DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
January 2011

Probability (Ph. D. version).

Instructions to the Student

(a) Answer all six questions. Each will be graded from 0 to 10

(b) Use a different booklet for each question. Write the problem number and your code number (**NOT YOUR NAME**) on the outside cover

(c) Keep scratch work on separate pages in the same booklet.

(d) If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use

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1. A person plays an infinite sequence of games He wins the n-th game with probability $1/\sqrt{n}$, independently of the other games.

(i) Prove that for any A, the probability is one that the player will accumulate A dollars if he gets a dollar each time he wins two games in a row.

(ii) Does the claim in part (i) hold true if the player gets a dollar only if he wins three games in a row? Prove or disprove it

2. There are 10 coins in a bag. Five of them are normal coins, one coin has two 'heads' and four coins have two 'tails'. You pull one coin out, look at one of its sides and see that it is a 'tail'. What is the probability that it is a normal coin?

3. Let X_n be a Markov chain with the state space $\mathbb{N}$, with the transition probabilities $P(z, z^2) = P(z, z-1) = 1/2$ for $z \geq 2$, $P(z, z+1) = 1$ for $z = 1$

(i) Find a strictly monotonically decreasing non-negative function $f : \mathbb{N} \to \mathbb{R}^+$ such that $f(X_n)$ is a supermartingale.

(ii) Prove that for each initial distribution $\mathrm{P}(\lim_{n\to\infty} X_n = +\infty) = 1$.

4. Let ξ_n be independent identically distributed random variables with $P(\xi_n = -1) = P(\xi_n = 1) = 1/2$.

(i) Prove that the series $\sum_{n=1}^{\infty} e^{-n}\xi_n$ converges with probability one

(ii) Prove that the distribution of $\xi = \sum_{n=1}^{\infty} e^{-n}\xi_n$ is singular, i. e., concentrated on a set of Lebesgue measure zero.

5. Let ξ_n be a sequence of independent random variables with ξ_n uniformly distributed on $[0, n^2]$ Find a_n and b_n such that $(\sum_{i=1}^{n} \xi_i - a_n)/b_n$ converges in distribution to a non-degenerate limit and identify the limit.

6. (i) Let X_t, $t > 0$, be random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Assuming that $E|X_t|^2 = E|X|^2 < \infty$ for all t, prove that $P(\lim_{t\to 0} X_t = X) = 1$ implies $\lim_{t\to 0} E|X_t - X|^2 = 0$, i. e., under the above assumption almost sure convergence implies convergence in mean square.

(ii) Let X_t, $t \in \mathbb{R}$, be a random process with the property that $\mathrm{E}X_t$ and $C(h) = \mathrm{E}(X_t X_{h+t})$ are finite and do not depend on t (such a process is called wide-sense stationary). Prove that the correlation function $C(h)$ is continuous if the trajectories of X_t are continuous.