Pricing American Options by Adaptive Finite Element Method

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Abstract

The evaluation of the price of an option is of considerable importance in finance. There is no general closed-form analytical solution for the price of American-style options. To solve this problem, people usually resort to numerical methods, whose improvement is still an active field of research. The American-style option pricing problem based on the Black-Scholes model can be written as a parabolic variational inequality. This reformulation is crucial to construct a successful numerical treatment of the problem, as suggested by Wilmott, Dewynne, and Howison in [23]. We focus on the issues related to the construction of an adaptive method to solve this problem efficiently and accurately.

1 Introduction

An *option* is a contract between the writer and the holder that gives the right, but not the obligation, to the holder to buy or sell a risky asset at a prespecified fixed price within a specified period [23, Chapter 1]. The underlying risky asset could be stocks, stock indices, futures, currencies, or commodities, etc. An option contract is a form of derivative instrument, which can be traded on exchanges or over the counter. A *call* (*put*) option allows its holder to buy (sell) the underlying asset at the *strike price* K. Option holders can only exercise their *European-style* options at the expiration or maturity date, T; in contrast, *American-style* options can be exercised any time before they expire.

Why are options more and more popular these days? Purchasing options offers you the ability to position yourself accordingly with your market expectations in a manner such that you can both profit and protect yourselves with limited risk. The decision as to what type of options to buy is dependent on whether your outlook for the respective security is positive (bullish) or negative (bearish). If your outlook is positive, buying a call option with lower strike price creates the opportunity to share in the upside potential of a stock without having to risk more than a fraction of its market value. Conversely, if you anticipate downward movement, buying a put option with high strike price will enable you to protect your investment against downside risk without limiting profit potential.

The option premium is the price at which the option contract trades. The premium is paid by the buyer to the writer of the option. In return, the writer of the call option is obligated to deliver the underlying security to an option buyer if the call is exercised or buy the underlying security if the put is exercised. The writer keeps the premium whether or not the option is exercised. Then it is nature to ask what is a fair price for an option. The *option pricing problem* is an important and fundamental financial problem. Generally speaking, there are two basic ways to solve option pricing problems: analytical methods and numerical methods. In their Nobel-winning work [3],

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Black and Scholes derived explicit pricing formulas for European call and put options on stocks which do not pay dividends. For American options, the Black-Scholes model results in a *free boundary problem*. Unfortunately, unlike in the European case, one can not find explicit closed-form solutions to the American option pricing problem in general. Johnson [14] and MacMillan [15] approximate American put options on a non-dividend paying stock analytically. Geske and Johnson [11] give an analytic solution in a series form for American options on dividend paying stocks.

When the formulas for the exact solutions are too difficult to be practically used, we resort to numerical methods, such as lattice methods, simulation-based methods, PDE-based methods, etc. We refer to the book by Wilmott, Dewynne, and Howison [23], the recent review by Broadie and Detemple [5], and the references therein for a review and comparison of many numerical strategies for valuating American options. Based on our experience on European-style option price evaluation, PDE-based methods are usually more efficient for problems in low dimension. In this project, we will consider the variational formulation of the problem and discretize it using the finite difference method in time and the finite element method in space with adaptive time-space mesh refinement.

The rest of the paper is organized as follows: In Section 2, we study an 1D model problem and rewrite it as a linear complementarity problem. Section 3 gives a general American option pricing problem in multi-dimension as a parabolic variational inequality. Section 4 will shortly review existing numerical methods for pricing American-style options and will mainly introduce the finite element method. In Section 5, we will introduce the idea of adaptive time-space mesh refinement and list several open questions in Section 6.

2 The Black-Scholes Model as A Free Boundary Problem

Now we take the pricing problem of an American put option on a non-dividend paying stock as a model problem. Let $S(t)$ denote the underlying stock price and $V(S,t)$ be the American put option price at time t. It is well-known that the price of an American option satisfies the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad \forall\, S > S_f(t) \text{ and } t \in [0,T], \tag{2.1}$$

where σ is the volatility of the underlying stock, r is the interest rate, and $S_f(t)$ denotes the free boundary at time t. We know that the price of an American option is no less than the pay-off function $H(S) := (K-S)_+$[1] because of the non-arbitrage assumption[2] and moreover

$$V(S,t) = H(S) \qquad \forall\, 0 \le S \le S_f(t) \text{ and } t \in [0,T]. \tag{2.2}$$

The final and boundary conditions are given by

$$\begin{cases} V(S,T) = H(S), \;\; S \ge 0, \\ V(S_f(t),t) = H(S_f(t)), \; V_S(S_f(t),t) = -1, \;\; 0 < t \le T, \\ \lim\limits_{S\to\infty} V(S,t) = 0, \;\; 0 \le t \le T. \end{cases} \tag{2.3}$$

In this way, we write the price of an American put option as the solution of a free boundary problem (2.1) to (2.3). Although this formulation is mathematically beautiful, a major difficulty under this setting is that one needs to solve for V along with the unknown free boundary S_f[3].

