The Langlands Program is a large and active area of research with a great deal of overlap with modern number theory and representation theory. One of its many goals is the generalization of class field theory to non-abelian Galois extensions of number fields. As we will describe below, this requires studying representation theory for all manner of reductive group. In particular, we will describe some results as they pertain to the matrix group $\text{Sp}_2(F)$ where $F$ is a $p$-adic local field.

Now let us describe the results from local class field theory that we seek to generalize. Let $F$ be a $p$-adic local field, $\mathcal{O}$ its ring of integers and $\mathcal{P}$ the maximal ideal of $\mathcal{O}$. Recall that $\mathcal{P}$ is generated by a some uniformizing element $\varpi$. Moreover $\mathcal{O}/\mathcal{P} \simeq \mathbb{F}_q$ the finite field with $q = p^m$ elements. It is a well known fact that the following sequence is exact

$$1 \to \mathcal{I}_F \to \text{Gal}(\bar{F}/F) \to \text{Gal}(\mathbb{F}_q/\mathbb{F}_q) \to 1.$$ 

Moreover, it is known that

$$\text{Gal}(\mathbb{F}_q/\mathbb{F}_q) \simeq \hat{\mathbb{Z}} \simeq \varprojlim \mathbb{Z}/n\mathbb{Z}.$$ 

So we see that $\mathbb{Z} \subset \text{Gal}(\mathbb{F}_q/\mathbb{F}_q)$; thus we define the Weil group $\mathcal{W}_F$ to be the full inverse image of $\mathbb{Z}$ in $\text{Gal}(\bar{F}/F)$ and $\text{Fr}_q$ to be a fixed element mapping to $1 \in \mathbb{Z}$.

According to some results of Artin, there exists an isomorphism

$$r_F : F^\times \simeq \mathcal{W}_F^{ab}$$

with $r_F(\varpi) = \text{Fr}_q$. Because the two groups are isomorphic, we also get a correspondence between their irreducible representations. In fact, for every (quasi)character

$$\sigma : \mathcal{W}_F^{ab} \to \mathbb{C}^\times$$

there is a (quasi)character

$$\chi : F^\times \to \mathbb{C}^\times$$

such that $L(s, \chi) := (1 - \chi(\varpi)q^{-s})^{-1} = (1 - \sigma(\text{Fr}_q)q^{-s})^{-1} =: L(s, \sigma)$.

In this talk, we will explore a generalization of this correspondence to general matrix groups with a special emphasis on the rank two symplectic group $\text{Sp}_2(F)$. Let $G$ be a (split) matrix group with entries in $F$. Further, we let $\mathcal{A}(G)$ denote the set of
equivalence classes of irreducible admissible representations of $G$. Finally, we denote as $\mathcal{G}(G)$ as the set of equivalence classes of pairs $\rho' = (\rho, N)$, where

$$\rho : \mathcal{W}_F \to \mathcal{L}G^0$$

is a semisimple representation from $\mathcal{W}_F$ to the complex group $\mathcal{L}G^0$; $N$ is a nilpotent element of $\text{Lie}(\mathcal{L}G^0)$ such that

$$\rho(\text{Fr}_q)N\rho(\text{Fr}_q)^{-1} = |\varpi| N.$$

The local Langlands correspondence predicts that for any $\rho' \in \mathcal{G}(G)$, there exists a $\Pi_{\rho'} \subset \mathcal{A}(G)$ with several properties, including the following two.

1. For $\rho'_1 \neq \rho'_2$, $\Pi_{\rho'_1} \cap \Pi_{\rho'_2} = \emptyset$.

2. For any $\pi \in \Pi_{\rho'}$, $L(s, \pi) = L(s, \rho')$.

While there are some general formulas for computing $L(s, \rho')$, there are no general methods for determining $\Pi_{\rho'}$ or computing $L(s, \pi)$. However, for a certain type of representation of $G$, Lusztig has some criteria to help determine $\Pi_{\rho'}$. In particular, the criteria applies to representations having a vector fixed by an Iwahori subgroup of $G$. Furthermore, we will discuss a method that computes $L$-factors for representations of groups $G$, where $G$ is defined as preserving a non-degenerate inner product. The method is called the doubling integral of Piatetski-Shapiro and Rallis. While the actual computations that will be discussed in this talk are achieved from a modified version of the doubling integral, we will discuss the more basic construction to highlight the general ideas and motivation underlying the method.

Ultimately, the research on which this talk is based partitioned the unramified principal series of $\text{Sp}_2(F)$ according to various $\rho' \in \mathcal{G}(G)$ and verified the equality of $L$-factors using a modified version of the doubling integral for $\text{Sp}_2(F)$. This version of the doubling integral was then shown to work with a topological covering group of $\text{Sp}_2(F)$ called the metaplectic cover of $\text{Sp}_2(F)$. It is conjectured that these representations and $L$-factors for this group correspond to specific representations and $L$-factors on the group $\text{SO}_5(F)$. This conjecture was supported by the computations mentioned above. However, due to time constraints, the metaplectic case will not be discussed in the talk.