An Overview of Leon Greenberg's Work in Numerical Analysis
by
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In the late 1980s, Leon started to attend Babuska's weekly numerical analysis seminar. He became interested in oscillation theorems and their application to the calculation of the counting function

$$N(\lambda) = \text{number of eigenvalues} < \lambda$$

for Sturm-Liouville problems. These theorems can be regarded as being, in some sense, generalizations from the matrix case to the ODE case of the well known Sturm Sequence Theorem for tridiagonal matrices.

It is perhaps a rather astonishing fact that as recently as the late 1980s, very little had been done on the counting function for Hamiltonian systems of ODEs,

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} S_{11}(x,\lambda) & S_{12}(x,\lambda) \\ S_{12}^*(x,\lambda) & S_{22}(x,\lambda) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

where the coefficients S_{ij} are all square matrices, as are 0 and I, while u and v are vector-valued functions of appropriate dimension. These systems arise abundantly in quantum mechanics as a result of semi-discretization of Schrödinger-type PDEs. The situation in the late 1980s was that an attempt had been made to obtain an expression for the counting function for the special case of a Sturm-Liouville equation with matrix coefficients, in a paper of Atkinson, Krall, Leaf and Zettl. The expression they obtained could be calculated by solving initial value problems. Unfortunately the paper contained a fundamental error which meant that the formula was wrong. After I solved this problem in my Ph.D., Leon set about generalizing this work to Hamiltonian systems arising from general $2n$-th order symmetric differential expressions with matrix coefficients.

The result was Leon's most important and substantial contribution to the spectral theory of Hamiltonian systems, written up in two substantial articles. It remains to this day the best available account of the oscillation theory for these systems and contains expressions for the counting function even in the case of coupled boundary conditions.

Leon always had a great admiration for F.V. Atkinson in Toronto, who also worked on similar problems. In my opinion, these two articles are at least as good as Atkinson's best work on Hamiltonian systems.

Leon submitted the first article to a journal who sent it to an ill-informed referee, who claimed (incorrectly) that Morse had already solved this problem in the 1930s. The paper was rejected. Leon seemed relatively untroubled by this rejection, and never bothered to submit either of the papers elsewhere. They can still be downloaded from his web page:

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www.math.umd.edu/~lng/prufer1.ps
www.math.umd.edu/~lng/prufer2.ps
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I spent a large part of the last 11 years trying to persuade Leon that these articles should be published, but without success. People are still publishing papers in top-rank journals in which they rediscover parts of what Leon has in those two articles.

Our initial collaboration was a spin-off from my Ph.D. work and from these two articles of Leon. We developed computer code to solve certain types of differential equation eigenproblem automatically, exploiting the best available methods. One of the approaches we used was based on matrix exponentials and the Baker-Campbell-Hausdorff formula, which was a cornerstone of a code written by Alex Dragt (Physics Dept., UMCP) in the 1970s. While inspired by this approach, our method relied crucially on good estimates of the generalized 'oscillation count' for the system, based on computing the count explicitly for a system with piecewise constant coefficients. These methods had the advantage that they could go all the way up the spectrum without any deterioration in accuracy and without 'skipping' any eigenvalues. [As an aside, I should mention that the Lie Algebra approach to numerical solution of ODEs, pioneered by people like Dragt in the physics literature, was regarded in the numerical analysis literature as outlandish. The importance of preserving important structural properties such as unitarity of solution matrices was not widely appreciated. Arieh Iserles (Cambridge) had made a brief and unsuccessful attempt to popularize these ideas among numerical analysts in 1984. By the time of Arieh's second (successful) attempt in the mid-to-late 1990s, part of the explosion of research on 'Geometric Integration', both Leon and I had moved on.]

Leon became more interested in problems arising from fluid dynamics, most of which are non-selfadjoint. We wrote a large code for non-selfadjoint problems. Back in the early 1970s, the so-called 'compound matrix method' had been used by Davies in Newcastle but had never really caught on, because there were no numerical algorithms for systems of ODEs which could really take advantage of linearity. Without such algorithms, Davey believed that it was better to use nonlinear formulations such as continuous orthonormalization, which had a lower complexity as a function of the dimension of the original system of ODEs. The work of the Cambridge group provided some new methods which were really fast on linear problems. We exploited this in writing our code. I still use the code regularly some six years after we completed it. It has been pressed into service for such diverse applications as resonance problems, Regge pole problems, stability curve problems for Orr-Sommerfeld equations, and even for investigating fairly theoretical issues such as spectral inclusion and spectral exactness for sequences of regularizations of a singular differential operator.

Leon visited me in Leicester five times: in 1994, 1996, 1997, 1998 and 2000. During the 1998 visit he went on a short tour of Germany. There he met Mennicken, who introduced him to block operator matrices. Leon became fascinated by two particular problems: the (simple ODE version of) the Hain-Lüst problem, and the Ekman problem. Leon was particularly fascinated by the Ekman problem because of the work of Faller, who had been at Maryland. We developed an oscillation theory for the Hain-Lüst problem and we also carried out a very thorough spectral analysis of the singular Ekman problem. The Hain-Lüst paper attracted a lot more interest than I had expected for such an unashamedly toy problem, which probably says something about mathematicians. The Ekman problem turned out to be our only major paper on theoretical aspects of a singular non-selfadjoint problem; the paper on that work will appear in Mathematical Proceedings of the Cambridge Philosophical Society early next year.

Leon only published medium to large papers, and only in very good journals.

Working with Leon was always demanding. We never worked on a problem where the solution was predictable; and, to make life even more interesting, we always seemed to choose problems where it was not clear whether or not we'd be clever enough to find a solution. The stress of working at our limit meant that we often had arguments, which could appear a great deal more animated than they really were. One of my other collaborators, upon witnessing such a confrontation, remarked that I seemed to be making a habit of working with crotchety old men. He never stopped to consider the deductions which might be made by applying this statement, taking into account his own bald pate and white hair.