The idea is to reformulate the problem such that the free boundary does not show up explicitly. For this purpose, we rewrite the free boundary problem (2.1)–(2.3) as a *linear complementarity*

[1] $(\cdot)_+ = \max(\cdot, 0)$.

[2] It simply means no one can make immediate risk-free profit.

[3] S_f is also called exercise boundary. For American option holders, they need to decide whether and when to exercise an option. This leads to an optimal exercise policy problem.

problem (LCP) (for details, see [23]). We use the time to maturity $\tilde{t} = T - t$ and $x = \log S$ as independent variables, then the function $u(x,\tilde{t}) := V(e^x, T-\tilde{t})$ satisfies the following LCP (we will write t instead of $\tilde{t}$ for time to maturity from now on):

$$\frac{\partial u}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial u}{\partial x} + ru \geq 0 \qquad \text{for } x \in \mathbb{R} \text{ and } 0 \leq t \leq T, \tag{2.4}$$

with the obstacle condition

$$u(x,t) \geq \chi(x) \qquad \text{for } x \in \mathbb{R} \text{ and } 0 \leq t \leq T \tag{2.5}$$

and the initial condition

$$u(x,0) = u_0(x) \qquad \text{for } x \in \mathbb{R}, \tag{2.6}$$

where $u_0(x) = \chi(x) = H(e^x)$ is the payoff function in the log of the asset price. Moreover, for each point $(x,t) \in \mathbb{R} \times [0,T]$, the *complementarity condition* has to be satisfied, i.e., there holds equality in at least one of (2.4) and (2.5). The set of points $(x,t) \in \mathbb{R} \times [0,T]$ where $u(x,t) = \chi(x)$ is called the *contact set*, and its complement the *non-contact set*. The boundary between the two sets is called the *free boundary* or exercise boundary since it indicates the optimal time to exercise the option.

The subject of complementarity problems and their applications in engineering and economics have received intensive attention over more than three decades. We refer to the review paper by Ferris and Pang [9] and the references therein for a comprehensive overview of the importance of linear and nonlinear complementarity problems in various of application areas.

3 Parabolic Variational Inequalities

The problem (2.4)-(2.6) has to be solved on the whole real line. For practical computations one uses a bounded interval $\Omega := (-R,R)$ and boundary conditions $u(-R,t) = \chi(-R)$, $u(R,t) = 0$. This domain truncation causes an additional error which decreases exponentially in R, and is therefore negligible for R large enough. For $v,w \in H^1(\Omega)$ we define

$$\langle v,w\rangle := \int_{-R}^{R} v(x)w(x)dx,$$
$$a(v,w) := \int_{-R}^{R} \left[\tfrac{\sigma^2}{2}v'(x)w'(x) + (\tfrac{\sigma^2}{2} - r)v'(x)w(x) + rv(x)w(x)\right] dx.$$

Then the resulting problem on $(0,T) \times \Omega$ can also be written in variational form as follows:

Problem 1 (1d Model Problem) *Find a weak solution $u(x,t)$ in $L^2(0,T;\mathcal{K}(\cdot)) \cap H^1(0,T;H^{-1}(\Omega))$ such that*

$$\langle \partial_t u, u-v\rangle + a(u,u-v) \leq \langle f, u-v\rangle, \qquad \forall v \in \mathcal{K}(t) := \{v \in H^1(\Omega) : v \geq \chi(t)\}, \quad a.e. \ t \in [0,T]. \tag{3.1}$$

Note that the solution $u(x,t)$ has a singular behavior in both space and time close to $t = 0$ and $x = \log K$ (i.e. time close to maturity, price close to strike price).

