THE 38th ANNUAL (2016) UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION

PART II  SOLUTIONS

1. The only such table is

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<td>216</td>
<td>280</td>
<td>441</td>
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</table>

2. Suppose \( X \) is a pdf. Note that if \( x \in X \), then \( x + 2, x + 3, x + 5, x + 7 \not\in X \). Also, from \( x + 1, x + 4 \) and \( x + 6 \) at most one can be in \( X \). Thus, among every eight consecutive integers at most 2 may be in \( X \). This implies \( X \) has at most \( \frac{2016}{4} = 504 \) elements.

The set \( X = \{4k \mid 1 \leq k \leq 504\} \) is a pdf, because if \( x, y \in X \), then \( |x - y| \) is a multiple of 4. Therefore \( |x - y| \) is not prime.

The answer is 504.

3. For \( i \in \{1, 2, \ldots, 14\} \), let \( M_i \) denote the midpoint of the line segment \( X_iX_{i+1} \). Suppose, for the sake of contradiction, that each of these points is within distance 1/2 from the origin. Then, by the triangle inequality the distance between any two points \( M_i \) and \( M_j \) is no more than 1 for all \( i, j \).

Note that \( M_1X_2 = X_1X_2/2, M_2X_2 = X_3X_2/2 \), and \( \angle M_1X_2M_2 = \angle X_1X_2X_3 \), and thus \( \triangle X_1X_2X_3 \) is similar to \( \triangle M_1X_2M_2 \) and \( M_1M_2 = X_1X_3/2 \). We have \( X_1X_3 = 2M_1M_2 \leq 2 \cdot 1 = 2 \). Repeating this reasoning, we find that the distances \( X_{2k-1}X_{2k+1} \) are all no more than 2, for \( k \in \{1, 2, \ldots, 7\} \). By the triangle inequality, the distance from \( X_1 \) to \( X_{15} \) is less than or equal to \( 7 \cdot 2 = 14 \). But it was assumed that \( X_1 = (10, 0) \) and \( X_{15} = (0, 10) \), and so the distance \( X_1X_{15} \) is actually \( \sqrt{10^2 + 10^2} = 10\sqrt{2} \). Since \( 14^2 = 196 < 200 = (10\sqrt{2})^2 \), we find that the distance \( X_1X_{15} \) exceeds 14, which is a contradiction. This completes the proof.

4. For every \( k \in \{1, 2, \ldots, 82\} \), there is an associated three-letter sequence \((s_k, s_{k+1}, s_{k+2})\). There are 27 distinct three-letter sequences that can be constructed from \( \{A, B, C\} \). Since \( 82 = 27 \cdot 3 + 1 \), the pigeonhole principle implies that some three-letter sequence \((r, s, t)\) must appear more than 3 times as a consecutive subsequence of \((s_1, \ldots, s_{84})\). One such occurrence must be at the end of \((s_1, s_2, \ldots, s_{84})\), since otherwise one of the three sequences \((r, s, t, A)\), \((r, s, t, B)\), \((r, s, t, C)\) would have to appear more than once as a consecutive subsequence of \((s_1, s_2, \ldots, s_{84})\). Similarly, one such occurrence must be at the beginning of \((s_1, s_2, \ldots, s_{84})\), since otherwise one of the sequences \((A, r, s, t)\), \((B, r, s, t)\), \((C, r, s, t)\) would have to appear more than once as a consecutive subsequence of \((s_1, s_2, \ldots, s_{84})\). Therefore, \((s_{82}, s_{83}, s_{84}) = (r, s, t) = (s_1, s_2, s_3) = (A, B, B)\). The correct answer is \( B \).

Remark. Although it was not required for credit, here is an example of a sequence \((s_1, s_2, \ldots, s_{84})\) satisfying the conditions of the problem. (This example comes from the multiplicative structure of the finite field of order 81.)
5. We claim that there is no such sequence. On the contrary assume $a_n$ is such a sequence. For any positive integer $n$, we use the AM-GM inequality to obtain

$$(\sum_{k=n+1}^{2n} a_k)(\sum_{k=n+1}^{2n} \frac{1}{a_k}) \geq n(\prod_{k=n+1}^{2n} a_k)^{1/n} \cdot n(\prod_{k=n+1}^{2n} \frac{1}{a_k})^{1/n} = n^2$$

On the other hand,

$$\sum_{k=n+1}^{2n} a_k \leq \sum_{k=1}^{2n} a_k \leq (2n)^2 = 4n^2$$

Combining these two inequalities we obtain

$$\sum_{k=n+1}^{2n} \frac{1}{a_k} \geq \frac{1}{4}$$

Adding up this inequality for $n = 2, 4, 8, \ldots, 2^m$, we obtain

$$\sum_{k=1}^{2^m} \frac{1}{a_k} \geq \frac{m}{4}$$

This is larger than 2016 when $m > 4 \cdot 2016$, which is a contradiction.