TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

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Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified. In problems with multiple parts, be sure to go on to the rest of the problem even if there is some part you cannot do. In working on any part, you may assume the answer to any previous part, even if you have not proved it.

Problem 1.

A continuous map between Hausdorff spaces is closed if the image of each closed set is closed. A continuous map is proper if the inverse image of each compact subset of the range is compact.

a) Let X, Y be metric spaces. Show every continuous proper map $f\colon X \to Y$ is closed.
b) Give an example of a map which is not closed and explain.

Problem 2.

In what follows, "surface" will mean a connected compact surface-with-boundary. χ denotes Euler characteristic. Prove or disprove the following statements (using standard results, which you can assume, of course).

(1) Two orientable surfaces A and B with $\partial(A) = \partial(B) = \emptyset$ are homeomorphic if and only if $\chi(A) = \chi(B)$.
(2) Let $E < 0$ be an integer. The number of homeomorphism classes of orientable surfaces A with $\chi(A) = E$ equals $1 - E$.
(3) If A is an orientable surface and B is a nonorientable surface then A cannot be homotopy-equivalent to B.
(4) If A is an orientable surface and B is a nonorientable surface then A cannot be homeomorphic to B.

Problem 3.

Let X be a locally path-connected space and

$$f_j : X \to S^1 = \{z \in \mathbb{C} \mid |z| = 1\}, \quad j = 0, 1,$$

be two continuous maps. Show f_0 and f_1 are homotopic if and only if there is a continuous function $\alpha : X \to \mathbb{R}$ such that $f_0(x) = f_1(x) \exp 2\pi i \alpha(x)$ for all $x \in X$.

Problem 4.

Fix $n \geq 1$, and let X_p denote the space obtained by attaching an $(n+1)$-cell to S^n by a map of degree p, where p is a prime.

a) Compute $H_*(X_p \times X_q, \mathbb{Z})$. Here p may or may not be equal to q.

b) If r is a prime (possibly equal to p) and $\mathbb{Z}_r$ is the cyclic group of order r, compute $H_*(X_p \times X_p, \mathbb{Z}_r)$.

Problem 5.

If X is a topological space, the *n-th symmetric power* S^nX of X is defined to be the quotient of $X^n = X \times X \times \cdots \times X$ (n factors) by the action of the symmetric group Σ_n, acting by permutation of the factors. The quotient space is given the quotient topology.

(1) Show that if $X = \mathbb{CP}^1 \cong S^2$, then S^nX is homeomorphic to $\mathbb{CP}^n$. (Hint: the elementary symmetric functions $\sigma_1(z_1, \cdots, z_n) = z_1 + \cdots + z_n, \cdots, \sigma_n(z_1, \cdots, z_n) = z_1 \cdots z_n$ give a homeomorphism from $S^n\mathbb{C}$ to $\mathbb{C}^n$ and you just need to extend it.)

(2) Compute (as explicitly as possible) the map of cohomology *rings* $H^*(\mathbb{CP}^n; \mathbb{Z}) \to H^*((S^2)^n; \mathbb{Z})$ induced by the quotient map $X^n \to S^nX, X = S^2$. In other words, give the structure of each cohomology ring (for this you can use standard results instead of computing from scratch), and then explain what each generator maps to.

Problem 6.

a) Let N^n be an orientable compact connected n-manifold without boundary. Show $H^n(N, \mathbb{Z}) \simeq \mathbb{Z}$.

b) Suppose M^n is another n-dimensional connected compact manifold without boundary and $f\colon M \to N$ is a continuous map such that

$$f_*\colon H_n(M, \mathbb{Z}) \to H_n(N, \mathbb{Z})$$

is onto. Show M is orientable and $f_*\colon H_r(M, \mathbb{Z}) \to H_r(N, \mathbb{Z})$ is onto for all $r \geq 0$.