Sherry E. Scott (Aug 2000) Advisor: J. Benedetto

Spectral Analysis of Fractal Noise in Terms of Wiener's Generalized Harmonic Analysis and Wavelet Theory

Wiener's Generalized Harmonic Analysis (GHA) extends harmonic analysis to functions not accessible to the L^1, L^2, and Fourier series theories. Even with this extension, the spectral analysis of a large class of functions, e. g., nonstationary processes and in particular, fractal noise, is not completely resolved. Using Wiener's work as a starting point, we apply harmonic analysis, distribution theory, and wavelet theory to adapt GHA to these functions.

Wiener defined the deterministic autocorrelation of a function on the real line where the analogous probability concept is the stochastic autocorrelation of a random process. The deterministic autocorrelation R_f of a function $f : R \to C$ is defined as

$$R_f(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t+u)\overline{f(u)}\, du. \quad (1)$$

The right side of (1) is the mean of the convolution $f * \overline{f}$, where $\overline{f}$ is the involution defined as $\overline{f}(t) = \overline{f(-t)}$. If the limit in (1) exists, the Herglotz-Bochner Theorem implies that there is a bounded Radon measure $\mathcal{S}$ such that the Fourier transform of R_f equals $\mathcal{S}$. In this setting, $\mathcal{S}$ is called the power spectrum of R_f. Conversely, the Wiener-Wintner theorem asserts that for a given positive bounded measure $\mathcal{S}$, there exists a function f such that the Fourier transform of the autocorrelation R_f of f is equal to $\mathcal{S}$.

My dissertation considers an extension of this relationship between f and its power spectrum to functions whose spectra behave like $\frac{1}{|\gamma|^k}$, where $0 < k < 1$. For $0 < k < 1$, $\frac{1}{|\gamma|^k}$ is a positive measure but is not bounded. Thus classical GHA and in particular, the Wiener-Wintner theorem do not apply. This $\frac{1}{f}$-type behavior occurs in the spectra of a family of processes commonly referred to as $\frac{1}{f}$ processes. As the spectral analysis of such processes has been an area of extensive statistically-based research, we first consider the statistical perspective of our problem by reviewing three relatively recent contributions to the area. We then turn to our deterministic approach where, specifically, the task is to find a function f such that

$$\widehat{R}_f(\gamma) = \frac{1}{|\gamma|^k} \quad (2)$$

We give a solution to (2) which generalizes techniques introduced by Wiener and Wintner[17] and extended to higher dimensions by Benedetto and Kerby [2,12]. We also briefly discuss the extension of our solution to higher dimensions. From a theoretical point of view, our work extends the Wiener-Wintner theorem to spectra $\mathcal{S}$ which are not positive bounded Radon measures, and in so doing, validates equation (2), a formula analogous to (1), but in a more general setting. From an applications point of view, our result provides information from a mathematical and deterministic perspective about the structure of signals which give rise to fractal noise, the widely occurring physical phenomena characterized by the $\frac{1}{|\gamma|^k}$ shape of its measured spectra, where here the term fractal refers to the spectra's power law decay of fractional order, e.g., see [9].