In practice, the underlying risky asset could be a basket of stocks. On the other hand, interest rate (or volatility or both) could be stochastic and one can formulate an American option pricing problem with stochastic parameters as a problem with only deterministic parameters but in higher dimension. These important applications motivate us to consider problems in multi-dimensions. From now on, we will formulate the evaluation of American options as a *parabolic variational inequality* in several variables on a bounded polyhedron Ω in $\mathbb{R}^d$. Let $\mathcal{Q} := \Omega \times (0,T)$ be the parabolic cylinder with the lateral boundary $\Gamma := \partial\Omega \times [0,T]$. Consider an obstacle $\chi \in H^1(\mathcal{Q})$ such that $\chi \leq 0$ on Γ and nonempty convex sets

$$\mathcal{K}(t) := \{v \in H_0^1(\Omega) : v \geq \chi(t)\} \qquad \text{a.e. } t \in [0,T], \tag{3.2}$$

where the Sobolev space $H_0^1(\Omega)$ is the usual Hilbert space of functions on Ω, vanishing on $\partial\Omega$ with bounded first derivatives in $L^2(\Omega)$. Consider the linear operator

$$\mathcal{L}v := -\text{div}(\mu^2 \nabla v) + \boldsymbol{b} \cdot \nabla v + cv \tag{3.3}$$

with coefficients $\mu^2 \in L^\infty(\Omega)$, $\boldsymbol{b} \in [W^{1,\infty}(\Omega)]^d$, $c \in L^\infty(\Omega)$, and

$$\mu^2(x) \geq \mu_0^2 > 0, \quad \|\boldsymbol{b}\|_{L^2(\Omega)} \leq b_0, \quad d(x) := -\frac{1}{2}\text{div}\boldsymbol{b}(x) + c(x) \geq d_0^2 \geq 0 \text{ a.e. } x \in \Omega,$$

for some positive constants μ_0, b_0, d_0.

We then consider the following parabolic variational inequality or obstacle problem:

Problem 2 (Variational Formulation: Continuous Version) *Given the data $f \in L^1(0,T;L^2(\Omega))$, $u_0 \in L^2(\Omega)$ and the convex sets, $\mathcal{K}(t)$, defined in* (3.2)*, find a* weak *solution, $u \in C([0,T];\mathcal{K}(\cdot))$, of*

$$\langle \partial_t u + \mathcal{L}u, u - v\rangle \;\leq\; \langle f, u - v\rangle, \quad \forall v \in \mathcal{K}(t) \;\; a.e. \;\; t \in [0,T] \tag{3.4}$$

with initial condition $u(0,x) = u_0(x)$ in Ω and boundary condition $u(t,x) = 0$ on Γ for a.e. $t \in [0,T]$. If $f \in L^2(0,T;L^2(\Omega))$, then $u \in H^1_{loc}(0,T;L^2(\Omega))$ is a strong *solution of (3.4) (see [4]).*

The existence and regularity results have been presented in the book by Brézis [4]. For a quick review of the numerical treatments of parabolic variational inequalities, we refer to the book by Glowinski [12]. As mentioned before, we consider the finite difference method in time and the finite element method in space for this time-dependent problem.

4 A Numerical Strategy for Variational Inequalities

In [5], Broadie and Detemple compared the efficiencies and accuracies of several existing numerical methods for valuing American options. The authors list several criteria for a good method in practice, like speed, accuracy, simplicity of implementation, ease of adaptability to other types of options, economic insights offered by the method, and availability of derivative information, etc., which could serve as the guideline of future research on numerical methods for valuing American options. In this note, we are going to consider backward Euler scheme in time and piecewise linear finite element method in space with space-time adaptivity:

We discretize the spatial domain Ω into simplices $S \in \mathcal{T}_h^n$, and partition the time domain $[0,T]$ into N subintervals, i.e. $0 = t_0 < t_1 < \cdots < t_N = T$. We denote the maximum time step and maximum space meshsize by τ and h, respectively. Let V_h^n be the usual conforming linear finite element subspace over the mesh $\mathcal{T}_h^n$ at time $t = t_n$ and $I^n : H_0^1(\Omega) \to V_h^n$ is the interpolation operator onto V_h^n. Consider the corresponding discrete convex set at time $t = t_n$

$$\mathcal{K}_h^n := \{v \in V_h^n \cap H_0^1(\Omega) : v \geq \chi_h^n\} \tag{4.1}$$

where the nth element of the sequence $\{\chi_h^n \in V_h^n\}_{n=0}^N$ is a piecewise linear approximation of the obstacle χ at time t_n.

