\mathcal{D}})$.
 (b) Prove or disprove: There exist constants $\gamma > 0, C > 0$ such that

$$\gamma (\|u\|_{L^2(\mathcal{D})}^2 + \|v\|_{L^2(\mathcal{D})}^2) + B \left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right] \geq C (\|u\|_{H^1(\mathcal{D})}^2 + \|v\|_{H^1(\mathcal{D})}^2),$$

for all $\begin{pmatrix} u \\ v \end{pmatrix} \in H^1(\mathcal{D}) \times H^1(\mathcal{D})$.

6. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be smooth, and define the domains $U_\pm \subset \mathbb{R}^2$ by $U_+ = \{y > f(x)\}$ and $U_- = \{y < f(x)\}$. Let $\Gamma = \{y = f(x)\}$ be the common boundary of these domains, and define the vector $N = (-f'(x), 1)$.

Consider functions $u_\pm : U_\pm \rightarrow \mathbb{R}$. Assume that u_\pm are (or have extensions) smooth up to the boundary and are harmonic. Define

$$u(x, y) = \begin{cases} u_-(x, y) & \text{if } y < f(x) \\ u_+(x, y) & \text{if } y \geq f(x) \end{cases}.$$

Prove or disprove: If $u_+ = u_-$ on Γ and $N \cdot \nabla u_+ = N \cdot \nabla u_-$ on Γ , then u is harmonic (i.e., u is $C^2(\mathbb{R}^2)$ and $\Delta u = 0$ in \mathbb{R}^2).