1. You are given three buckets with a capacity to hold 8, 5, and 3 quarts of water, respectively. Initially, the first bucket is filled with 8 quarts of water, while the remaining two buckets are empty. There are no markings on the buckets, so you are only allowed to empty a bucket into another one or to fill a bucket to its capacity using the water from one of the other buckets.

(a) Describe a procedure by which we can obtain exactly 6 quarts of water in the first bucket.
(b) Describe a procedure by which we can obtain exactly 4 quarts of water in the first bucket.

2. A point in the plane is called a lattice point if its coordinates are both integers. A triangle whose vertices are all lattice points is called a lattice triangle. In each case below, give explicitly the coordinates of the vertices of a lattice triangle \( T \) that satisfies the stated properties.

(a) The area of \( T \) is \( 1/2 \) and two sides of \( T \) have length greater than 2011.
(b) The area of \( T \) is \( 1/2 \) and the three sides of \( T \) each have length greater than 2011.

3. Alice and Bob play several rounds of a game. In the \( n \)th round, where \( n = 1, 2, 3, \ldots \), the loser pays the winner \( 2^{n-1} \) dollars (there are no ties). After 40 rounds, Alice has a profit of \$2011\) (and Bob has lost \$2011\). How many rounds of the game did Alice win, and which rounds were they? Justify your answer.

4. Each student in a school is assigned a 15-digit ID number consisting of a string of 3’s and 7’s. Whenever \( x \) and \( y \) are two distinct ID numbers, then \( x \) and \( y \) differ in at least three entries. Show that the number of students in the school is less than or equal to 2048.

5. A triangle \( ABC \) has the following property: there is a point \( P \) in the plane of \( ABC \) such that the triangles \( PAB \), \( PBC \) and \( PCA \) all have the same perimeter and the same area. Prove that:

(a) If \( P \) is not inside the triangle \( ABC \), then \( ABC \) is a right-angled triangle.
(b) If \( P \) is inside the triangle \( ABC \), then \( ABC \) is an equilateral triangle.