DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
January, 2011

Statistics (Ph. D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (**NOT YOUR NAME**) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

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1. Let $(x_1, \ldots, x_n)$ be a sample from a population with pdf

$$f(x;\theta) = \frac{1}{\sqrt{2\pi}\theta} \exp \frac{-(x-\theta)^2}{2\theta^2}$$

with $\theta > 0$ as a parameter.
(i) Find the minimal sufficient statistic and justify its minimality and incompleteness.
(**Hint**: Suffice to construct two unbiased estimators of θ or of θ^2 that are functions of the minimal sufficient statistic.)
(ii) Calculate the Fisher information on θ in the sample and the efficiency of $\bar{x}$ as an estimator of θ.
(**Hint**: You need to recall the values of the third and fourth moments of the standard normal random variable.)

2. Let $X_1, \ldots, X_n$ be iid random variables uniformly distributed on $(\theta,\, 2\theta)$ with $\theta > 0$ as a parameter.
(i) Show that the family of pdf's of θ,

$$\pi(\theta; \alpha,\, a,\, b) = K(\alpha,\, a,\, b)\theta^{\alpha},\; a < \theta < b$$

with $\alpha \in (0, \infty)$, $0 < a < b < \infty$ as parameters and $K(\alpha,\, a,\, b) = (\int_a^b \theta^{\alpha} d\theta)^{-1}$ the normalizing factor, is a conjugate family of prior pdf's.
(ii) If the parameters of the prior are $\alpha = 1$, $a = 2$, $b = 3$ and $n = 4$, calculate the Bayes estimators of θ for the loss functions $L(\tilde{\theta},\, \theta) = (\tilde{\theta} - \theta)^2$ and $L(\tilde{\theta},\, \theta) = |\tilde{\theta} - \theta|$.

3. Let $(x_1, \ldots, x_m)$, $(y_1, \ldots, y_n)$ be two independent samples from normal distributions $N(\mu_1, \sigma^2)$ and $N(\mu_2, 2\sigma^2)$, respectively, with $(\mu_1,\, \mu_2,\, \sigma^2)$ as parameters.
(i) Find the minimal sufficient statistic for $(\mu_1,\, \mu_2,\, \sigma^2)$.
(ii) Construct the uniformly minimum variance unbiased estimator (UMVUE) of σ^2.

4. Let $(x_1, \ldots, x_n)$ be a sample from a population with density

$$f(x; \theta) = \frac{\theta e^{\theta x}}{e^{\theta} - 1},\; 0 < x < 1.$$

with $\theta > 0$ as a parameter.
(i) Construct a uniformly most powerful (UMP) test of size α of

$$H_0 : \theta \le 1 \text{ vs } H_1 : \theta > 1.$$

(ii) Assuming n large and using the CLT, find (approximately) the threshold in the rejection region of the UMP test for $\alpha = .05$ and approximate its power function via the standard normal Φ

5. Let $(x_1, \ldots, x_m)$, $(y_1, \ldots, y_n)$ be two independent samples from populations with pdf's
$f_1(x; \theta) = \exp\{-(x - \theta)\}$, $x > \theta$ and $f_2(y; \theta) = 2\exp\{-2(y - \theta)\}$, $y > \theta$, respectively, with θ as a parameter.
(i) Find the maximum likelihood estimator $\hat{\theta}_{m,n}$ of θ and calculate its mean.
(ii) Assuming $m = cn(1 + o(1))$, $c > 0$ and $n \to \infty$ find appropriate

$\{a_n, n = 1, 2, \ldots\}$ and the nondegenerate limiting distribution of $a_n(\hat{\theta}_{m,n} - \theta)$.

6. Let $(x_1, \ldots, x_m)$, $(y_1, \ldots, y_n)$ be two independent samples of sizes m and n from normal populations $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively, with all four parameters unknown.

(i) Based on the sufficient statistics, construct a pivot for σ_1/σ_2.

(ii) Express the distribution of the pivot in terms of an F-distribution and construct a confidence interval of level $1 - \alpha$ for σ_1/σ_2.