27^{th} ANNUAL UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART II
November 30, 2005, 1:00–3:00

NO CALCULATORS
2 hours

1. The three little pigs are learning about fractions. They particularly like the number $x = 1/5$, because when they add the denominator to the numerator, add the denominator to the denominator, and form a new fraction, they obtain $6/10$, which equals $3x$ (so each little pig can have his own x). The 101 Dalmatians hear about this and want their own fraction. Your job is to help them.

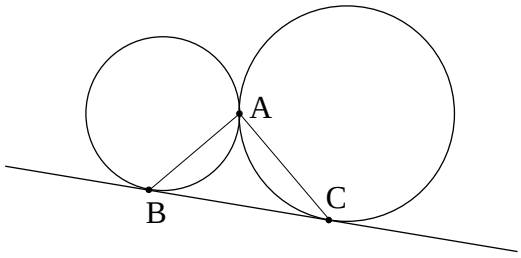
   (a) Find a fraction y such that when the denominator is added to the numerator and also added to the denominator, the result is $101y$.

   (b) Prove that the fraction y (put into lowest terms) in part (a) is the only fraction in lowest terms with this property.

2. A small kingdom consists of five square miles. The king, who is not very good at math, wants to divide the kingdom among his 9 sons. He tells each son to mark out a region of 1 square mile. Prove that there are two sons whose regions overlap by at least $1/9$ square mile.

3. Let $\pi(n)$ be the number of primes less than or equal to n. Sometimes n is a multiple of $\pi(n)$. It is known that $\pi(4) = 2$ (because of the two primes 2, 3) and $\pi(64540) = 6454$. Show that there exists an integer n, with $4 < n < 64540$, such that $\pi(n) = n/8$.

4. Two circles of radii R and r are externally tangent at a point A. Their common external tangent is tangent to the circles at B and C. Calculate the lengths of the sides of triangle ABC in terms of R and r.



5. There are 2005 people at a meeting. At the end of the meeting, each person who has shaken hands with at most 10 people is given a red T-shirt with the message "I am unfriendly." Then each person who has shaken hands only with people who received red T-shirts is given a blue T-shirt with the message "All of my friends are unfriendly." (Some lucky people might get both red and blue T-shirts, for example, those who shook no one's hand.) Prove that the number of people who received blue T-shirts is less than or equal to the number of people who received red T-shirts.