STAT 650 Take home exam March 9 - 16, 2017

1. [20+25] Any integer n is either a multiple of 3 in which case $n = 3m$ for some integer m or has a form $n = 3m + 1$ or $n = 3m + 2$.
Let $\{X(t),\ t \geq 0\}$ be a pure birth continuous time Markov chain. Assume that

$$P\{\text{an event occurs in } (t, t+h) | X(t) = 3m\} = \lambda_0 h + o(h)$$
$$P\{\text{an event occurs in } (t, t+h) | X(t) = 3m+1\} = \lambda_1 h + o(h)$$
$$P\{\text{an event occurs in } (t, t+h) | X(t) = 3m+2\} = \lambda_2 h + o(h)$$

and the probabilities that more than one event occurs are $o(h),\ h \to 0$.

(i) Derive the differential equations for

$$P_0(t) = P\{X(t) = 3m \text{ for some } m\}$$
$$P_1(t) = P\{X(t) = 3m+1 \text{ for some } m\}$$
$$P_2(t) = P\{X(t) = 3m+2 \text{ for some } m\}.$$

.
(ii) Find the stationary distribution $\{p_0 = \lim_{t\to\infty} P_0(t),\ p_1,\ p_2\}$.

2. [20] Let $\{X_i(t),:\ \ t \geq 0\}$ be a Poisson process with $X_i(0) = 0$ and rate $\lambda_i,\ i = 1, 2$.
Prove that if $\lambda_1 > \lambda_2$, then for any $N \geq 1$ and any $t > 0$

$$P\{X_1(t) \geq N\} > P\{X_2(t) \geq N\}.$$

3. [15] Let $\{X_n,\ n = 0, 1, 2, \ldots\}$ be a discrete time Markov chain with a finite state space S and a transition matrix P.
Show that if the chain is irreducible with one aperiodic recurrent class, then for some m all the elements of P^m are positive.

4. [10+10] Let $\{X(t),\ Y(t)\}$ be two independent Poisson processes.
(i) Prove or disprove that $Z(t) = \sqrt{X(t) + Y(t)}$ is a Markov process.
(ii) Set $V(t) = |X(t) - Y(t)|$. Find if the distribution of $V(t+h)$ is determined by the value of $V(t)$ for small $h,\ h \to 0$. Prove or disprove that $V(t)$ is a Markov process.