

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARYLAND  
GRADUATE WRITTEN EXAMINATION  
January, 2011

Probability (M. A. Version)

*Instructions to the Student*

- a. Answer all six questions. Each will be graded from 0 to 10
  - b. Use a different booklet for each question. Write the problem number and your code number (NOT YOUR NAME) on the outside cover.
  - c. Keep scratch work on separate pages in the same booklet
  - d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use
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1. Let  $X$  and  $Y$  be two independent random variables, both having geometric distribution with parameter  $p$ . Set

$$S = \min(X, Y), \quad T = |X - Y|$$

- (i) Find the joint probability mass function of the random pair  $(S, T)$ .
- (ii) Find the marginal probability mass functions of  $S$  and  $T$ . Are  $S$  and  $T$  independent?

2. Two players,  $A$  and  $B$  have their urns containing white and red balls. Let  $p_A$  and  $p_B$  be the proportions of white balls in the urns, respectively. They take turns in drawing a ball at random with replacement from their urns. The winner is the first player who gets a white ball.

- (i) Find the probability that the player who plays first wins.
- (ii) For what values of  $p_A$  and  $p_B$  the game is fair, i. e.  $P(A \text{ wins}) = P(B \text{ wins})$ ?

3. Let  $X$  be a random variable having symmetric distribution (i. e.,  $-X$  has the same distribution as  $X$ ), such that  $\mathbb{E}(X^2) < \infty$ , and let  $Z$  be a Bernoulli random variable with parameter  $p$ , independent of  $X$ . Set

$$Y = \begin{cases} X, & \text{if } Z = 1, \\ -X & \text{if } Z = 0. \end{cases}$$

- (i) Prove that  $Y$  and  $Z$  are independent and  $Y$  has the same distribution as  $X$ .
- (ii) For what values of  $p$  are  $X$  and  $Y$  (a) uncorrelated, (b) independent?

4. Let  $N, X_1, X_2, \dots$  be a sequence of independent random variables with  $N$  having a geometric distribution with parameter  $p$  and  $X_1, X_2, \dots$  having the same exponential distribution with parameter  $\lambda$ . Set

$$Y = \min\{X_1, \dots, X_N\}$$

- (i) Find the distribution function of  $Y$ .
- (ii) Calculate  $\mathbb{E}(Y)$ .

5. There are 10 coins in a bag. Six of them are normal coins, one coin has two 'heads' and three coins have two 'tails'. You draw at random a coin, look at one of its sides and see that it is a 'tail'. Find the probability that the drawn coin is normal.

6. Let  $U$  and  $V$  be two independent random variables, both uniformly distributed on  $(0, 1)$ . Set

$$T = 2\pi U, \quad R = \sqrt{-2 \log V}$$

- (i) Find the marginal probability density function (pdf) of  $T$  and the joint pdf of  $(T, R)$ .
- (ii) Prove that the random variables

$$X = R \cos T, \quad Y = R \sin T$$

are independent each having standard normal distribution.