Homework #5 October 25 - 30, 2007

1. Let $(x_1, \ldots, x_n)$ be a sample from a population with finite variance σ^2.
(i) Show that the sample variance $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$is an unbiased estimator of σ^2.
(ii) Prove that s^2 is a consistent estimator of σ^2, i. e., s^2 converges in probability to σ^2 as $n \to \infty$, and s is a consistent estimator of σ.

2. Let Y_1, $Y_2, \ldots$ be independent observations with $E(Y_i) = \mu$, $\mathrm{var}(Y_i) = \sigma_i^2$.
(i) Find the minimum variance linear unbiased estimator of μ from the first n observations and compare its variance with that of $\bar{Y} = (Y_1 + \ldots + Y_n)/n$.
(ii) Assuming additionally that all Y_i have normal distribution, find a necessary and sufficient condition for consistency of the minimum variance linear unbiased estimator. Is it possible that this estimator is consistent while $\bar{Y}$ is not?

3. Let $(x_1, \ldots, x_n)$ be a sample from a population with density function

$$f(x;\theta) = (\theta + 1)x^{\theta},\ 0 < x < 1$$

with $\theta > 0$ as a parameter.
Find the maximum likelihood estimator of θ and prove its consistency.

4. Let ϵ_1, ϵ_2, ϵ_3 be independent standard normal random variables and

$$Y_1 = \epsilon_1,\ Y_2 = 2\epsilon_1 + \epsilon_2,\ Y_3 = 3\epsilon_1 + 2\epsilon_2 + \epsilon_3.$$

(i) Find $E(Y_3|Y_1,\ Y_2)$ and $E(Y_3|Y_2)$.
(ii) Which of the two variances is bigger, $\mathrm{var}\{E(Y_3|Y_1, Y_2)\}$ or $\mathrm{var}\{E(Y_3|Y_2)\}$?