Problem 3 (Variational Formulation: Discrete Version) *Given the data $\{F^n\}_{n=1}^N \subset L^2(\Omega)$, which is an approximation of f, and initial guess $U_h^0 \in \mathcal{K}_h^0$, find an approximate solution $U_h^n \in \mathcal{K}_h^n$ for $1 \leq n \leq N$ such that*

$$\langle \frac{U_h^n - I^n U_h^{n-1}}{\tau_n} + b \cdot \nabla U_h^n + cU_h^n - F^n, U_h^n - v_h\rangle + \langle \mu^2 \nabla U_h^n, \nabla(U_h^n - v_h)\rangle \leq 0 \quad \forall v_h \in \mathcal{K}_h^n, \tag{4.2}$$

with homogenous boundary conditions.

The finite element method (FEM) has a long history in practical use and is widely applied to lots of physics and engineering problems (see [19]). It has been proved to be very successful in many areas, like structural mechanics and computational fluid dynamics. Many researchers have applied the finite element method for parabolic obstacle problem, but relatively few papers [13, 22, 2] deal with the *a priori* error analysis. The main difficulties in the error analysis for parabolic obstacle problems are handling the unilateral constraint and the lack of regularity: solutions do not possess high regularity even if all the data are smooth (see [4]). Note that, for the American option pricing problem, the obstacle χ and initial condition u_0 are only in $H^{\frac{3}{2}-\varepsilon}(\Omega)$.

5 Adaptive Mesh Refinement

Adaptive mesh refinement is an important tool to deal with the multiscale phenomena and to reduce the size of the linear systems that arise from the finite element method. We refer to the book by Verfürth [21] for a review of a posteriori error estimates and adaptive techniques for elliptic PDE's.

Generally, the adaptive FEM for static problems will generate graded meshes and iterations in the form

$$\boxed{\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE/COARSEN.}} \tag{5.3}$$

The ESTIMATE procedure determines in which part of the domain the error is still big. Since we can not compute the exact error of the solution, we need to find some kind of computable local error estimator to estimate the local error of the approximate solution we get from the SOLVE part. To design a good adaptive finite element method, finding a reliable and efficient *a posteriori* error estimator is essential and is one of the main goals of our future study. Recently, a lot of efforts have been made for a posteriori error analysis for elliptic variational inequalities [1, 6, 20, 10, 17, 18]; but for parabolic variational inequalities, to the best of our knowledge, there is nothing has been done for a posteriori error estimates, except for [16], in which the authors study the time discretization of a general class of evolution problems.

Inspired by the work of Fierro and Veeser [7], Nochetto, Siebert and Veeser [18] present a fully localized a posteriori error estimator for elliptic obstacle problems and the numerical experiments have demonstrated its advantages. The idea of localization, which is originally introduced in [7], is that the error estimator should vanish in the contact set except for the obstacle resolution. This idea can also be applied to the American option pricing problem to improve efficiency.

6 List of Open Questions

We conclude this note with a list of open questions which require further study:

- To the best of our knowledge, the best result in terms of a priori error estimate for the numerical method we use is the work by Johnson [13] back in 1976, in which the author proves a sub-optimal convergence rate in time $O((\log\tau^{-1})^{1/4}\tau^{3/4}+h)$ for the $L^2(0,T;H^1(\Omega))$ error, under the assumption of the initial solution $u_0 \in W^{2,\infty}(\Omega)$. In [16], the authors prove the optimal convergence rate in time $O(\tau)$ for the error in the $L^2(0,T;H^1(\Omega))$ norm for time-discrete problem. The optimal convergence rate of the fully-discrete problem has been observed in many numerical experiments, but can we prove this theocratically?
- In [18], the numerical experiments have demonstrated the advantages of the fully localized error estimator for elliptic obstacle problems. The idea of localization is that the error estimator should vanish in the contact set except for the obstacle resolution. Can we generalize this idea and get a fully localized error estimator for parabolic obstacle problems?
- In practice, to determine the early exercise boundary is as important as to evaluate the option premium. Interesting questions to ask include how to approximate the exercise boundary and how good the numerical approximation is.

- In each iteration of the adaptive FEM, we need to solve a finite-dimensional linear complementarity problem. There are lots of general purpose discrete LCP solvers (we refer to the monograph [8]), which have been applied to price American options. An efficient discrete LCP solver is desirable, along with suitable analysis of its performance.
- The other interesting question to consider is the convergence of the adaptive method for this time-dependent problems.

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